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UNBOUNDED REPRESENTATIONS OF $*$-ALGEBRAS

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Basic results on unbounded operator algebras are given, a general class of representations, called adjointable representations is introduced and irreducibility of representations is considered. A characterization of self-adjointness for closed, strongly cyclic $*$-representations is presented.

1. Introduction. Algebras of unbounded operators and unbounded representations of $*$-algebras have been important in quantum field theory [**1**, **3**, **9**, **10**] and certain studies of Lie algebras [**5**, **7**]. The present paper proceeds along the lines initiated and developed by Robert Powers [**6**, **7**] and much of the notation and definitions follow [**6**]. In §2, we present some basic results concerning unbounded operator algebras, introduce a class of representations called adjointable representations, and consider irreducibility of representations. Section 3 characterizes the self-adjointness of closed, strongly cyclic $*$-representations.

2. Adjointable representations. Let M and N be subspaces (linear manifolds) in a Hilbert space H. Let $L(M,N)$ and $L_c(M,N)$ denote the collection of linear operators and closable linear operators, respectively with domain M and range in N. For simplicity we use the notation $L(M)=L(M,M)$ and $L_c(M)=L_c(M,M)$. Notice that $L_c(H)$ is the set of bounded linear operators on H. We denote the domain of an operator A by $D(A)$ and if A is closable we denote the closure of A by $\bar{A}$. A collection of operators $\mathscr{B}$ is an *op-algebra* if there exists a subspace M such that $\mathscr{B}\subseteq L(M)$ and $A,B\in\mathscr{B}$ implies $AB,(\alpha A+B)\in\mathscr{B}$ for all $\alpha\in\mathbf{C}$. A set $\mathscr{B}\subseteq L(M)$ is *symmetric* if M is dense and $A\in\mathscr{B}$ implies $D(A^*)\supseteq M$ and $A^*|M\in\mathscr{B}$. A symmetric op-algebra $\mathscr{B}\subseteq L(M)$ that contains $I|M$ is called an *op*-algebra*. It is easy to see that if $\mathscr{B}\subseteq L(M)$ is an op*-algebra, then the map $A\to A^*|M$ is an involution so $\mathscr{B}$ is a $*$-algebra. Also, if π is a representation of a $*$-algebra $\mathscr{A}$, then $\pi(\mathscr{A})=\{\pi(A)\colon A\in\mathscr{A}\}$ is an op-algebra and if π is a $*$-representation of $\mathscr{A}$, then $\pi(\mathscr{A})$ is an op*-algebra (we always assume that a $*$-algebra contains an identity I).

A set $\mathscr{B}\subseteq L(M,N)$ is *directed* if for any $B_1,B_2\in\mathscr{B}$ there exists a $B_3\in\mathscr{B}$ such that $\|B_1x\|,\|B_2x\|\leqq\|B_3x\|$ for all $x\in M$. For example, if $\mathscr{B}\subseteq L_c(H)$ and $\{\lambda I\colon\lambda\geqq 0\}\subseteq\mathscr{B}$, then $\mathscr{B}$ is directed. Indeed, just let $B_3=(\|B_1\|+\|B_2\|)I$. For an example of an unbounded directed set, let $\mathscr{B}\subset L(M,H)$ and suppose $B_1,B_2\in\mathscr{B}$ implies $B_3=$

$I \mid M + B_1^* B_1 + B_2^* B_2 \in \mathcal{B}$. Then for any $x \in M$ we have

$$\begin{aligned} \| B_3 x \|^2 &= \| x \|^2 + \| (B_1^* B_1 + B_2^* B_2 x \|^2 + 2\langle (B_1^* B_1 + B_2^* B_2)x, x \rangle \\ &\geqq 2(\| B_1 x \|^2 + \| B_2 x \|^2) \\ &\geqq \| B_1 x \|^2, \| B_2 x \|^2. \end{aligned}$$

In particular, any op $*$-algebra is directed.

An *extension* $\mathcal{B}_1$ of $\mathcal{B} \subseteq L(M, N)$ is a set of operators $\mathcal{B}_1 \subseteq L(M_1, N_1)$ where $M \subseteq M_1$, $N \subseteq N_1$ and for which there exists a bijection $\phi: \mathcal{B} \to \mathcal{B}_1$ such that $\phi(B) \mid M = B$ for every $B \in \mathcal{B}$. If $\mathcal{B} \subseteq L(M, N)$, the *$\mathcal{B}$-topology* on M is the topology generated by the set of seminorms $\{\| x \|, \| Bx \|: B \in \mathcal{B}\}$. The completion of M in the $\mathcal{B}$-topology is denoted by $\hat{M}_{\mathcal{B}}$ or simply $\hat{M}$ if no confusion can arise. We say that $\mathcal{B} \subseteq L(M, N)$ is *collectively closed* if for any net $x_\alpha \in M$ satisfying $x_\alpha \to x \in H$, $Bx_\alpha \to y(B) \in H$ for every $B \in \mathcal{B}$, then $x \in M$ and $Bx = y(B)$. Clearly if all $B \in \mathcal{B}$ are closed then $\mathcal{B}$ is collectively closed; the converse need not hold.

THEOREM 1.

(1) *$\mathcal{B} \subset L(M, N)$ is collectively closed if and only if $M = \hat{M}_{\mathcal{B}}$.*

(2) *If $\mathcal{B} \subseteq L_c(M, N)$, then the set $\mathcal{B}_1 = \{\bar{B} \mid M_1: B \in \mathcal{B}\}$ where $M_1 = \cap\{D(\bar{B}): B \in \mathcal{B}\}$ is collectively closed.*

(3) *If $\mathcal{B} \subseteq L_c(M, N)$, then the set $\bar{\mathcal{B}} = \{\bar{B} \mid \hat{M}_{\mathcal{B}}: B \in \mathcal{B}\}$ is the minimal collectively closed extension of $\mathcal{B}$. Moreover, if $\mathcal{B} \subseteq L_c(M)$ and $A, B \in \mathcal{B}$ implies $AB \in \mathcal{B}$, then $\bar{\mathcal{B}} \subseteq L_c(\hat{M}_{\mathcal{B}})$.*

(4) *If $\mathcal{B} \subseteq L_c(M, N)$ is directed, then $\hat{M}_{\mathcal{B}} = \cap\{D(\bar{B}): B \in \mathcal{B}\}$.*

(5) *If $\mathcal{B} \subseteq L_c(M)$ is an op-algebra, then $\bar{\mathcal{B}}$ is an op-algebra. If $\mathcal{B} \subseteq L_c(M)$ is an op $*$-algebra, then $\bar{\mathcal{B}}$ is an op $*$-algebra and $\hat{M}_{\mathcal{B}} = \cap\{D(\bar{B}): B \in \mathcal{B}\}$.*

Proof.

(1) Suppose $\mathcal{B} \subseteq L(M, N)$ is collectively closed and $x_\alpha \in M$ is a Cauchy net in the $\mathcal{B}$-topology. Then x_α and Bx_α are Cauchy in H so there exist $x, y(B) \in H$ such that $x_\alpha \to x$, $Bx_\alpha \to y(B)$ in H for every $B \in \mathcal{B}$. Since $\mathcal{B}$ is collectively closed, $x \in M$ and $Bx_\alpha \to Bx$, so $x_\alpha \to x$ in the $\mathcal{B}$-topology and M is complete in the $\mathcal{B}$-topology. Hence $M = \hat{M}_{\mathcal{B}}$. Conversely, suppose $M = \hat{M}_{\mathcal{B}}$, and x_α is a net in M such that $x_\alpha \to x$ and $Bx_\alpha \to y(B)$ in H for every $B \in \mathcal{B}$. Then x_α is Cauchy in the $\mathcal{B}$-topology. Since M is complete in the $\mathcal{B}$-topology there exists an $x' \in M$ such that $x_\alpha \to x'$ and $Bx_\alpha \to Bx'$ in H for every $B \in \mathcal{B}$. Hence $x = x' \in M$ and $Bx = Bx' = y(B)$.

(2) This is straightforward.

(3) It is clear that

$\hat{M}_{\mathcal{B}} = \{x \in \cap\{D(\bar{B}): B \in \mathcal{B}\}: M \ni x_\alpha \to x, \quad Bx_\alpha \to \bar{B}x$ for all $B \in \mathcal{B}\}$. We now show that $\hat{M}_{\mathcal{B}}$ is complete in the $\bar{\mathcal{B}} \mid \hat{M}_{\mathcal{B}}$-topology. Suppose $x_\alpha \in \hat{M}_{\mathcal{B}}$ is Cauchy in the $\bar{\mathcal{B}}$ $\hat{M}_{\mathcal{B}}$-topology. Then x_α and Bx_α are Cauchy in H for every $B \in \mathcal{B}$. Hence there exists an $x \in H$ such that $x_\alpha \to x$ and $\bar{B}x_\alpha \to \bar{B}x$ for every $B \in \mathcal{B}$. Since $x_\alpha \in \hat{M}_{\mathcal{B}}$ there exists a net $x_{\alpha\beta} \in M$ such that $x_{\alpha\beta} \to x_\alpha$ and $Bx_{\alpha\beta} \to \bar{B}x_\alpha$ in H for every $B \in \mathcal{B}$. Now $x_{\alpha\beta}$ is a net in M and $x_{\alpha\beta} \to x$, $Bx_{\alpha\beta} \to \bar{B}x$ in H for every $B \in \mathcal{B}$. Hence $x \in \hat{M}_{\mathcal{B}}$. It follows from (1) that $\bar{\mathcal{B}}$ is collectively closed. Clearly $\bar{\mathcal{B}}$ is an extension of $\mathcal{B}$. Moreover, $\bar{\mathcal{B}}$ is a minimal collectively closed extension since any collectively closed extension of $\mathcal{B}$ must contain $\hat{M}_{\mathcal{B}}$ in its domain. Now suppose $\mathcal{B} \subseteq L_c(M)$ and $A, B \in \mathcal{B}$. If $x \in \hat{M}_{\mathcal{B}}$, then there exists a net $x_\alpha \in M$ such that $x_\alpha \to x$ and $Bx_\alpha \to \bar{B}x$ for every $B \in \mathcal{B}$. For fixed $A \in \mathcal{B}$ we have $A\,\underline{x_\alpha} \in M$ and $Ax_\alpha \to \bar{A}x$ and for every $B \in \mathcal{B}$, since $BA \in \mathcal{B}$, $BAx_\alpha \to \overline{BA}x = \bar{B}\bar{A}x$. Hence $\bar{A}x \in \hat{M}_{\mathcal{B}}$ and $\bar{\mathcal{B}} \subseteq L_c(\hat{M}_{\mathcal{B}})$.

(4) Suppose that $\mathcal{B} \subseteq L_c(M, N)$ is directed. We have seen that $\hat{M}_{\mathcal{B}} \subseteq \cap\{D(\bar{B}): B \in \mathcal{B}\}$. If $x \in \cap\{D(\bar{B}): B \in \mathcal{B}\}$, then for each $B \in \mathcal{B}$ there exists a sequence $x(B, i) \in M$ such that $x(B, i) \to x$ and $Bx(B, i) \to \bar{B}x$. For each $B \in \mathcal{B}$ and for each integer $n > 0$ there exists an integer $n_B > 0$ such that $\|x(B, n_B) - x\| < n^{-1}$ and $\|Bx(B, n_B) - \bar{B}x\| < n^{-1}$. For A, $B \in \mathcal{B}$, define the order $(A, n_A) < (B, m_B)$ if $\|Az\| \leqq \|Bz\|$ for every $z \in M$ and $n < m$. Since $\mathcal{B}$ is directed, $\{(B, n_B)\}$ is a directed partially ordered set and $x(B, n_B)$ is a net. Notice that if $\|Az\| \leqq \|Bz\|$ for every $z \in M$ then $\|\bar{A}y\| \leqq \|\bar{B}y\|$ for every $y \in \cap\{D(\bar{B}): B \in \mathcal{B}\}$. Indeed let $z_i \in M$ be a sequence such that $z_i \to y$ and $Bz_i \to \bar{B}y$. Since $\|Az_i - Az_j\| \leqq \|Bz_i - Bz_j\|$, Az_i is Cauchy and hence $Az_i \to \bar{A}y$. Therefore,

$$\|\bar{A}y\| = \lim\|Az_i\| \leqq \lim\|Bz_i\| = \|\bar{B}y\|.$$

Clearly, $x(B, m_B) \to x$ and to show that $Ax(B, m_B) \to \bar{A}x$ let $\epsilon > 0$ and let $n > 0$ be an integer such that $n^{-1} < \epsilon$. Then for $(B, m_B) > (A, n_A)$ we have

$$\begin{aligned} \|Ax(B, m_B) - \bar{A}x\| &= \|\bar{A}x(B, m_B) - \bar{A}x\| \\ &\leqq \|\bar{B}x(B, m_B) - \bar{B}x\| \\ &< m^{-1} < n^{-1} < \epsilon\,. \end{aligned}$$

It follows that $x \in \hat{M}_{\mathcal{B}}$.

(5) This is a straightforward consequence of (2) and (3).

In the work of R. Powers [6] only hermitian representations are considered. But there are important representations that are not hermitian. For example, even if π is hermitian, π^* need not be. We therefore treat a larger class of representations, which we call adjointable, that includes π^* whenever π is hermitian.

Let $\mathscr{A}$ be a $*$-algebra and let π, π_1 be two representations of $\mathscr{A}$ with domains $D(\pi), D(\pi_1) \subseteq H$. We say that π and π_1 are *adjoint* and write $\pi\, a\, \pi_1$, if $\langle \pi(A)x, y\rangle = \langle x, \pi_1(A^*)y\rangle$ for every $A \in \mathscr{A}$ and $x \in D(\pi)$, $y \in D(\pi_1)$. Notice that a is a symmetric relation; that is $\pi\, a\, \pi_1$ if and only if $\pi_1\, a\, \pi$. Also, $\pi\, a\, \pi$ if and only if π is hermitian. Furthermore, if $\pi\, a\, \pi_1$ and $\pi_1\, a\, \pi_2$ then $\pi(A) = \pi_2(A)$ on $D(\pi) \cap D(\pi_2)$ for every $A \in \mathscr{A}$ and if $D(\pi) = D(\pi_2)$ then $\pi = \pi_2$. We say that a representation π is *adjointable* if there exists a representation π_1 such that $\pi\, a\, \pi_1$.

If π is a representation of a $*$-algebra $\mathscr{A}$, we define $D(\pi^*) = \cap\{D(\pi(A)^*)\colon A \in \mathscr{A}\}$ and $\pi^*(A) = \pi(A^*)^* \mid D(\pi^*)$ for all $A \in \mathscr{A}$. (To save parentheses we use the notation $\pi(A)^* = [\pi(A)]^*$.) In general, π^* need not be a representation since, for one thing, $D(\pi^*)$ need not be dense. If π is hermitian, then π^* is a representation [6]. Hence, if π is hermitian, then

$$\langle \pi(A)x, y\rangle = \langle x, \pi(A)^* y\rangle = \langle x, \pi^*(A^*)y\rangle$$

for every $A \in \mathscr{A}$ and $x \in D(\pi)$, $y \in D(\pi^*)$ so $\pi\, a\, \pi^*$ and each is adjointable.

THEOREM 2.

(1) *π is adjointable if and only if $D(\pi^*)$ is dense.*

(2) *If π is adjointable, then π^* is a closed representation and is the largest representation adjoint to π.*

(3) *Suppose $\pi \subset \pi_1$. If $\pi_1\, a\, \pi_2$, then $\pi\, a\, \pi_2$. If π_1 is adjointable, then so is π and $\pi_1^* \subset \pi^*$.*

(4) *If π is adjointable, then there exists a smallest closed representation $\bar{\pi}$ which extends π. If $\pi\, a\, \pi_1$, then $\bar{\pi}\, a\, \pi_1$.*

(5) *If π is adjointable, then π^*, $\bar{\pi}$ are adjointable, π^{**} is a closed representation and $\pi \subset \bar{\pi} \subset \pi^{**}$, $\pi^{***} = \pi^*$, $\bar{\pi}^* = \pi^*$.*

(6) *If π is hermitian and π_1 is an hermitian extension of π, then $\pi \subset \pi_1 \subset \pi^*$.*

(7) *If π is hermitian, then π^{**} and $\bar{\pi}$ are hermitian and $\pi \subset \bar{\pi} \subset \pi^{**} \subset \pi^*$.*

Proof.

(1) If π is adjointable and $\pi a \pi_1$ then $\pi_1(A^*) \subset \pi(A)^*$ for every $A \in \mathscr{A}$ so $D(\pi_1) \subseteq D(\pi^*)$ and $D(\pi^*)$ is dense. Conversely, suppose

$D(\pi^*)$ is dense. For $x \in D(\pi)$, $y \in D(\pi^*)$ we have

$$\begin{aligned}\langle \pi(A^*)x, \pi^*(B)y\rangle &= \langle \pi(A^*)x, \pi(B^*)^* y\rangle \\ &= \langle \pi(B^*)\pi(A^*)x, y\rangle \\ &= \langle \pi(B^*A^*)x, y\rangle \\ &= \langle x, \pi(B^*A^*)^* y\rangle .\end{aligned}$$

Hence $\pi^*(B)y \in D(\pi(A^*)^*)$ and $\pi(A^*)^*\pi^*(B)y = \pi(B^*A^*)^* y$ for every $A, B \in \mathscr{A}$. If follows that $\pi^*(B): D(\pi^*) \to D(\pi^*)$ and $\pi^*(A)\pi^*(B) = \pi((AB)^*)^* = \pi^*(AB)$. Moreover, π^* is linear since for $x \in D(\pi)$, $y \in D(\pi^*)$ we have

$$\begin{aligned}\langle \pi^*(\alpha A + B)y, x\rangle &= \langle \pi(\bar\alpha A^* + B^*)^* y, x\rangle \\ &= \langle y, \bar\alpha\pi(A^*)x\rangle + \langle y, \pi(B^*)x\rangle \\ &= \langle [\alpha\pi^*(A) + \pi^*(B)]y, x\rangle.\end{aligned}$$

It follows that π^* is a representation and $\pi\, a\, \pi^*$.

(2) It was shown in (1) that π^* is a representation if π is adjointable. It follows from Theorem 1 (2) that π^* is closed. If $\pi\, a\, \pi_1$ then $\langle \pi(A)x, y\rangle = \langle x, \pi_1(A^*)y\rangle$ for all $x \in D(\pi)$, $y \in D(\pi_1)$. Hence, $D(\pi_1) \subseteq D(\pi^*)$ and $\pi_1(A^*) \subset \pi(A)^* = \pi^*(A^*)$ for every $A \in \mathscr{A}$ so $\pi_1 \subset \pi^*$.

(3) Suppose $\pi \subset \pi_1$ and $\pi_1\, a\, \pi_2$. Then for every $x \in D(\pi)$, $y \in D(\pi_2)$ we have $\langle \pi(A)x, y\rangle = \langle \pi_1(A)x, y\rangle = \langle x, \pi_2(A^*)y\rangle$. Hence $\pi\, a\, \pi_2$. For all $x \in D(\pi)$, $y \in D(\pi_1^*)$ we have $\langle \pi(A)x, y\rangle = \langle x, \pi_1^*(A^*)y\rangle$. Hence $\pi\, a\, \pi_1^*$ and by (2) we have $\pi_1^* \subset \pi^*$.

(4) If π is adjointable, then by (1), $D(\pi^*)$ is dense. Then $D(\pi(A)^*)$ is dense so $\pi(A)$ is closable for every $A \in \mathscr{A}$. Define $D(\bar\pi) = \hat D(\pi)_{\mathscr{B}}$ where $\mathscr{B} = \{\pi(A): A \in \mathscr{A}\}$ and $\bar\pi(A) = \overline{\pi(A)} \mid D(\bar\pi)$. It follows from Theorem 1 (3) that $\{\bar\pi(A): A \in \mathscr{A}\}$ is the minimal collectively closed extension of $\mathscr{B}$. It is straightforward to show that $\bar\pi$ is a representation and that $\pi\, a\, \pi_1$ implies $\bar\pi\, a\, \pi_1$.

(5) If π is adjointable then so is π^* and from (2) π^{**} is a closed representation. If $x \in D(\pi)$, $y \in D(\pi^*)$ then for all $A \in \mathscr{A}$ we have $\langle \pi^*(A^*)y, x\rangle = \langle \pi(A)^* y, x\rangle = \langle y, \pi(A)x\rangle$. Hence $x \in \cap\{D[\pi^*(A^*)^*]: A \in \mathscr{A}\} = D(\pi^{**})$ and $\pi^{**}(A)x = \pi^*(A^*)^* x = \pi(A)x$ so $\pi \subset \pi^{**}$. Since $\pi \subset \bar\pi$ we have by (3) that $\bar\pi^* \subset \pi^*$. Since $\pi\, a\, \pi^*$ from (4) we have $\bar\pi\, a\, \pi^*$. Hence by (2) $\pi^* \subset \bar\pi^*$ so $\pi^* = \bar\pi^*$. By (3) $\pi^{***} \subset \pi^*$. Since $\pi^*\, a\, \pi^{**}$, by (2) we have $\pi^* \subset \pi^{***}$ so $\pi^{***} = \pi^*$.

(6) For all $x \in D(\pi)$, $y \in D(\pi_1)$, $A \in \mathscr{A}$ we have

$$\langle \pi(A)x, y\rangle = \langle \pi_1(A)x, y\rangle = \langle x, \pi_1(A^*)y\rangle .$$

Hence $\pi_1 a \pi$ and by (2) $\pi_1 \subset \pi^*$.

(7) It is shown in [6] that $\bar{\pi}$ is hermitian if π is hermitian. Since π is hermitian we have $\pi \subset \pi^*$. Applying (3) twice gives $\pi^{**} \subset \pi^{***}$ so π^{**} is hermitian. Since π^{**} is closed we have from (2) that $\pi \subset \bar{\pi} \subset \pi^{**}$ and from (6) $\pi^{**} \subset \pi^*$.

We now show that the extensions in (7) can be distinct. Let $\mathscr{A}$ be the free commutative $*$-algebra on one hermitian generator A. Define the representation π of $\mathscr{A}$ on the Hilbert space $H = L^2[0,1]$ as follows: $D(\pi) = \{f \in C^\infty [0,1]: f^{(n)}(0) = f^{(n)}(1) = 0, n = 0, 1, 2, \dots\}$ $\pi(A) = -i\,d/dt$. It is straightforward to show that π is hermitian and that $\pi = \bar{\pi} = \pi^{**} \subsetneqq \pi^*$ [8]. Now let π_1 be the representation of $\mathscr{A}$ on H defined by:

$$D(\pi_1) = \{f \in C^\infty[0,1]: f(0) = f(1), f^{(n)}(0) = f^{(n)}(1), n = 1, 2, \dots\}$$
$$\pi_1(A) = -i\,d/dt .$$

It is straightforward to show that π_1 is hermitian and that $\pi_1 = \bar{\pi}_1 \subsetneqq \pi_1^{**} = \pi_1^*$ [8].

We now consider commutants and irreducibility. If $\pi a \pi_1$, define $C(\pi, \pi_1)$ to be the set of operators $C \in L_c(H)$ satisfying $\langle C\pi(A)x, y\rangle = \langle Cx, \pi_1(A^*)y\rangle$ for every $x \in D(\pi)$, $y \in D(\pi_1)$, $A \in \mathscr{A}$. The proof of the following lemma is straightforward.

LEMMA 3.

(1) *$C(\pi, \pi_1)$ is a weakly closed subspace of $L_c(H)$ containing I.*

(2) *$C(\pi, \pi_1)$*
$= \{C \in L_c(H): C: D(\pi) \to D(\pi_1^*), C\pi(A) = \pi_1^*(A)C \mid D(\pi)\}$.

(3) *$C \in C(\pi, \pi_1)$ if and only if $C^* \in C(\pi_1, \pi)$.*

The *commutant* of a $*$-representation π is defined as $\pi(\mathscr{A})' = C(\pi, \pi)$. It follows from Lemma 3 that $\pi(\mathscr{A})'$ is a weakly closed, symmetric subspace of $L_c(H)$ containing I. However, $\pi(\mathscr{A})'$ need not be a von Neumann algebra [6]. If π is self-adjoint then $\pi(\mathscr{A})'$ is a von Neumann algebra [6]. If π is a $*$-representation, the *strong commutant* is defined by

$$\pi(\mathscr{A})'_s = \{C \in \pi(\mathscr{A})': C: D(\pi) \to D(\pi)\}.$$

Hence

$$\pi(\mathscr{A})'_s = \{C \in L_c(H): C: D(\pi) \to D(\pi), C\pi(A) = \pi(A)C \mid D(\pi), \forall A \in \mathscr{A}\}.$$

It is easy to see that $\pi(\mathscr{A})'_s$ is an op-algebra in $L_c(H)$ containing I and if π is closed, then $\pi(\mathscr{A})'_s$ is weakly closed **[1]**. Again $\pi(\mathscr{A})'_s$ need not be a von Neumann algebra but if π is self-adjoint, then $\pi(\mathscr{A})'_s$ is a von Neumann algebra and $\pi(\mathscr{A})'_s = \pi(\mathscr{A})'$.

LEMMA 4. *A $*$-representation π is self-adjoint if and only if $\pi(\mathscr{A})' = \pi(\mathscr{A})'_s$ and $D(\pi^*) = \cup\{Cx: x \in D(\pi), C \in \pi(\mathscr{A})'\}$.*

Proof. Necessity follows from our previous observations. For sufficiency, if $\pi(\mathscr{A})' = \pi(\mathscr{A})'_s$ then $C: D(\pi) \to D(\pi)$ for all $C \in \pi(\mathscr{A})'$. Hence $D(\pi^*) = \cup\{Cx: x \in D(\pi), C \in \pi(\mathscr{A})'\} \subset D(\pi)$.

For a bounded $*$-representation π of a $*$-algebra $\mathscr{A}$ on a Hilbert space H the following conditions are equivalent **[2, 4]**.

(i) $\pi(\mathscr{A})' = \{\lambda I: \lambda \in C\}$.

(ii) The only invariant closed subspaces of H are $\{0\}$ and H.

(iii) Every nonzero vector in $H = D(\pi)$ is cyclic.

A bounded $*$-representation π is said to be *irreducible* if π satisfies any one (and hence all) of these three conditions.

For unbounded self-adjoint representations one can give examples **[6, 8]** which show that no two of the above conditions are equivalent. Also, there is more than one natural way to extend some of the above conditions for unbounded self-adjoint representations. Let π be a self-adjoint representation. We say that a subspace M is a *self-adjoint invariant subspace* for π if M is invariant and $\pi \mid M$ is self-adjoint. The following are natural conditions that one might use to define irreducibility for a self-adjoint representation π of a $*$-algebra $\mathscr{A}$ with domain $D(\pi) \subseteq H$.

(1) $\pi(\mathscr{A})' = \{\lambda I: \lambda \in C\}$.

(2) The only invariant subspaces for π which are complete in the $\pi(\mathscr{A})$-topology are $\{0\}$ and $D(\pi)$.

(2′) The only self-adjoint invariant subspaces for π are $\{0\}$ and $D(\pi)$.

(3) Every nonzero vector in $D(\pi)$ is strongly cyclic.

(3′) Every nonzero vector in $D(\pi)$ is cyclic.

THEOREM 5. *If π is self-adjoint representation of the $*$-algebra $\mathscr{A}$ on the Hilbert space H, then*

$$\begin{array}{ccc} (2') & \leftrightarrow (1) \leftarrow & (2) \\ \uparrow & & \updownarrow \\ (3') & \longleftarrow & (3) \end{array}$$

Proof. $(2)\to(3)$. Suppose (2) holds, $0\neq\phi\in D(\pi)$ and $M=\{\pi(A)\phi\colon A\in\mathscr{A}\}$. Clearly, $M\neq\{0\}$ and M is an invariant subspace of H for π. Let $\hat{M}$ be the completion of M in the $\pi(\mathscr{A})$-topology. Since π is closed, $\hat{M}\subseteq D(\pi)$ and clearly $\hat{M}$ is a subspace of H. We now show that $\hat{M}$ is invariant under π. If $x\in\hat{M}$, then there exists a net $x_\alpha\in M$ such that $x_\alpha\to x$ in the $\pi(\mathscr{A})$-topology. Fix an $A\in\mathscr{A}$. Then for every $B\in\mathscr{A}$ we have

$$\pi(B)\pi(A)x_\alpha=\pi(BA)x_\alpha\to\pi(BA)x=\pi(B)\pi(A)x.$$

Hence $\pi(A)x_\alpha\to\pi(A)x$ in the $\pi(\mathscr{A})$-topology so $\pi(A)x\in\hat{M}$ and $\pi(A)\hat{M}\subseteq\hat{M}$. Since (2) holds, $\hat{M}=D(\pi)$. Hence M is dense in $D(\pi)$ in the $\pi(\mathscr{A})$-topology so ϕ is a strongly cyclic vector for π.

$(3)\to(2)$. Suppose (2) does not hold. Then there exists a $\pi(\mathscr{A})$-complete invariant subspace M of H with $M\neq\{0\}, D(\pi)$. If $0\neq\phi\in M$, then clearly ϕ is not a strongly cyclic vector for π

$(1)\to(2')$. Suppose $(2')$ does not hold. Then there exists a nontrivial self-adjoint invariant subspace M for π. Now M is not dense in H since otherwise $\pi\mid M$ is a $*$-representation of $\mathscr{A}$ on $\bar{M}=H$ with domain $M\subseteq D(\pi)$. Then $\pi\mid M\subset\pi=\pi^*\subset(\pi\mid M)^*$. Since $\pi\mid M$ is self-adjoint, $\pi\mid M=\pi$ and $D(\pi)=M$ which is a contradiction. By Theorem 4.7 **[6]** the projection E on $\bar{M}$ satisfies $E\in\pi(\mathscr{A})'$. Since $E\neq 0$, I, (1) does not hold.

$(2')\to(1)$. Suppose (1) does not hold. Since π is self-adjoint, $\pi(\mathscr{A})'$ is a von Neumann algebra so there exists a nontrivial projection $E\in\pi(\mathscr{A})'$. By Theorem 4.7 **[6]**, $ED(\pi)$ is a nontrivial self-adjoint invariant subspace for π. Thus $(2')$ does not hold.

$(3')\to(1)$. Suppose $(3')$ holds. Let $0\neq E\in\pi(\mathscr{A})'$ be a projection. By Theorem 4.7 **[6]**, $ED(\pi)=M$ is a self-adjoint invariant subspace for π. Let $0\neq\phi\in M$. Since ϕ is cyclic and $\{\pi(A)\phi\colon A\in\mathscr{A}\}\subseteq M$, M is dense in H. As in $(1)\to(2')$ above, $M=D(\pi)$ and hence $E=I$. Since 0 and I are the only projections in $\pi(\mathscr{A})'$, we have $\pi(\mathscr{A})'=\{\lambda I\colon\lambda\in C\}$.

$(3)\to(3')$. This is trivial. $(2)\to(1)$. Since $(2)\to(2')$ trivially, this follows from $(2')\to(1)$ above.

3. Closed strongly cyclic $*$-representations. In this section we shall mainly be concerned with characterizing selfadjointedness for closed strongly cyclic $*$-representations. Let π be a $*$-representation of a $*$-algebra $\mathscr{A}$ with domain $D(\pi) \subseteq H$. The *unbounded commutant* $\pi(\mathscr{A})^c$ of π is defined as the set of operators $C \in L(D(\pi), H)$ such that $\langle C\pi(A)x, y\rangle = \langle Cx, \pi(A^*)y\rangle$ for all $x, y \in D(\pi)$ and $A \in \mathscr{A}$. The *strong unbounded* commutant is defined by $\pi(\mathscr{A})^c_s = \{C \in \pi(\mathscr{A})^c : C\,D(\pi) \to D(\pi)\}$. Notice that $\pi(\mathscr{A})' \mid D(\pi) \subset \pi(\mathscr{A})^c$ and $\pi(\mathscr{A})'_s \mid D(\pi) \subseteq \pi(\mathscr{A})^c_s$. In fact,

$$\pi(\mathscr{A})' = \{\bar{C} : C \in \pi(\mathscr{A})^c,\ C \text{ bounded}\}$$

$$\pi(\mathscr{A})'_s = \{\bar{C} : C \in \pi(\mathscr{A})^c_s,\ C \text{ bounded}\}.$$

We say that a net $B_\alpha \in L(M, N)$ *converges weakly* to $B \in L(M, N)$ if $\langle B_\alpha x, y\rangle \to \langle Bx, y\rangle$ for every $x, y \in M$. Moreover, $\mathscr{B} \subseteq L(M, N)$ is *weakly closed* if for any net $B_\alpha \in \mathscr{B}$ which converges weakly to some $B \in L(M, N)$ we have $B \in \mathscr{B}$. The proof of the next lemma is straightforward.

LEMMA 6.

(1) *If π is self-adjoint, then $\pi(\mathscr{A})^c = \pi(\mathscr{A})^c_s$.*

(2) $\pi(\mathscr{A})^c = \{C \in L(D(\pi), D(\pi^*)) : C\pi(A) = \pi^*(A)C, \forall A \in \mathscr{A}\}$.

(3) $\pi(\mathscr{A})^c_s = C \in L(D(\pi)) : C\pi(A) = \pi(A)C, \forall A \in \mathscr{A}\}$.

(4) *$\pi(\mathscr{A})^c$ is a weakly closed subspace of $L(D(\pi), D(\pi^*))$ containing $I \mid D(\pi)$.*

(5) *$\pi(\mathscr{A})^c_s$ is an op-algebra in $L(D(\pi))$.*

(6) *$\pi(\mathscr{A})^c = \pi(\mathscr{A})^c_s$ if and only if $\pi(\mathscr{A})^c$ is an op-algebra.*

Let $\mathscr{A}$ be a $*$-algebra and let π, π_1 be $*$-representation of $\mathscr{A}$ on Hilbert spaces H, H_1, respectively. We say that π and π_1 are *equivalent*, and write $\pi \cong \pi_1$, if there exists a unitary transformation V from H onto H_1 such that $VD(\pi) = D(\pi_1)$ and $\pi(A) = V^*\pi_1(A)V$ for every $A \in \mathscr{A}$.

Let ω be a state on $\mathscr{A}$. Then by the *GNS* construction for $*$-algebras [**6**], there exists a closed, strongly cyclic $*$-representation π_ω of $\mathscr{A}$ with strongly cyclic vector x_0 such that $\omega(A) = \langle \pi_\omega(A)x_0, x_0\rangle$ for every $A \in \mathscr{A}$. Moreover, if π is any closed, strongly cyclic $*$-representation of $\mathscr{A}$ with strongly cyclic vector y_0 such that $\langle \pi(A)y_0, y_0\rangle = \omega(A)$ for every $A \in \mathscr{A}$ then $\pi \cong \pi_\omega$ [**6**].

We now characterize states ω such that π_ω is self-adjoint. A linear functional $F: \mathscr{A} \to C$ is *ω-bounded* if for every $B \in \mathscr{A}$ there exists an $M_B \geqq 0$ such that $|F(BA)| \leqq M_B\omega(A^*A)^{1/2}$ for every $A \in \mathscr{A}$. For example, if $A_\alpha \in \mathscr{A}$ is a net such that $\omega(A^*_\alpha A^*AA_\alpha)$ is Cauchy for every $A \in \mathscr{A}$, then the functional $F(A) = \lim \omega(A^*_\alpha A)$ is ω-bounded. Indeed, for every $B \in \mathscr{A}$ we have

$$\begin{aligned}F(BA)| &= \lim|\omega(A_\alpha^*BA)| = \lim|\omega(A^*B^*A_\alpha)| \\ &\leqq \omega(A^*A)^{1/2}\lim\omega(A_\alpha^*BB^*A_\alpha)^{1/2}.\end{aligned}$$

If every ω-bounded linear functional has the above form, then we call ω a *Riesz state.*

THEOREM 7. *Let ω be a state on the $*$-algebra $\mathscr{A}$. Then π_ω is self-adjoint if and only if ω is a Riesz state.*

Proof. Recall that π_ω is constructed as follows. Let $\mathscr{I}$ be the left ideal $\mathscr{I} = \{A \in \mathscr{A}: \omega(A^*A) = 0\}$ and let H_0 be the inner product space consisting of equivalence classes $[A]$ in $\mathscr{A}/\mathscr{I}$ with inner product $\langle [A],[B]\rangle = \omega(B^*A)$. Let H be the Hilbert space completion of H_0. Define a $*$-representation π_0 of $\mathscr{A}$ with domain $D(\pi_0) = H_0$ by $\pi_0(A)[B] = [AB]$. If $\pi_\omega = \bar{\pi}_0$, then π_ω is a closed, strongly cyclic $*$-representation with domain $D(\pi_\omega) = \hat{H}_{0\pi_0(\mathscr{A})}$ and strongly cyclic vector $[I]$. Now suppose π_ω is self-adjoint and $F: \mathscr{A} \to C$ is ω-bounded. If $\omega(A^*A) = 0$, then $F(A) = 0$ so $F: \mathscr{I} \to 0$. Hence F can be considered as a linear functional on H_0. Since $|F([A])| \leqq M_I\|[A]\|$, F is a continuous linear functional on H_0 and by the Riesz theorem there exists a $z \in H$ such that $F([A]) = \langle [A], z\rangle$ for every $[A] \in H_0$. Now for every $B \in \mathscr{A}$ we have

$$\begin{aligned}\langle \pi_0(B)[A], z\rangle| &= |\langle [BA], z\rangle| = |F([BA])| \\ &= |F(BA)| \leqq M_B\|[A]\|.\end{aligned}$$

Hence $z \in D(\pi_0^*) = D(\pi_\omega^*) = D(\pi_\omega)$, so there exists a net $[A_\alpha] \in H_0$ which converges to z in the $\pi_0(\mathscr{A})$-topology. Thus $[AA_\alpha]$ is Cauchy for every $A \in \mathscr{A}$. Finally, for every $A \in \mathscr{A}$ we have

$$F(A) = \lim\langle [A],[A_\alpha]\rangle = \lim\omega(A_\alpha^*A).$$

Conversely, suppose ω is a Riesz state and $x \in D(\pi_\omega^*)$. Define the linear functional $F: \mathscr{A} \to C$ by $F(A) = \langle [A], x\rangle$. Then for every $A, B \in \mathscr{A}$ we have

$$|F(BA)| = |\langle \pi(B)[A], x\rangle| \leqq M_B\|[A]\| = M_B\omega(A^*A)^{1/2}$$

so F is ω-bounded. Hence there exists a net $A_\alpha \in \mathscr{A}$ such that $\omega(A_\alpha^*A^*AA_\alpha)$ is Cauchy for every $A \in \mathscr{A}$ and $F(A) = \lim\omega(A_\alpha^*A)$ for every $A \in \mathscr{A}$. It follows that $[A_\alpha]$ is Cauchy in the $\pi_0(\mathscr{A})$-topology and hence there exists a $y \in D(\pi_\omega)$ such that $[A_\alpha] \to y$. Furthermore, for every $A \in \mathscr{A}$ we have $F(A) = \lim\omega(A_\alpha^*A) = \lim\langle [A],[A_\alpha]\rangle = \langle [A], y\rangle$. Hence $x = y \in D(\pi_\omega)$ and π_ω is self-adjoint.

COROLLARY. *A closed, strongly cyclic $*$-representation π with strongly cyclic vector x_0 is self-adjoint if and only if the state $A \to \langle \pi(A)x_0, x_0 \rangle$ is a Riesz state.*

A state ω is *faithful* if $\omega(A^*A) = 0$ implies $A = 0$. A vector $x_0 \in D(\pi)$ is *separating* if $\pi(A)x_0 = 0$ implies $\pi(A) = 0$. If ω is faithful then the strongly cyclic vector x_0 for π_ω is separating. Conversely, if x_0 is separating, then $\omega(A^*A) = 0$ implies $\pi_\omega(A) = 0$. A representation π of $\mathscr{A}$ is *ultra-cyclic* if there exists an $x_0 \in D(\pi)$ such that $D(\pi) = \{\pi(A)x_0 : A \in \mathscr{A}\}$. We then call x_0 an *ultra-cyclic vector.* Ultra-cyclic representations are important because of the following result.

LEMMA 8. *π is a closed, strongly cyclic $*$-representation if and only if π is the closure of an ultra-cyclic $*$-representation π^0.*

Proof. Suppose π is a closed, strongly cyclic $*$-representation of $\mathscr{A}$ with strongly cyclic vector x_0. Define $D(\pi^0) = \{\pi(A)x_0 : A \in \mathscr{A}\}$ and $\pi^0(B)\pi(A)x_0 = \pi(BA)x_0$. Then π^0 is an ultra-cyclic $*$-representation and $\bar{\pi}^0 = \pi$. Conversely, if π is the closure of an ultra-cyclic $*$-representation π^0 with ultra-cyclic vector x_0, then π is a closed $*$-representation. Moreover, since $D(\pi)$ is the completion of $D(\pi^0)$ in the $\pi(\mathscr{A})$-topology, x_0 is a strongly cyclic vector for π.

We call π^0 in the proof of Lemma 8 the *underlying* ultra-cyclic $*$-representation for π. We can obtain information about π by studying the simpler representation π^0. For example, a condition characterizing the essential self-adjointness of π^0 characterizes the self-adjointness of π. Moreover, $\pi^{0*} = \pi^*$ and $\pi^0(\mathscr{A})' = \pi(\mathscr{A})'$.

Let π be an arbitrary ultra-cyclic $*$-representation of $\mathscr{A}$ with a separating ultra-cyclic vector x_0. For $x \in D(\pi^*)$ define $\pi^c(x) \in L(D(\pi), D(\pi^*))$ by $\pi^c(x)\pi(A)x_0 = \pi^*(A)x$. This is a well-defined operator since $\pi(A)x_0 = \pi(B)x_0$ implies $\pi(A) = \pi(B)$. Then for every $y, z \in D(\pi)$ we have

$$\langle \pi(A^*)y, z \rangle = \langle y, \pi(A)z \rangle = \langle y, \pi(B)z \rangle = \langle \pi(B^*)y, z \rangle.$$

Hence $\pi(A^*) = \pi(B^*)$, so $\pi(A^*)^* = \pi(B^*)^*$ and finally

$$\pi^*(A) = \pi(A^*)^* \,|\, D(\pi^*) = \pi(B^*)^* \,|\, D(\pi^*) = \pi^*(B).$$

It is straightforward to see that $D(\pi)$ is a $*$-algebra with identity x_0 under the product $(\pi(A)x_0) \circ (\pi(B)x_0) = \pi(AB)x_0$ and involution $(\pi(A)x_0)^* =$

$\pi(A^*)x_0$. Moreover, for every $x, y, z \in D(\pi)$ we have $\langle x \circ y, z\rangle = \langle y, x^* \circ z\rangle$.

THEOREM 9. *Let π be an ultra-cyclic $*$-representation of $\mathscr{A}$ with a separating, ultra-cyclic vector x_0.*

(1) *π^c is a weakly continuous linear bijection from $D(\pi^*)$ into $\pi(\mathscr{A})^c$.*

(2) *The following statements are equivalent.*

(a) *$Cx_0 \in D(\pi)$ for every $C \in \pi(\mathscr{A})^c$.*

(b) *$\pi(\mathscr{A})^c$ is an op-algebra.*

(c) *π is self-adjoint.*

(3) *$\pi(\mathscr{A})^c$ is an op$*$-algebra if and only if π is self-adjoint and there exists an involution b on the $*$-algebra $D(\pi)$ satisfying*

$$\langle x^*, y\rangle = \langle y^b, x\rangle \tag{3.1}$$

for every $x, y \in D(\pi)$.

(4) *If $\pi(\mathscr{A})^c$ is an op$*$-algebra, then π^c is a weakly continuous ^{b}-anti-isomorphism of $D(\pi)$ onto $\pi(\mathscr{A})^c$.*

Proof.

(1) Clearly, π^c is linear. To show that π^c maps $D(\pi^*)$ into $\pi(\mathscr{A})^c$, for $x \in D(\pi^*)$, $A \in \mathscr{A}$, $z \in D(\pi)$ and $y = \pi(B)x_0 \in D(\pi)$ we have

$$\begin{aligned}\langle \pi^c(x)\pi(A)y, z\rangle &= \langle \pi^c(x)\pi(AB)x_0, z\rangle \\ &= \langle \pi^*(AB)x, z\rangle = \langle \pi^*(B)x, \pi(A^*)z\rangle \\ &= \langle \pi^c(x)\pi(B)x_0, \pi(A^*)z\rangle = \langle \pi^c(x)y, \pi(A^*)z\rangle.\end{aligned}$$

To show that π^c is surjective, let $C \in \pi(\mathscr{A})^c$. Then $Cx_0 \in D(\pi^*)$ and for any $y = \pi(A)\, x_0 \in D(\pi)$ we have

$$\begin{aligned}\pi^c(Cx_0)y &= \pi^c(Cx_0)\pi(A)x_0 = \pi^*(A)Cx_0 \\ &= C\pi(A)x_0 = Cy.\end{aligned}$$

To show that π^c is injective, suppose that $x, x_1 \in D(\pi^*)$ and $\pi^c(x) = \pi^c(x_1)$. Then

$$x = \pi^*(1)x = \pi^c(x)x_0 = \pi^c(x_1)x_0 = \pi^*(1)x_1 = x_1.$$

To show that π^c is weakly continuous, suppose that $x_i, x \in D(\pi^*)$ and

$x_i \to x$ in norm. Then for any $y = \pi(B)x_0 \in D(\pi)$ and $z \in D(\pi)$ we have

$$\begin{aligned}\lim\langle \pi^c(x_i)y, z\rangle &= \lim\langle \pi^*(B)x_i, z\rangle\\ &= \lim\langle x_i, \pi(B^*)z\rangle = \langle x, \pi(B^*)z\rangle\\ &= \langle \pi^*(B)x, z\rangle = \langle \pi^c(x)\pi(B)x_0, z\rangle\\ &= \langle \pi^c(x)y, z\rangle.\end{aligned}$$

(2) (a)$\to$(b). Suppose that (a) holds and $C \in \pi(\mathscr{A})^c, y = \pi(A)x_0 \in D(\pi)$. We then have $Cy = C\pi(A)x_0 = \pi^*(A)Cx_0 = \pi(A)Cx_0 \in D(\pi)$. Hence, by Lemma 6(6), $\pi(\mathscr{A})^c$ is an op-algebra.

(b)$\to$(c). If $x \in D(\pi^*)$, then by (1) $\pi^c(x) \in \pi(\mathscr{A})^c$. If $\pi(\mathscr{A})^c$ is an op-algebra, then $x = \pi^*(1)x = \pi^c(x)x_0 \in D(\pi)$. Hence $D(\pi^*) = D(\pi)$ and π is self-adjoint.

(c)$\to$(a). If π is self-adjoint, then $\pi(\mathscr{A})^c \subseteq L(D(\pi))$.

(3) Suppose $\pi(\mathscr{A})^c$ is an op$*$-algebra. Then, by (2), π is self-*adjoint. If* $C \in \pi(\mathscr{A})^c$, then $D(\pi) \subseteq D(C^*)$ and $C^* | D(\pi) \in \pi(\mathscr{A})^c$ so $C^*: D(\pi) \to D(\pi)$. For $x \in D(\pi)$, by (1) $\pi^c(x) \in \pi(\mathscr{A})^c$ so $x^b \equiv \pi^c(x)^* x_0 \in D(\pi)$. For $x = \pi(A)x_0 \in D(\pi)$ and $y \in D(\pi)$ we have

$$\begin{aligned}\langle y^b, x\rangle &= \langle \pi^c(y)^* x_0, x\rangle = \langle x_0, \pi^c(y)\pi(A)x_0\rangle\\ &= \langle x_0, \pi(A)y\rangle = \langle \pi(A^*)x_0, y\rangle = \langle x^*, y\rangle\end{aligned}$$

so (3.1) holds. That ^{b} is an involution now follows from (3.1). For example,

$$\begin{aligned}\langle (y \circ z)^b, x\rangle &= \langle x^*, y \circ z\rangle = \langle y^* \circ x^*, z\rangle\\ &= \langle z^b, x \circ y\rangle = \langle x^* \circ z^b, y\rangle = \langle y^b, z^{b^*} \circ x\rangle\\ &= \langle z^b \circ y^b, x\rangle.\end{aligned}$$

The other properties of an involution follow in a similar way. Conversely, suppose π is self-adjoint and there exists an involution ^{b} on $D(\pi)$ satisfying (3.1). Then by (2), $\pi(\mathscr{A})^c$ is an op-algebra. If $C \in \pi(\mathscr{A})^c$, then for any $x = \pi(B)x_0 \in D(\pi)$ and $y = \pi(A)x_0 \in D(\pi)$ we have

$$\begin{aligned}\langle Cy, x\rangle &= \langle C\pi(A)x_0, x\rangle = \langle Cx_0, \pi(A^*)x\rangle\\ &= \langle Cx_0, \pi(A^*B)x_0\rangle = \langle Cx_0, [\pi(B^*A)x_0]^*\rangle\\ &= \langle \pi(B^*A)x_0, (Cx_0)^b\rangle = \langle \pi(A)x_0, \pi(B)[(Cx_0)^b]\rangle\\ &= \langle y, \pi^c[(Cx_0)^b]x\rangle.\end{aligned}$$

Hence $D(\pi) \subseteq D(C^*)$, $C^* | D(\pi) = \pi^c[(Cx_0)^b] \in \pi(\mathscr{A})^c$ and so $\pi(\mathscr{A})^c$ is an op$*$-algebra.

(4) Suppose $\pi(\mathcal{A})^c$ is an op*-algebra. It follows from (1) that $\pi^c: D(\pi) \to \pi(\mathcal{A})^c$ is a weakly continuous linear bijection. For $x = \pi(A)x_0 \in D(\pi)$ and $y = \pi(B)x_0 \in D(\pi)$ we have

$$\pi^c(x)y = \pi^c(x)\pi(B)x_0 = \pi(B)x = \pi(B)x_0 \circ \pi(A)x_0 = y \circ x.$$

It is now clear that π^c is an anti-isomorphism. To show that π^c is a ^{b}-anti-isomorphism, for $x \in D(\pi)$ and $y = \pi(A)x_0 \in D(\pi)$ we have

$$\begin{aligned}\pi^c(x^b)y &= y \circ x^b = y \circ [\pi^c(x)^* x_0] = \pi(A)\pi^c(x^*)x_0 \\ &= \pi^c(x)^*\pi(A)x_0 = \pi^c(x)^* y.\end{aligned}$$

COROLLARY. *Let π be a closed, strongly cyclic *-representation of $\mathcal{A}$ with separating, strongly cyclic vector x_0 and let π^0 be the underlying ultra-cyclic representation. Then π is self-adjoint if and only if $Cx_0 \in D(\pi)$ for every $C \in \pi^0(\mathcal{A})^c$.*

Proof. If π is self-adjoint and $C \in \pi^0(\mathcal{A})^c$, then $Cx_0 \in D(\pi^{0*}) = D(\pi^*) = D(\pi)$. Conversely, suppose $Cx_0 \in D(\pi)$ for every $C \in \pi^0(\mathcal{A})^c$. If $x \in D(\pi^*)$, then $x \in D(\pi^{0*})$ so by Theorem 9(1), $\pi^c(x) \in \pi^0(\mathcal{A})^c$. Hence $x = \pi^c(x)x_0 \in D(\pi)$, so $D(\pi) = D(\pi^*)$ and π is self-adjoint.

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