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MEASURE ALGEBRAS OF SEMILATTICES WITH FINITE BREADTH

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The main result of this paper is that if S is a locally compact semilattice of finite breadth, then every complex homomorphism of the measure algebra $M(S)$ is given by integration over a Borel filter (subsemilattice whose complement is an ideal), and that consequently $M(S)$ is a P-algebra in the sense of S. E. Newman. More generally it is shown that if S is a locally compact Lawson semilattice which has the property that every bounded regular Borel measure is concentrated on a Borel set which is the countable union of compact finite breadth subsemilattices, then $M(S)$ is a P-algebra. Furthermore, complete descriptions of the maximal ideal space of $M(S)$ and the structure semigroup of $M(S)$ are given in terms of S, and the idempotent and invertible measures in $M(S)$ are identified.

In earlier work Baartz and Newman have shown that if S is the finite product of totally ordered locally compact semilattices, then every complex homomorphism is given by integration over a Borel subsemilattice whose complement is an ideal [1, Th. 3.15], and consequently, the structure semigroup of $M(S)$ in the sense of Taylor [10] is itself a semilattice [9, Th. 3]. In both papers it is shown that such results do not hold for the infinite dimensional cube $S = I^{\omega}$, and Newman conjectures that what is needed for these results to hold is a "finite dimensionality" condition. In this paper it is shown that these results hold provided the locally compact semilattice in question has "finite breadth"; i.e., satisfies a finite dimensionality condition familiar from the theory of compact semilattices.

The paper is organized as follows. Section 1 contains generalities on semilattices and the notion of breadth. Section 2 is devoted to the proof of our main result for finite breadth semilattices. In § 3 we discuss the extension of these results to a more general setting and give examples to show how our hypotheses differ from those of Newman.

1. Semilattices.

A semilattice is a commutative idempotent semigroup. We may also (equivalently) describe a semilattice as a partially ordered set in which every two elements have a greatest lower bound. Thus the product of two elements is their greatest lower bound. The reader should note that this convention differs from that in [1, 9], in which the product of two elements in a semilattice is viewed as their least upper bound.

A semilattice on a Hausdorff space S is a topological (semitopological) semilattice if the multiplication function which sends (x, y) to xy from $S \times S$ to S is jointly (separately) continuous. It is known that a compact semitopological semilattice is actually a topological semilattice [8].

If S is a semilattice and $A \subset S$ we define the upper and lower sets of A by

$$\uparrow A = \{y \in S: x \leqq y \quad \text{for some} \quad x \in A\}$$

and

$$\downarrow A = \{z \in S: z \leqq x \quad \text{for some} \quad x \in A\}\ .$$

For singleton sets we adopt the notation $\uparrow x$ and $\downarrow x$ instead of $\uparrow \{x\}$ and $\downarrow \{x\}$. We call a subset I of S an *ideal* if $\downarrow I = I$. Equivalently I is an ideal if $x \in S$, $y \in I$ implies $xy \in I$. A subset F of S is a *filter* if $\uparrow F = F$ and F is a subsemilattice of S. Note that a subsemilattice $F \subseteqq S$ is a filter if and only if $S \backslash F$ is an ideal of S.

If A is a nonempty subset of a semilattice S we denote the greater lower bound of A by $\wedge A$ ($\wedge A$ exists for all finite sets A, and also for all infinite sets if S is a compact topological semilattice). A fiinite set A is said to be *meet-irredundant* if $\wedge A < \wedge B$ for any proper subset B of A. A semilattice S is said to have *breadth* n (denoted $\mathrm{br}(S) = n$) if n is the greatest cardinality of the meet-irredundant subsets of S. Equivalently S has breadth n if and only if n is the smallest integer such that any finite subset J of S of cardinality $m > n$ has a subset L of cardinality n such that $\wedge J = \wedge L$, and this is equivalent to n being the smallest integer such that any finite subset J of cardinality $n + 1$ has a subset L of cardinality n such that $\wedge J = \wedge L$. We adopt the convention that a singleton semilattice has breadth 0.

A subset A of a semilattice S is *bounded above* if there exists $p \in S$ such that $p \geqq a$ for all $a \in A$. The bounded breadth of S is n (denoted $\mathrm{bbr}(S) = n$) if n is the greatest cardinality of any meet-irredundant set bounded above.

PROPOSITION 1.1. *Let S be a semilattice of finite breadth. Then* $\mathrm{bbr}(S) \leqq \mathrm{br}(S) \leqq \mathrm{bbr}(S) + 1$. *If S has an identity, then* $\mathrm{bbr}(S) = \mathrm{br}(S)$.

Proof. Clearly $\mathrm{bbr}(S) \leqq \mathrm{br}(S)$ and the two agree if S has an identity (since in the latter case every set is bounded above). If $\{x_1, \cdots, x_n\}$ is a meet-irredundant set in S, let $y_i = x_i x_n$ for $i = 1, \cdots, n-1$. Then it is straightforward to verify that $\{y_1, \cdots, y_{n-1}\}$ is a meet-irredundant set bounded by x_n. Hence it follows that $\mathrm{br}(S) \leqq \mathrm{bbr}(S) + 1$.

PROPOSITION 1.2. *Let S be a topological semilattice such that $\mathrm{bbr}(S) \leqq n$, where $n \geqq 1$. If I is a dense ideal of S, then for any subsemilattice T contained in $S \backslash I$,* $\mathrm{bbr}(T) \leqq n - 1$.

Proof. Let $x_1, \cdots, x_n \in T$ and let $b \in T$ such that $x_i \leqq b$ for $i = 1, \cdots, n$, where $n \geqq 2$. Let y_α be a net in I converging to b. Then $z_\alpha = y_\alpha b$ is a net in I (since I is an ideal) converging to $bb = b$. Since $\mathrm{bbr}(S) \leqq n$ for each α there exists $u_\alpha \in \{x_1, \cdots, x_n, z_\alpha\} = F_\alpha$ such that $\wedge F_\alpha = \wedge (F_\alpha \backslash \{u_\alpha\})$. But $\wedge F_\alpha \in I$ since $z_\alpha \in I$; hence $u_\alpha \neq z_\alpha$ since $x_i \in T$ for $1 \leqq i \leqq n$ and T is a subsemilattice. By picking subnets and renaming, we may assume $u_\alpha = x_1$ for each α. Then $x_1 \cdots x_n = \lim x_1 \cdots x_n z_\alpha = \lim x_2 \cdots x_n z_\alpha = x_2 \cdots x_n$. Hence $\mathrm{bbr}\, T \leqq n-1$.

Now suppose $n = 1$. Let $x < b$ be two elements of T. Again let $\{z_\alpha\}$ be a net in I such that $z_\alpha \leqq b$ for all α and $z_\alpha \rightarrow b$. Since S has bounded breadth 1 and I is an ideal, we see that $z_\alpha < x$ for all α. Therefore $b = \lim z_\alpha \leqq x$, a contradiction. So $\mathrm{bbr}(T) = 0$.

PROPOSITION 1.3. *Let S be a topological semilattice and let $A = \{x_1, \cdots x_n\}$ be a meet-irredundant subset of S of cardinality n. Then there exist open sets $U_1, \cdots, U_n$ such that $x_j \in U_j$ for $j = 1, \cdots, n$ and if $y_j \in U_j$ for $j = 1, \cdots, n$, then $\{y_1, \cdots, y_n\}$ is a meet-irredundant set of distinct elements.*

Proof. Suppose not. Then there exists a net $(y_{1\alpha}, \cdots, y_{n\alpha})$ converging to $(x_1, \cdots, x_n)$ in $\prod_{j=1}^n S$ such that for each α, there exists $i, 1 \leqq i \leqq n$, such that $\bigwedge_{j=1}^n y_{j\alpha} = \wedge \{y_{j\alpha}: 1 \leqq j \leqq n,\ j \neq i\}$. By picking subnets and renumbering if necessary, we may assume that $y_{1\alpha}$ is always the omitted. Then $\bigwedge_{j=1}^n x_j = \lim \bigwedge_{j=1}^n y_{j\alpha} = \lim \bigwedge_{j=2}^n y_{j\alpha} = \bigwedge_{j=2}^n x_j$. However, this conclusion contradicts the hypothesis that A is meet-irredundant.

In the following T^* denotes the closure of T.

COROLLARY 1.4. *Let T be a subsemilattice of finite breadth of a topological semilattice S. Then* $\mathrm{br}(T) = \mathrm{br}(T^*)$.

Proof. Suppose $\mathrm{br}(T^*) = n$. Then there exists a meet-irredundant set $\{x_1, \cdots, x_n\}$ of cardinality n in T^*. By Proposition 1.3 there exist open sets $U_1, \cdots, U_n$ with $x_j \in U_j$ for $1 \leqq j \leqq n$ such that if y_j is chosen in $U_j \cap T$, then $\{y_1, \cdots, y_n\}$ is meet-irredundant. Hence $\operatorname{br}(T) \geqq n$. But since $T \subset T^*$, $\operatorname{br}(T) \leqq n$.

Let X be a compact Hausdorff space and let $P(X)$ denote the set of nonempty compact subsets of X. It is well known that $P(X)$ is a compact Hausdorff space when endowed with the topology of open sets generated by the subbasis

$$N(U, V) = \{A \in P(X): A \subset U \quad \text{and} \quad A \cap V \neq \emptyset\}$$

where U and V are arbitrary open subsets of X. A net K_α of compact subsets of of X converges to $K \in P(X)$ if and only if $K = \limsup K_\alpha = \liminf K_\alpha$. We call this topology the Vietoris topology.

PROPOSITION 1.5. *Let K_α converge to K in $P(S)$ where S is a compact semilattice. If each K_α is a compact topological subsemilattice of S such that $\operatorname{bbr}(K_\alpha) \leqq n$, then K is also a compact subsemilattice and $\operatorname{bbr}(K) \leqq n$. Hence the collection of compact subsemilattices of bounded breadth less than or equal to n is a closed subset of $P(S)$.*

Proof. It is known that $P(S)$ endowed with the operation $AB = \{ab: a \in A,\ b \in B\}$ is a compact topological semigroup [4]. Since K_α converges to K, and $K_\alpha K_\alpha = K_\alpha$ for each α, by continuity $KK = K$, i.e., K is a subsemilattice.

Suppose $\operatorname{bbr}(K) > n$. Then there exists a meet-irredundant set $\{y_1, \cdots, y_{n+1}\}$ of distinct elements in K and a $p \in K$ such that $y_j \leqq p$ for $1 \leqq j \leqq n+1$. By Proposition 1.3 there exist open sets $U_1, \cdots, U_{n+1}$ with $y_j \in U_j$ for all j such that if a point is chosen from each U_j, the set of elements obtained is meet-irredundant. Pick by continuity of multiplication an open set V, $p \in V$, and open sets $V_1, \cdots, V_{n+1}$, $y_j \in V_j$ for $1 \leqq j \leqq n+1$, such that $VV_j \subset U_j$. By the definition of the Vietoris topology, there exists K_α such that $K_\alpha \cap V \neq \emptyset$ and $K_\alpha \cap V_j \neq \emptyset$ for $1 \leqq j \leqq n+1$. Choose $q \in K_\alpha \cap V$ and $w_j \in K_\alpha \cap V_j$. Then $z_j = qw_j \in K_\alpha \cap U_j$ for $j = 1, \cdots, n+1$. Now $\{z_1, \cdots, z_{n+1}\}$ is a meet-irredundant set and it is bounded in K_α by q. This is in contradiction to the hypothesis that $\operatorname{bbr}(K_\alpha) \leqq n$.

PROPOSITION 1.6. *Let S be a compact topological semilattice. Then $\{\downarrow s: s \in S\}$ is a closed subset of $P(S)$.*

Proof. The set of all singletons $\{\{s\}: s \in S\}$ is homeomorphic to S and hence a compact subset of $P(S)$. Since $\downarrow s = Ss$, we have that $\{\downarrow s: s \in S\}$ is simply a translate of the compact set of singletons in the topological semigroup $P(S)$, and hence is compact.

A subset A of a topological space X is said to be *locally closed* if A is the intersection of an open set and a closed set. Equivalently A is locally closed if it is open in its closure.

PROPOSITION 1.7. *Let S be a topological semilattice of finite breadth. Then any dense filter in S is open. Hence every filter in S is locally closed.*

Proof. We first assume S has an identity. Let F be a dense filter in S. If 1 is not in the interior of F, then there exists a net x_α in the ideal $I = S\backslash F$ converging to 1. Since for any $y \in S$, $yx_\alpha \in I$ and yx_α converges to $y1 = y$, I is dense in S. Hence by Propositions 1.1 and 1.2, $\mathrm{br}(F) < \mathrm{br}(S)$. But by Corollary 1.4 $\mathrm{br}(F) = \mathrm{br}(S)$. Hence it must be the case that 1 is in the interior of F.

We now drop the assumption that S has an identity and let F be a dense filter in S. If $x \in F$ is not in the interior of F, then there exists a net x_α in the complement of F converging to x. Then also the net $y_\alpha = xx_\alpha$ is not in F and converges to x.

Since F is a filter and $x \in F$, $xF = \downarrow x \cap F$. Since F is dense in S, xF is dense in $xS = \downarrow x$. Hence $\downarrow x \cap F$ is a dense filter in the subsemilattice $\downarrow x$ which has x for an identity. Hence by the first part of the proof x is in the interior of $\downarrow x \cap F$ in $\downarrow x$. But the net y_α converges to x in $\downarrow x$ and is not in $\downarrow x \cap F$. This contradiction implies that x must have been in the interior of F in S. Hence F is open.

Since any filter in S is a dense filter in its closure, the last statement of the proposition follows from what has just been proved.

We conclude this section with some remarks about compact 0-dimensional semilattices and discrete semilattices. Let $\boldsymbol{S}$ be the category of discrete semilattice monoids and identity preserving semilattice morphisms, and $\boldsymbol{Z}$ the category of compact 0-dimensional semilattice monoids and continuous identity preserving semilattice morphisms. Then, clearly $2 = \{0, 1\}$, the unique two point semilattice, is both an $\boldsymbol{S}$-object and a $\boldsymbol{Z}$-object. Moreover, as is described at great length in [3], $\boldsymbol{S}$ and $\boldsymbol{Z}$ are dual categories under the functors $D: \boldsymbol{S} \to \boldsymbol{Z}^{op}$ and $E: \boldsymbol{Z}^{op} \to \boldsymbol{S}$ given by $D(S) = \boldsymbol{S}(S, 2)(=\hat{S})$ and $E(T) = \boldsymbol{Z}(T, 2)(=\hat{T})$, and their obvious extension to the morphisms. Thus, for any $\boldsymbol{Z}$-object T, the morphisms $\boldsymbol{Z}(T, 2)$ separate the points, and, in fact, $T \simeq \hat{\hat{T}}$.

DEFINITION 1.8. If $T \in \boldsymbol{Z}$, then $k \in T$ is a local minimum in T

if $\uparrow k$ is open in T. $K(T)$ denotes the set of all local minima of the semilattice T.

If $f: T \to 2$ is a $\boldsymbol{Z}$-morphism then, $f^{-1}(1)$ is a clopen subsemilattice of T, and so it has a minimum, k. Moreover, since $f^{-1}(1)$ is a filter, $\uparrow k = f^{-1}(1)$. Thus, $f = \chi_{\uparrow k}$, the characteristic function of $\uparrow k$. Clearly, $\chi_{\uparrow k} \in \boldsymbol{Z}(T, 2)$ for any $k \in K(T)$, and so $\hat{T} = \{\chi_{\uparrow k}: k \in K(T)\}$. Moreover, if $k_1, k_2 \in K(T)$, then $\chi_{\uparrow k_1} \cdot \chi_{\uparrow k_2} \in \hat{T}$, and $\chi_{\uparrow k_1} \cdot \chi_{\uparrow k_2} = \chi_{\uparrow k_3}$, where $k_3 = k_1 \vee k_2$. Thus $(K(T), \vee)$ is a semilattice with 0 as identity, and this semilattice is isomophic to $\hat{T}$.

Conversely, if $S \in \boldsymbol{S}$, then clearly $\chi_{\uparrow s} \in \hat{S}$ for each $s \in S$. However, for $s_1, s_2 \in S$, $s_1 \vee s_2$ may not be defined in S. Thus, there are more semicharacters in $\hat{S}$ than just those generated by some $s \in S$. However, if $f \in \hat{S}$, then $f^{-1}(1)$ is a filter on S, and for $f, g \in \hat{S}$, $f \cdot g = \chi_F$, where $F = f^{-1}(1) \cap g^{-1}(1)$. Thus, if $(\mathscr{F}(S), \cap)$ is the semilattice of all filters on S under intersection, then $(\mathscr{F}(S), \cap) \simeq \hat{S}$ (algebraically), and so we topologize $\mathscr{F}(S)$ with the topology from $\hat{S} \subseteq 2^S$. Therefore, if $S \in \boldsymbol{S}$, we can refer to $\hat{S}$ as the filter semilattice on S.

If S is a compact 0-dimensional semilattice, then S^1, the semilattice S with an identity adjoined as an isolated point, is clearly a $\boldsymbol{Z}$-object. Similarly, if S is a discrete semilattice, then $S^1 \in \boldsymbol{S}$. Moreover, for a semilattice S (discrete or compact 0-dimensional) $\hat{S}^1 = \hat{S} \cup \{\chi_{\{1\}}\}$, so the structure of $\hat{S}^1$ is completely determined by that of $\hat{S}$.

2. Locally compact semilattices with finite breadth. In this section, S is a locally compact semilattice with finite breadth, and $M(S)$ is the Banach algebra of all bounded Borel measures on S under convolution. We will show that for every complex homomorphism h of $M(S)$, there is a filter $F \subset S$ such that $h(\mu) = \mu(F)$ for all $\mu \in M(S)$. Recall that a semicharacter of a semigroup is a homomorphism of the semigroup into the unit disk in $\boldsymbol{C}$ under multiplication. Since the semicharacters of S are precisely the characteristic functions of the filters in S, we will have shown that every homomorphism of $M(S)$ is given by integration against a semicharacter.

We begin with a simple measure-theoretic lemma.

LEMMA 2.1. *Let $\mathscr{F}$ be a family of Borel sets on the locally compact space X. Let μ be a positive bounded Borel measure on X. Then $\mu = \mu_0 + \sum_{n=1}^{\infty} \mu_n$, where $\mu_0(F) = 0$ for all $F \in \mathscr{F}$ and each $\mu_n (n \geqq 1)$ is concentrated on some $F_n \in \mathscr{F}$. Moreover, $\mu_0, \mu_1, \mu_2, \cdots$ are pairwise mutually singular positive bounded Borel measures. Finally, μ_0 and $\mu - \mu_0$ are uniquely determined.*

Proof. Let $l = \sup\{\mu(F) \mid F \in \mathscr{F}\}$; then $l < \infty$. Choose $F_1 \in$

$\mathscr{F} \ni \mu(F_1) \geqq 1/2\, l$. Let $\mu_1 = \mu \mid F_1$ and $\nu_1 = \mu | F_1^c$. Let $l_1 = \sup\{v_1(F) | F \in \mathscr{F}\}$. For each $n > 1$ choose $F_n \in \mathscr{F} \ni \nu_{n-1}(F_n) \geqq 1/2\, l_{n-1}$. Let $\mu_n = v_{n-1} \mid F_n$ and $\nu_n = \nu_{n-1} \mid F_n^c$. Let $l_n = \sup\{\nu_n(F) \mid F \in \mathscr{F}\}$. This defines inductively the sequences $\{\mu_n\}$, $\{v_n\}$, and $\{l_n\}$. Note $\mu \geqq \nu_1 \geqq \nu_2 \geqq \cdots \geqq \nu_n \geqq \nu_{n+1} \geqq \cdots \geqq 0$ and in fact $\mu = \nu_N + \sum_{n=1}^{N} \mu_n$ for all N. We also have $l \geqq l_1 \geqq l_2 \geqq \cdots \geqq l_n \geqq l_{n+1} \geqq \cdots$ for all n. Note that the μ_n are pairwise mutually singular, since if $m < n$, $\mu_n \leqq \nu_m \perp \nu_n$. Now the μ_n are mutually singular and dominated by μ, so $\sum_{n=1}^{\infty} \mu_n$ exists and is dominated by μ. Set $\mu_0 = \mu - \sum_{n=1}^{\infty} \mu_n$. Clearly $\mu_0 \leqq \nu_N$ for all N, and so for each $F \in \mathscr{F}$, $\mu_0(F) \leqq \nu_N(F) \leqq l_N \leqq 2\nu_N(F_{N+1}) = 2\|\mu_{N+1}\| \longrightarrow 0$ as $N \longrightarrow \infty$ since $\sum_{n=1}^{\infty} \|\mu_n\| = \|\sum_{n=1}^{\infty} \mu_n\| < \infty$. Thus $\mu_0(F) = 0$ for all $F \in \mathscr{F}$, and clearly each μ_n is concentrated on F_n.

It remains to check the uniqueness. Let L be the closed linear span of all bounded Borel measures concentrated on some element of $\mathscr{F}$. Clearly L is an L-subspace of $M(X)$ in the sense of [10], and so by the Lebesgue decomposition theorem $M(S) = L \oplus L^\perp$, where $L^\perp = \{\eta \mid \eta \perp \nu$ for all $\nu \in L\}$. Since $\mu_0 \in L^\perp$ and $\mu - \mu_0 \in L$, the uniqueness follows.

We now consider a compact semilattice S.

PROPOSITION 2.2. *Let S be a compact semilattice, and let $\mathscr{F}$ be a family of compact subsets of S such that $\mathscr{F}$ is a closed subset of $P(S)$. If μ is a probability measure on S such that $\mu(F) = 0$ for all $F \in \mathscr{F}$, then there exists a metric quotient $f: S \longrightarrow S'$ of S such that*

$$\mu(f^{-1}(f(F))) = 0 \quad \textit{for all} \quad F \in \mathscr{F}\ .$$

Proof. Let $\varepsilon > 0$. Suppose for each neighborhood $\mathscr{U}$ of the diagonal $\Delta \subset S \times S$, there exists $F_{\mathscr{U}} \in \mathscr{F}$ such that $\mu(\mathscr{U}[F_{\mathscr{U}}]) \geqq \varepsilon$ where

$$\mathscr{U}[A] = \{b \in S: (b, a) \in \mathscr{U} \quad \text{for some} \quad a \in A\}\ .$$

Since $\mathscr{F}$ is closed and hence compact in $P(S)$, some subnet F_α of the net $\{F_{\mathscr{U}}: \Delta \subset \text{interior}\ (\mathscr{U})\}$ converges to $F \in \mathscr{F}$. Since $\mu(F) = 0$, by outer regularity there exists an entourage $\mathscr{U}$ (i.e., a neighborhood of the diagonal) such that $\mu(\mathscr{U}[F]) < \varepsilon$. Pick an entourage $\mathscr{V}$ such that $\mathscr{V} \circ \mathscr{V} \subset \mathscr{U}$ and $\mathscr{V}$ is symmetric. Since $F_\alpha \longrightarrow F$, there exists a β such that $F_\beta \subset \mathscr{V}[F]$ and $\mu(\mathscr{V}[F_\beta]) \geqq \varepsilon$. Then

$$\varepsilon \leqq \mu(\mathscr{V}[F_\beta]) \leqq \mu(\mathscr{V} \circ \mathscr{V}[F]) \leqq \mu(\mathscr{U}[F]) < \varepsilon\ ,$$

a contradiction. Hence there exists an entourage $\mathscr{U}_\varepsilon$ such that $\mu(\mathscr{U}_\varepsilon[F]) < \varepsilon$ for all $F \in \mathscr{F}$.

Now using the uniform continuity of multiplication on S, we choose inductively for each n a compact entourage $\mathscr{U}_n$ satisfying

(1) $\Delta \wedge \mathscr{U}_n \subset \mathscr{U}_n$ (products taken coordinatewise),

(2) $\mathscr{U}_n = \mathscr{U}_n^{-1}$,

(3) $\mu(\mathscr{U}_n(F)) < 1/n$ for all $F \in \mathscr{F}$,

(4) $\mathscr{U}_n \circ \mathscr{U}_n \subset \mathscr{U}_{n-1}$.

It is now standard that $\rho = \cap \{\mathscr{U}_n : n \in \omega\}$ is a closed congruence on S (see e.g. [4], Proposition 8.6, p. 49) and $S' = S/\rho$ is metrizable. That $\mu(f^{-1}(f(F))) = 0$ for all $F \in \mathscr{F}$ follows easily from property (3).

PROPOSITION 2.3. *Let S be a compact semilattice of finite breadth. Suppose h is a complex homomorphism on $M(S)$ which annihilates the discrete measures. Then $h = 0$.*

Proof. Clearly it suffices to show $h(\mu) = 0$ for every probability measure μ. Let $n = \mathrm{bbr}(S)$. We argue by induction on n, and clearly it holds for $n = 0$. We show that if the proposition is valid for $\mathrm{bbr}(S) < n$, it is true for $\mathrm{bbr}(S) = n$. Let μ be a probability measure on S. Let $\mathscr{F}_1$ denote the collection of principal ideals $\downarrow s$, $s \in S$, let $\mathscr{F}_2$ denote the collection of compact subsemilattices of $bbr \leqq n-1$, and let $\mathscr{F} = \mathscr{F}_1 \cup \mathscr{F}_2$. By 2.1, $\mu = \mu_0 + \sum_{k=1}^{\infty} \mu_k$ where μ_0 annihilates every member of $\mathscr{F}$ and each μ_k, $k \geqq 1$, is concentrated on some member of $\mathscr{F}$.

If μ_k is concentrated on some member of $\mathscr{F}_2$, our inductive hypothesis gives $h(\mu_k) = 0$. If μ_k is concentrated on $\downarrow x_k$, then $\mu_k = \mu_k * \delta_{x_k}$ (the unit point mass at x_k), so $h(\mu_k) = h(\mu_k * \delta_{x_k}) = h(\mu_k) \cdot 0 = 0$. Thus we need only show that $h(\mu_0) = 0$, and so we assume without loss of generality that μ annihilates every member of $\mathscr{F}$.

By 2.1 we can write the convolution power $\mu^{n+2} = \nu_0 + \nu_1$ where ν_0 annihilates every member of $\mathscr{F}$ and ν_1 lives on a countable union of members of $\mathscr{F}$.

By Propositions 1.5 and 1.6 $\mathscr{F}_1$, $\mathscr{F}_2$ and hence $\mathscr{F}$ are closed in $P(S)$. Hence by 2.2 there are closed congruences ρ_1 and ρ_2 on S such that $S_1 = S/\rho_1$ and $S_2 = S/\rho_2$ are metrizable semilattices and $\mu(f_1^{-1}(f_1(F))) = 0$ for all $F \in \mathscr{F}_1$, $\nu_0(f_2^{-1}(f_2(F))) = 0$ for all $F \in \mathscr{F}_2$, where $f_1 : S \rightarrow S_1$ and $f_2 : S \rightarrow S_2$. Let $\rho = \rho_1 \cap \rho_2$, and $f : S \rightarrow S' = S/\rho$.

Since S' embeds as a subdirect product of $S_1 \times S_2$, S' is metrizable. Furthermore since $\rho \subset \rho_1$ and $\rho \subset \rho_2$, $\mu(f^{-1}(f(F))) = 0$ for all $F \in \mathscr{F}$ and $\nu_0(f^{-1}(f(F))) = 0$ for all $F \in \mathscr{F}$.

Choose a countable dense set $\{y_k\} \subset S'$; we have $I' = \bigcup_k \downarrow y_k$ is a dense Borel ideal of S'. We have for $y \in S'$ and the induced measure $f(\mu)$ on S' that

$$f(\mu)(\downarrow y) = \mu(f^{-1}(\downarrow y)) = \mu f^{-1}(f(\downarrow x)) = 0 \quad \text{where} \quad f(x) = y\ .$$

Similarly $f(\nu_0)(\downarrow y)=0$. Hence $f(\mu)(I')=0=f(\nu_0)(I')$. Thus $I=f^{-1}(I')$ is a Borel ideal of S such that $\mu(I) = f(\mu)(I') = 0 = f(\nu_0)(I') = \nu_0(I)$. Hence $\mu = \mu \mid (S\backslash I)$ and $\nu_0 = \nu_0 \mid (S\backslash I)$.

Let $R = S\backslash I$. If T is a compact subsemilattice of S contained in R, we claim $\mu(T) = 0$.

Note first of all that $f(T) = f(f^{-1}(f(T)) \cap I^*)$. One containment is obvious. Conversely let $y = f(t) \in f(T)$ where $t \in T$. Since $I' = f(I)$ is dense in S', there exists a net $\{y_\alpha\} \subset I'$ converging to y in S'. Pick $x_\alpha \in I$ such that $f(x_\alpha) = y_\alpha$. By compactness some subnet of x_α converges to $x \in I^*$. By continuity $f(x_\alpha)$ converges to $f(x) = y$. Hence $x \in f^{-1}(f(T)) \cap I^*$ and so $y \in f(f^{-1}(f(T)) \cap I^*)$.

Now $f^{-1}(f(T)) \cap I^* = P$ is a subsemilattice of S contained in $I^*\backslash I$ (since $T \cap I = \varnothing$). By Proposition 1.2 $\operatorname{bbr}(P) \leqq n-1$. Hence $P \in \mathscr{F}$, and thus $\mu(P) = 0$. Hence by the way S' was chosen $\mu(f^{-1}(f(P))) = 0$. But $f^{-1}(f(P)) = f^{-1}f(f^{-1}(f(T)) \cap I^*) = f^{-1}f(T) \supset T$. Hence $\mu(T) = 0$. Thus the claim is completed. Note in particular if $x \in S\backslash I$, then $\mu(\uparrow x) = 0$.

Now we claim $\mu^{n+2}(R) = 0$. In fact, for $1 \leqq i \leqq n+2$, let $E_i = \{(x_1, \cdots, x_{n+2}) \in R^{n+2} \mid x_i \geqq x_1 \cdots \hat{x}_i \cdots x_{n+2} \in R\}$, and let $F = \{(x_1, \cdots, x_{n+1}) \in R^{n+1} \mid x_1 \cdots x_{n+1} \in R\}$. (Here $\hat{x}_i$ means x_i is to be omitted.) Then using the Fubini Theorem we have

$$\begin{aligned}
(\mu \times \cdots \times \mu)(E_i) &= \int_{R^{n+2}} \chi_{E_i}(x_1, \cdots, x_{n+2}) d(\mu \times \cdots \times \mu)(x_1, \cdots, x_{n+2}) \\
&= \int_{R^{n+1}} \int_R \chi_{E_i}(x_1, \cdots, x_{n+2}) d\mu(x_i) d(\mu \times \cdots \times \mu)(x_1, \cdots, \hat{x}_i, \cdots, x_{n+2}) \\
&= \int_F \int_R \chi_{E_i}(x_1, \cdots, x_{n+2}) d\mu(x_i) d\mu \times \cdots \times \mu(x_1, \cdots, \hat{x}_i, \cdots, x_{n+2}) \\
&= \int_F \int_R \chi_{\uparrow x_1 \cdots \hat{x}_i \cdots x_{n+2}}(x_i) d\mu(x_i) d(\mu \times \cdots \times \mu)(x_1, \cdots, \hat{x}_i, \cdots, x_{n+2}) \\
&= \int_F \mu(\uparrow x_1 \cdots \hat{x}_i \cdots x_{n+2}) d(\mu \times \cdots \times \mu)(x_1, \cdots, \hat{x}_i, \cdots, x_{n+2}) = 0 .
\end{aligned}$$

Since $\operatorname{bbr}(S) = n$, $\operatorname{br}(S) \leqq n+1$ and so $\{(x_1, \cdots, x_{n+1}) \in R^{n+2} \mid x_1 \cdots x_{n+2} \in R\} \subset \bigcup_{i=1}^{n+2} E_i$. Thus $\mu^{n+2}(R) = (\mu \times \cdots \times \mu)(\{x_1, \cdots, x_{n+2}) \in R^{n+2} \mid x_1 \cdots x_{n+2} \in R\}) \leqq (\mu \times \cdots \times \mu)(\bigcup_{i=1}^{n+2} E_i) = 0$. So μ^{n+2} is concentrated on I. But $\nu_0(I) = 0$ and so $\mu^{n+2} = \nu_1$; i.e., μ^{n+2} lives on a countable union of elements of $\mathscr{F}$. It follows that $h(\mu^{n+2}) = 0$, whence $h(\mu) = 0$.

PROPOSITION 2.4. *Let S be a locally compact semilattice of finite breadth and suppose h is a complex homomorphism of $M(S)$ which annihilates the discrete measures. Then $h = 0$.*

Proof. We assume without loss of generality that μ is a probability measure on S, and show $h(\mu) = 0$. Let $\varepsilon > 0$, and choose a compact set $K \subset S$ such that $\mu(S \backslash K) < \varepsilon$. If n is the breadth of S, K^n is a compact subsemilattice of S and $\mu(S \backslash K^n) < \varepsilon$. By 2.3, since h annihilates every discrete measure in $M(K^n)$, $h(\mu | K^n) = 0$. So $|h(\mu)| = |h(\mu | S \backslash K^n)| \leqq \|\mu | S \backslash K^n\| < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, $h(\mu) = 0$.

THEOREM 2.5. *Let S be a locally compact semilattice of finite breadth. Let $h \in \Delta M(S)$. Let $F_h = \{s \in S \mid h(\delta_s) = 1\}$. Then F_h is a locally closed* (hence Borel) *filter, and for all $\mu \in M(S)$, $h(\mu) = \mu(F_h)$.*

Proof. Clearly F_h is a filter. By 1.7, F_h is locally closed, and hence Borel. We observe first that if $\mu \in M(S)$, $h(\mu) = h(\mu \mid F_h)$. Let $I_h = S \backslash F_h$, and let $K \subset I_h$ be an arbitrary compact subsemilattice. Then h annihilates every discrete measure living on K (since for $x \in K$, $h(\delta_x) = 0$) and so by 2.3, $h(\mu \mid K) = 0$. Using the regularity of μ we see that $h(\mu \mid I_h) = 0$. Thus $h(\mu) = h(\mu \mid F_h)$.

Now let $\varepsilon > 0$ and assume without loss of generality that μ is positive. Choose K a compact subset of F_h such that $\mu(F_h \backslash K) < \varepsilon$. Then if $n = br(S)$, K^n is a compact subsemilattice of F_h and $\mu(F_h \backslash K^n) < \varepsilon$. Let $k = \wedge K^n$. Then $k \in K^n$ and $h(\mu | K^n) = h(\delta_k) h(\mu | K^n) = h(\delta_k * \mu | K^n) = h(\mu(K^n)\delta_k) = \mu(K^n) h(\delta_k) = \mu(K^n)$. Hence $|h(\mu) - \mu(F_h)| \leqq |h(\mu) - h(\mu \mid F_h)| + |h(\mu | F_h) - h(\mu | K^n)| + |h(\mu | K^n) - \mu(K^n)| + |\mu(K^n) - \mu(F_h)| \leqq 2\mu(F_h \backslash K^n) < 2\varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have $h(\mu) = \mu(F_h)$.

If S is a locally compact semilattice of finite breadth, then so is S^1. Moreover, for any locally compact semigroup S, $M(S^1) \simeq M(S) \oplus C$, and so $\Delta M(S^1) \simeq \Delta M(S) \cup \{0\}$, where 0 is the 0-homomorphism of $M(S)$, and $\Delta M(S^1)$ is the one-point compactification of $\Delta M(S)$. Thus, if we determine the structure of $\Delta M(S^1)$, we have also determined the structure of $\Delta M(S)$, and conversely. The difference is that $\Delta M(S^1)$ is always a semigroup [10], whereas this is not true if S has no identity. Thus, throughout the rest of this section, we assume S is a locally compact semilattice with identity having finite breadth.

If S is such a semilattice, then according to Theorem 2.5, each complex homomorphism h of $M(S)$ is given by integration over some filter $F \subseteq S$. Thus, $\Delta M(S)$ is $\mathscr{F}(S)$, the set of all filters on S. Moreover, it is clear that the product of two homomorphisms of $M(S)$ corresponds to the intersection of their associated filters. Hence, algebraically, $\Delta M(S) \simeq (\mathscr{F}(S), \cap)$. Now, $\Delta M(S)$ is a semitopological semilattice in the weak topology [10]. But $\Delta M(S)$ is also compact and a semilattice, and so $\Delta M(S)$ is a compact topological semilattice in the weak topology [8].

From our discussion at the end of § 1, we know that $(\mathscr{F}(S), \cap) \simeq$

$\hat{S}_d$ is a compact 0-dimensional topological semilattice, where S_d is the semilattice S with the discrete topology. But, the topology on a compact semilattice is uniquely determined by the algebraic structure [7]. Hence, $\Delta M(S) \simeq \hat{S}_d$.

One of the key results in Taylor's work is the determination that, for any convolution measure algebra M, the so-called critical points in ΔM carry the cohomology of ΔM. A critical point is an element $x \in \Delta M$ such that $\uparrow x$ is open and $x \geqq 0$. For the semilattice $\Delta M(S)$, a critical point is what we referred to in § 1 as a local minimum. Hence the critical points of $\Delta M(S)$ are $K(\Delta M(S)) = K(\hat{S}_d) = K(\mathscr{F}(S), \cap)$. However, the critical points of $\mathscr{F}(S)$ are precisely the principal filters on S, i.e., the filters $F \subset S$ of the form $F = \uparrow s$ for some $s \in S$ [3]. Hence, $K(\Delta M(S)) = \{h: h^{-1}(1) = \uparrow s$ for some $s \in S\}$ is the set of critical points of $\Delta M(S)$. Identifying s with the principal filter $\uparrow s$ we have a natural correspondence between S and the critical points of ΔM.

For a semisimple convolution measure algebra M, Taylor defines the structure semigroup T of M to be the unique compact abelian monoid T such that there is an isometric L-isomorphism $f: M \rightarrow M(T)$ such that $f(M)$ is weak *-dense in $M(T)$, each complex homomorphism on M is given by integration against some semicharacter $h \in \hat{T}$, and $\hat{T}$ separates the points of T. In general, knowing the algebraic semigroup $\hat{T}$ does not determine the semigroup T uniquely. If we return now to the situation where $M = M(S)$ for some locally compact semilattice S with identity and having finite breadth, then $\Delta M(S)$ is a semilattice, and Taylor's work shows that the structure semigroup for $M(S)$ has the discrete semigroup $\Delta M(S)$ as its semilattice of semicharacters. Since the structure semigroup T for $M(S)$ always has enough semicharacters to separate points and since in this case each semicharacter is idempotent, it follows that T must be idempotent and hence a semilattice. Since the semicharacters of a semilattice have range $\{0, 1\}$, T can be embedded in a product of the two-point semilattice and hence must be totally disconnected. According to the duality between $\boldsymbol{S}$ and $\boldsymbol{Z}$ discussed in § 1, we must then have that $T \simeq \widehat{\Delta M(S)_d}$, where again $\Delta M(S)_d$ is the discrete semilattice $\Delta M(S)$. But, we know $\Delta M(S) \simeq \hat{S}_d$, so we conclude $T \simeq \widehat{(\hat{S}_d)_d}$. We summarize our results in the following:

THEOREM 2.6. *Let S be a compact semilattice with identity and having finite breadth. Then $\Delta M(S) \simeq \hat{S}_d$ is a compact 0-dimensional semilattice, and the critical points of $\Delta M(S)$ are precisely those complex homomorphisms h of $M(S)$ of the form $h = \chi_{\uparrow s}$ for some $s \in S$. Moreover, the structure semilattice of $M(S)$ is $\widehat{(\hat{S}_d)_d}$.*

As we remarked above, Taylor shows that $H^*(\Delta M(S))$ is the direct sum of the cohomology of the maximal subgroups $H(e)$ as e ranges over the critical points of $\Delta M(S)$. However, since $\Delta M(S)$ is a semilattice, $H(e)=\{e\}$ for all $e \in \Delta M(S)$, so $H^*(\Delta M(S)) \simeq \bigoplus_{s \in S_d} H(\{s\})$, where S_d represents the critical points of $\Delta M(S)$. We draw two conclusions.

First, the Shilov Idempotent Theorem states that the idempotents in $M(S)$ are in one-to-one correspondence with $H^0(\Delta M(S)) \simeq \bigoplus_{s \in S_d} H^0(\{s\})$. Hence, since the correspondence is given by the Gelfand transform, so that $\hat{\sigma}_s \sim H^0(\{s\}) \forall s \in S$, we conclude that $\mu \in M(S)$ is idempotent if and only if $\mu = \sum_{k=1}^n \alpha_k \delta_{s_k}$ for some $s_1, \cdots, s_n \in S$, where $(\forall s \in S) \sum_{s \leqq s_k} \alpha_k = \begin{cases} 0 \\ 1 \end{cases}$ (this latter follows from the fact that $\mu(\uparrow s) \in \{0, 1\}$ as $\mu(\uparrow s) = h(\mu)$ where $h = \chi_{\uparrow s} \in \Delta M(S)$).

Our second conclusion is as follows. Since $\Delta M(S)$ is 0-dimensional, $H^n(\Delta M(S)) = 0$ for $n \geqq 1$, so that, according to the Arens-Royden theorem, the group of invertible elements in $M(S)$ is precisely the group of exponential measures.

3. P-algebras. In [9], Newman defines a P-algebra to be a semisimple convolution measure algebra M such that whenever μ is positive element of M and $h \in \Delta M$, the $h(\mu) \geqq 0$. He shows (Theorem 1 of [9]) that these are precisely the semisimple convolution measure algebras whose structure semigroups are in fact semilattices. It is easily checked that the equivalence $(1) \Leftrightarrow (2) \Leftrightarrow (3)$ of Theorem 1 of [9] is true without assuming semisimplicity, so we shall define a P-algebra to be a convolution measure algebra M such that $h(\mu) \geqq 0$ for all $h \in \Delta M$ and all $\mu \in M$ such that $\mu \geqq 0$. Thus we have shown (Theorem 2.5) that if S is a locally compact semilattice with finite breadth, then $M(S)$ is a P-algebra. We shall see that this is true in a somewhat more general setting, but first we give a general condition which insures that $M(S)$ is a semisimple convolution measure algebra.

DEFINITION 3.1. *A locally compact semilattice is said to be Lawson if it has a neighborhood basis of compact subsemilattices. Equivalently, S is Lawson if the semilattice homomorphisms into* $([0, 1], \wedge)$ *separate the points of S.* (See [6].)

THEOREM 3.2. *If S is Lawson, then $M(S)$ is semisimple.*

Proof. Let $I = ([0, 1], \wedge)$. Baartz showed in [1] that if $n < \infty$, then $M(I^n)$ is semisimple. Now if α is any cardinal number, $M(I^\alpha) = \text{proj lim}_{n<\infty} M(I^n)$, where the maps $M(I^\alpha) \to M(I^n)$ are quotient maps, and where a measure $\mu \in M(I^\alpha)$ is zero iff its image in every $M(I^n)$ is zero. Since the complex homomorphisms of each $M(I^n)$ separate the points, it follows that the same is true of $M(I^\alpha)$.

Now clearly if S is a Lawson semilattice there is a continuous, injective semilattice morphism $f: S \to I^\alpha$ for some cardinal α. Every nonzero measure μ in $M(S)$ lives on a σ compact set T_μ, and $f: T_\mu \to f(T_\mu)$ is a Borel isomorphism. In particular $f(\mu) \neq 0$. So $f: M(S) \to M(I^\alpha)$ is an injection, and it follows that $M(S)$ is semisimple.

COROLLARY 3.3. *If S is a locally compact semilattice with finite breadth, then $M(S)$ is a semisimple P-algebra.*

Proof. Immediate from 2.5, 3.2, and the observation that S must be Lawson.

We can actually assert that $M(S)$ is a P-algebra for somewhat more general S.

THEOREM 3.4. *Let S be a locally compact semilattice. Suppose every $\mu \in M(S)$ is concentrated on a Borel set which is the countable union of compact subsemilattices of finite breadth. Then $M(S)$ is a P-algebra.*

Note. We are not asserting here that $M(S)$ is semisimple.

Proof. Straightforward from 2.5.

Here are two examples of compact finite breadth semilattices which cannot be imbedded in finite dimensional cubes and are thus not dealt with by Newman's methods, and a third to show that semilattices with nonfinite breadth, but still satisfying the hypotheses of Theorem 3.4, do exist.

EXAMPLE 3.5 (cf. Exercise 1.12 of [2]). Let S be the Rees quotient $I^2/(I \times \{0\} \cup \{0\} \times I)$. S has breadth 2.

EXAMPLE 3.6. Let $S = \{(x_n) \in I^\infty \mid x_n = 0$ for all but at most one $n\}$. Then S has breadth 2, but cannot be imbedded in a finite-dimensional cube I^n. In fact, S contains an infinite set $\{x_n\}$ of elements which annihilate each other pair-wise, and for any finite n, it is not difficult to see that no such infinite sets exist in I^n, (no matter what element of I^n is the image of the 0 in S).

EXAMPLE 3.7. For each $n \geqq 1$, let I^n denote the n-fold product of the semilattice $([0, 1], \wedge)$ with itself. Let $S_1 = \bigcup_{n \geqq 1} I^n$, and let $S = S_1/R$, where R is the relation on S_1 which identifies the identity of I^n with the zero of I^{n+1} for each $n = 1, 2, \cdots$. Then S is a locally compact semilattice satisfying the hypotheses of Theorem 3.4, but $\mathrm{br}(S) = \infty$. If S^1 denotes S with an identity added as the one point compactification of S, then S is a compact semilattice with identity satisfying the hypotheses of 3.4 which fails to have finite breadth.

We remark that the investigations of this paper can be carried out in a more general setting. Let S be a locally compact semilattice, and let $N(S)$ be the norm closure in $M(S)$ of all measures with support a subset of a compact finite breadth subsemilattice of S. Then $N(S)$ is a norm-closed L-subalgebra of $M(S)$ which consists precisely of all measures in $M(S)$ which vanish outside of a countable union of compact semilattices with finite breadth.

THEOREM 3.8. *Let S be a locally compact semilattice, let $h \in \Delta N(S)$, and let $F_h = \{s \in S: h(\delta_s) = 1\}$. Then for all $\mu \in N(S)$, $h(\mu) = \mu(F_h \cap K)$ where K is any countable union of compact subsemilattices of finite breadth such that μ vanishes outside of K.*

Proof. Let $\mu \in N(S)$ which vanishes outside of $K = \bigcup_{j=1}^{\infty} K_j$ where each K_j is a compact subsemilattice of finite breadth. We may assume the K_j tower up (by defining a new sequence with nth element $K_1 \cdot K_2 \cdots \cdot K_n$ if necessary). Let $\mu_j = \mu \mid K_j$. Then $\{\mu_j\}$ norm converges to μ. By Theorem 2.5 we have $h(\mu_j) = \mu(F_h \cap K_j)$ for each j. Hence $h(\mu) = \lim h(\mu_j) = \lim \mu(F_h \cap K_j) = \mu(F_h \cap K)$.

Using this theorem we deduce as before that $\Delta(N(S)) \cong \hat{S}_d$ and the structure semigroup T of $N(S)$ is $\widehat{(\hat{S}_d)_d}$. Other analogous remarks that were made about $M(S)$ before can now be made about $N(S)$.

In general any filter F on a locally compact semilattice S gives rise to a homomorphism $h_F \in \Delta(M(S))$ defined by $h_F(\mu)$ is the inner μ measure of F. Hence $\hat{S}_d$ always sits in $\Delta(M(S))$. It is conjectural that $\hat{S}_d$ is $\Delta(M(S)$ if and only if $M(S) = N(S)$. (We have shown the "if" direction.)

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