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SCHUR INDICES OVER THE 2-ADIC FIELD

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In this paper it is proved that if G is a finite group with abelian Sylow 2-subgroups, then the Schur index of any character of G over the 2-adic numbers Q_2 is equal to 1. Examples are given so as to show that this statement is false for each odd prime p.

The problem of determining the Schur index of a character of a finite group was reduced by R. Brauer and E. Witt to the case of handling hyper-elementary groups at q, q being a prime. Each of these groups has a cyclic normal subgroup with a factor group which is a q-group. Let p be a prime and Q_p the p-adic numbers. Let G be a hyper-elementary group at q and χ an irreducible character of G. It follows from a result of Witt [1] that if $p = q \neq 2$ then the Schur index $m_{Q_p}(\chi)$ of χ over Q_p is equal to 1. This statement is false for the case $p = q = 2$, because the quaternion group of order 2^3 has an irreducible character χ with $m_{Q_2}(\chi) = 2$.

The purpose of this paper is to show that the above statement also holds for the case $p = q = 2$, provided the Sylow 2-subgroups of a hyper-elementary group at 2 are abelian. In fact, we will prove more generally the following theorem.

THEOREM. *Let G be a finite group with abelian Sylow 2-subgroups. Let χ be any irreducible character of G. Then $m_{Q_2}(\chi) = 1$, that is the Schur index of χ over the 2-adic numbers Q_2 is equal to* 1.

Proof. It is well-known that $m_{Q_2}(\chi) = 1$ or 2 (cf. [1]), so $m_{Q_2}(\chi)$ equals its 2-part. Let n be the exponent of G and let L be the subfield of $Q_2(\zeta_n)$, ζ_n a primitive nth root of unity, such that $L \supset Q_2(\chi)$, $2 \nmid [L\colon Q_2(\chi)]$ and $[Q_2(\zeta_n)\colon L]$ is a power of 2. By the Brauer-Witt theorem [3, p. 31] there is an L-elementary subgroup H of G with respect to 2 and an irreducible character θ of H with the following properties: (1) there is a normal subgroup N of H and a linear character ψ of N such that $\theta = \psi^H$; (2) $H/N \cong \mathrm{Gal}\,(L(\psi)/L)$, in particular, H/N is a 2-group; (3) $L(\theta) = L$; (4) $m_L(\theta) = m_L(\chi) = m_{Q_2(\chi)}(\chi) = m_{Q_2}(\chi)$; (5) for every $h \in H$ there is a $\tau(h) \in \mathrm{Gal}\,(L(\psi)/L)$ such that $\psi(hnh^{-1}) = \tau(h)(\psi(n))$ for all $n \in N$; (6) $m_L(\theta)$ is the index of the crossed product $(\beta, L(\psi)/L)$ where, if D is a complete set of coset representatives of N in H $(1 \in D)$ with $hh' = n(h, h')h''$ for $h, h', h'' \in D$, $n(h, h') \in N$, then $\beta(\tau(h), \tau(h')) = \psi(n(h, h'))$. Since ψ is a linear character of N, the values of the factor set β are roots of unity.

Denote by N_0 the kernel of ψ. Then the factor group N/N_0 is cyclic. Put $2^r t = |N/N_0|$, $(2, t) = 1$. It is easy to see that there exist elements a, b of N such that $N/N_0 = \langle aN_0\rangle \times \langle bN_0\rangle$, $a^{2^r} \in N_0$, $b^t \in N_0$ and that the order of a is a power of 2. We have $\psi(a) = \zeta_{2^r}$, $\psi(b) = \zeta_t$, and $Q_2(\psi) = Q_2(\zeta_{2^r}, \zeta_t)$, where ζ_{2^r} and ζ_t are some primitive 2^rth and tth roots of unity, respectively. Let P be a Sylow 2-subgroup of H, which contains a. Since H/N is a 2-group, we may clearly assume that $D \subset P$. By assumption, P is abelian. Hence for each $x \in D$, $xax^{-1} = a$, and so

$$\theta(a) = \psi^H(a) = \sum_{x \in D} \psi(xax^{-1}) = |D|\,\psi(a) = |D|\,\zeta_{2^r} .$$

Consequently, $\zeta_{2^r} \in L = L(\theta)$.

Since $L(\psi) = L(\zeta_{2^r}, \zeta_t) = L(\zeta_t)$, $(2, t) = 1$, it follows that the extension $L(\psi)/L$ is unramified. Recall that the values of the factor set β are roots of unity. Hence the crossed product $(\beta, L(\psi)/L)$ is similar to L, i.e., $(\beta, L(\psi)/L) \sim L$ (cf. [3, Lemma 4.2]). This implies $m_{Q_2}(\chi) = m_L(\theta) = 1$, and the theorem is proved.

If p is an odd prime, then Witt [1] determined that $m_{Q_p}(\chi)$ divides $p-1$ for an irreducible character χ of a finite group G. Let d be a natural number that divides $p-1$. We now give an irreducible character χ of a finite group G with abelian Sylow p-subgroups such that $m_{Q_p}(\chi) = d$: The group G is generated by the elements x, y with defining relations

$$x^p = 1 , \qquad y^{d(p-1)} = 1 , \qquad xyx^{-1} = x^r ,$$

where r is a primitive root modulo p. (This group was dealt with in Appendix of [2].)

Now put $H = \langle x\rangle \times \langle y^{p-1}\rangle$. Then H is a normal, cyclic subgroup of G of order pd, the factor group G/H is cyclic of order $p-1$, and $G = H \cup Hy \cup \cdots \cup Hy^{p-2}$. Let ψ be the faithful linear character of H given by $\psi(x) = \zeta_p$, $\psi(y^{p-1}) = \zeta_d$. For each $i = 1, \cdots, p-2$, the character ψ^{y^i} of H defined by $\psi^{y^i}(z) = \psi(y^i z y^{-i})$, $z \in H$, is algebraically conjugate to ψ over the field $Q_p(\zeta_d)$, and $\psi^{y^i} \neq \psi$. It follows that the induced character $\chi = \psi^G$ is irreducible and that the simple component of the group algebra $Q_p[G]$ which corresponds to χ is isomorphic to the cyclic algebra $B = (\zeta_d, Q_p(\zeta_d, \zeta_p)/Q_p(\zeta_d), \sigma)$, where $\langle\sigma\rangle = \mathrm{Gal}\,(Q_p(\zeta_d, \zeta_p)/Q_p(\zeta_d))$, $\sigma(\zeta_p) = \zeta_p^r$, $\sigma(\zeta_d) = \zeta_d$ (cf. Propositions 3.4, 3.5 of [3]). Since $p \equiv 1 \pmod{d}$, then $Q_p(\zeta_d) = Q_p$, so $B = (\zeta_d, Q_p(\zeta_p)/Q_p, \sigma)$. It is easy to see that the index of this cyclic algebra is equal to d (see also Theorem 4.3 of [3]). Thus we conclude that $m_{Q_p}(\chi) = d$.

The above example shows that the similar statement to the theorem for each odd prime p does not hold.

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