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THREE DIMENSIONAL HOMOGENEOUS ALGEBRAS

J. A. MACDOUGALL AND L. G. SWEET

An algebra A is homogeneous if its automorphism group acts transitively on the set of one dimensional subspaces of A. In this paper the structure of all three dimensional homogeneous algebra is determined. These fall into three classes: (1) truncated quaternion algebras over formally real Pythagorean fields; (2) an algebra over GF(2) in which $x^2 = x$ for all x in A, and (3) two algebras over GF(2) which are generated by each of their nonzero elements. The automorphism group is determined in each case.

All algebras considered are assumed to be finite dimentional but not necessarily associative. If A is an algebra we denote its group of algebra automorphisms by $\mathrm{Aut}(A)$. An algebra A is said to be homogeneous if $\mathrm{Aut}(A)$ acts transitively on the set of one dimensional subspaces of A. The reader is referred to a paper by one of the authors [3] for a discussion of arbitrary homogeneous algebras and a bibliography of the related literature. The purpose of this paper is to determine the structure of all three dimensional homogeneous algebras.

Throughout this paper we assume that A is a nonzero three dimensional homogeneous algebra. Such algebras can be divided into four types in the following way. Let a be any nonzero element of a and let $\langle a \rangle$ denote the algebra generated by a. Then the four types are as follows:

Type 1. $a^2 = 0$
Type 2. $a^2 = \lambda a$, λ a nonzero scalar
Type 3. $\dim \langle a \rangle = 2$
Type 4. $\dim \langle a \rangle = 3$.

We now investigate each type separately.

Type 1. $a^2 = 0$.

Since $a^2 = 0$ the homogeneity condition implies that $x^2 = 0$ for all $x \in A$ and this implies that A is anti-commutative. Clearly A is not a quasi division algebra and so it follows from the results of Shult [1] and Gross [2] that the underlying field K must be infinite.

Let a be any nonzero element of A. Suppose we can find a nonzero $b \in A$, $b \neq \lambda a$ such that

$$ab = \lambda_1 a + \lambda_2 b\ .$$

If $\lambda_2 = 0$ then $\mathrm{tr}\, L_a = 0$ (see [3]) implies that L_a is nilpotent. But then the homogeneity condition implies that L_x and R_x are nilpotent for all x and so A is a special nil algebra. Since K is infinite we may use Theorem 2 of [4] to conclude that $A^2 = 0$. If $\lambda_2 \neq 0$ then extend $\{a, ab\}$ to a basis for A. It is now easy to contradict the fact that L_a and L_{ab} must be projectively similar. Hence ab never depends on a and b when a and b are independent.

Now choose a basis a, b, ab for A. Then

$$L_a = \begin{bmatrix} 0 & 0 & \alpha_1 \\ 0 & 0 & \alpha_2 \\ 0 & 1 & 0 \end{bmatrix} \quad L_b = \begin{bmatrix} 0 & 0 & \beta_1 \\ 0 & 0 & \beta_2 \\ -1 & 0 & 0 \end{bmatrix} \quad L_{ab} = \begin{bmatrix} -\alpha_1 & -\beta_1 & 0 \\ -\alpha_2 & -\beta_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

If necessary we can choose a new b to force $\alpha_1 = 0$ and then $\mathrm{tr}\, L_{ab} = 0$ implies that $\beta_2 = 0$. So we assume we have a basis a, b and ab such that

$$L_a = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha \\ 0 & 1 & 0 \end{bmatrix} \quad L_b = \begin{bmatrix} 0 & 0 & \beta \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad L_{ab} = \begin{bmatrix} 0 & -\beta & 0 \\ -\alpha & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

If $\alpha = 0$ then the fact that L_a and L_b are projectively similar forces $\beta = 0$ and then $L_{ab} = 0$ which is impossible. Hence $\alpha \neq 0$ and similarly $\beta \neq 0$.

We now show that A is a homogeneous algebra under the following conditions:

(i) $-\alpha$ is a nonzero square in K.

(ii) β is a nonzero square in K.

(iii) K has the property that the sum of nonzero squares is always a nonzero square (such a field is called a formally real Pythagorean field).

Let $\sigma \in \mathrm{Aut}(A)$. By considering $\sigma(xy) = \sigma(x)\sigma(y)$ as x and y run through the basis ab, a, b it is easy to show that

$$\sigma = \begin{bmatrix} C_{11} & -\beta/\alpha\, C_{21} & \beta C_{13} \\ -\alpha/\beta\, C_{21} & C_{22} & -\alpha C_{23} \\ 1/\beta\, C_{31} & -1/\alpha\, C_{31} & C_{33} \end{bmatrix}$$

where C_{ij} is the cofactor of the ij entry. Conversely any invertible matrix of this form represents an automorphism of A. It remains to be shown under what conditions $\mathrm{Aut}(A)$ actually acts transitively on the one dimensional subspaces of A. By considering compositions of automorphisms it is easy to see that A is homogeneous if and only if there exists a $\sigma \in \mathrm{Aut}(A)$ such that $\sigma(a) = \gamma(\lambda_1 a + \lambda_2 b + \lambda_3 ab)$ for any nonzero triple $(\lambda_1, \lambda_2, \lambda_3)$. If such an automorphism exists then $\sigma L_a = \gamma L_{\lambda_1 a+\lambda_2 b+\lambda_3 ab}\sigma$ and this implies that

$$-\alpha = \gamma^2(-\alpha\lambda_1^2 + \beta\lambda_2^2 - \beta\alpha\lambda_3^2) .$$

Since $\alpha \neq 0$ this equation forces conditions (1), (2), and (3). On the other hand suppose conditions (1), (2), and (3) are true. We wish to construct an automorphism σ of the form

$$\sigma = \begin{bmatrix} \gamma\lambda_1 & x_1 & x_2 \\ \gamma\lambda_2 & x_3 & x_4 \\ \gamma\lambda_3 & x_5 & x_6 \end{bmatrix} = \begin{bmatrix} C_{11} & -\beta/\alpha\, C_{12} & \beta C_{13} \\ -\alpha/\beta\, C_{21} & C_{22} & -\alpha C_{23} \\ 1/\alpha\, C_{31} & -1/\alpha\, C_{32} & C_{33} \end{bmatrix}.$$

This gives us a homogeneous linear system of the form $Bx = 0$ where $x = (x_1, x_2, x_3, x_4, x_5, x_6)$ and

$$B = \begin{bmatrix} 1 & 0 & 0 & \beta/\alpha\,\gamma\lambda_3 & 0 & -\beta/\alpha\,\gamma\lambda_2 \\ 0 & 1 & \beta\gamma\lambda_3 & 0 & -\beta\gamma\lambda_2 & 0 \\ 0 & \gamma\lambda_3 & 1 & 0 & 0 & -\gamma\lambda_1 \\ \alpha\gamma\lambda_3 & 0 & 0 & 1 & -\alpha\gamma\lambda_1 & 0 \\ 0 & \gamma/\alpha\,\lambda_2 & 0 & -\gamma/\alpha\,\lambda_1 & 1 & 0 \\ \gamma\lambda_2 & 0 & -\gamma\lambda_1 & 0 & 0 & 1 \end{bmatrix}$$

and also another system of quadratic equations

$$\gamma\lambda_1 = x_3x_6 - x_4x_5$$

$$\gamma\lambda_2 = \frac{\alpha}{\beta}(x_1x_6 - x_2x_5)$$

$$\gamma\lambda_3 = \frac{1}{\beta}(x_1x_4 - x_2x_3) .$$

If $\lambda_1 = \lambda_2 = 0$ then it can be checked that $x_1 = 0$, $x_2 = -\beta\gamma\lambda_3$, $x_3 = 1$, $x_4 = 0$, $x_5 = 0$, $x_6 = 0$ is a solution of both systems if we take $\gamma = 1/\sqrt{u}\,\lambda_3$. Suppose λ_1 and λ_2 are not both zero. Let $d = \sqrt{-\alpha\lambda_1^2 + \beta\lambda_2^2}$. Then it can be checked that

$$x_1 = \frac{\beta\gamma\lambda_2}{\alpha d}, \qquad x_1 = \frac{-\beta\gamma^2\lambda_1\lambda_3}{d}, \qquad x_3 = \frac{\gamma\lambda_1}{d}, \qquad x_4 = \frac{-\beta\gamma^2\lambda_2\lambda_3}{d},$$

$x_5 = 0$, $x_6 = d$ is a solution of both systems if we take

$$\gamma = \sqrt{\frac{-\alpha}{-\alpha\lambda_1^2 + \beta\lambda_2^2 - \alpha\beta_3^2}} .$$

Hence A is homogeneous.

So if A is a three dimensional homogeneous algebra of Type 1 we can choose a basis a, b, ab so that the multiplication table becomes

	a	b	ab
a	0	ab	αb
b	$-ab$	0	βa
ab	$-\alpha b$	$-\beta a$	0

where

(i) $-\alpha$ is a nonzero square in K.

(ii) b is a nonzero square in K.

(iii) K is a formally real Pythagorean field.

These algebras are related to the so-called quaternion algebras. Let K be any field and V be a 4-dimensional vector space over K with basis $1, x_1, x_2, x_3$. Now define a multiplication on V by using the following table where α, β are any nonzero scalars

	1	x_1	x_2	x_3
1	1	x_1	x_2	x_3
x_1	x_1	$\alpha 1$	x_3	αx_2 .
x_2	x_2	$-x_3$	$\beta 1$	$-\beta x_1$
x_3	x_3	$-\alpha x_2$	βx_1	$-\alpha\beta 1$

Then V is called a quaternion algebra with parameters α and β. We can now define a 3-dimensional algebra over K by deleting the top row, the left-most column and replacing the main diagonal of the above table with zeros. The resulting algebra is called the truncated algebra of pure quaternions.

We have shown that if A is a 3-dimensional homogeneous algebra of Type 1 over a field K then A is a truncated quaternion algebra with parameters $\alpha, -\beta$ where $-\alpha$ and β are nonzero squares and K is formally real Pythagorean field. It is interesting to note that all such algebras over a given field are actually isomorphic. In particular consider A_1, the usual vector cross product with basis i, j, k where $ij = -ji = k$, $ik = -ki = -j$, $jk = -kj = i$, $i^2 = j^2 = k^2 = 0$. Suppose A_2 is a homogeneous algebra with $ab = -ba = c$, $ac = -ca = \alpha b$, $bc = -cb = \beta a$, $a^2 = b^2 = c^2 = 0$ where $-\alpha$ and β are squares. Define a linear map $\sigma: A_2 \to A_1$, by extending $\sigma(a) = -\sqrt{-\alpha} i$, $\sigma(b) = \sqrt{\beta} j$ and $\sigma(c) = -\sqrt{-\alpha\beta} k$. Then it is easily checked that σ is an algebra isomorphism. So we have shown that a 3-dimensional homogeneous algebra of Type 1 is isomorphic to the usual vector cross product algebra over a formally real Pythagorean field K.

Type 2. $a^2 = \lambda a$, λ a nonzero scalar.

In this case the homogeneity condition implies that $x^2 = \lambda x$ for all $x \in A$ where λ is a nonzero scalar which may depend on x. Clearly A must be power associative. It was shown in [3] that K must be GF(2). But Gross showed [1] that the only nonzero homogeneous algebras over GF(2) are always commutative, quasi division algebras.

Choose $a, b \in A$ with $b \neq a$. Then $ab \neq a$ and $ab \neq b$. Also since $\operatorname{tr} L_a = 0$ but $\det L_a = 1$ it follows that $ab \neq a + b$. Hence a, b, ab form a basis for A. With respect to this basis we have

$$L_a = \begin{bmatrix} 1 & 0 & \alpha_1 \\ 0 & 0 & \alpha_2 \\ 0 & 1 & 1 \end{bmatrix} \quad L_b = \begin{bmatrix} 0 & 0 & \beta_1 \\ 0 & 1 & \beta_2 \\ 1 & 0 & 1 \end{bmatrix} \quad L_{ab} = \begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ 1 & 1 & \alpha_1 + \beta_2 \end{bmatrix}.$$

Since $\det L_a = \det L_b = 1$ we must have $\alpha_2 = \beta_1 = 1$. If necessary replace b by $a + b$ to force $\alpha_1 = 1$. Since $\det L_{ab} = 1$ we must have $\beta_2 = 0$. Also $\det (L_a + L_{ab}) = 1$ forces $\gamma_2 = 0$ and finally $\det (L_b + L_{ab}) = 1$ forces $\gamma_1 = 0$.

Hence A is of the form

	a	b	ab
a	a	ab	$a + b + ab$
b	ab	b	$a + ab$
ab	$a + b + ab$	$a + ab$	ab

It is easily checked that A is a homogeneous algebra. In fact $\operatorname{Aut}(A)$ is the group generated by

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Type 3. $\dim \langle a \rangle = 2$.

In this case $\langle a \rangle$ is a 2-dimensional subalgebra for each nonzero a in A. Fix $a \in A - (0)$ and choose $b \in A - \langle a \rangle$. Then $\langle a \rangle$ and $\langle b \rangle$ are two distinct 2-dimensional subalgebras of the 3-dimensional subalgebra A and so $\langle a \rangle \cap \langle b \rangle$ is a 1-dimensional subalgebra $\langle c \rangle$, contradicting the first line of this paragraph. Hence there are no three dimensional homogeneous algebras of *Type* 3.

Type 4. $\dim \langle a \rangle = 3$.

In this case we have $\langle a \rangle = A$. We first assume that A is commutative. The either a, a^2, aa^2 or a, a^2, a^2a^2 must form a basis. We consider the two cases separately.

(a) Suppose a, a^2, aa^2 do not form a basis. Then $aa^2 = \gamma_1 a + \gamma_2 a^2$ and a, a^2, a^2a^2 does form a basis. The homogeneity condition now implies that for any $\lambda \in K$

$$(a + \lambda a^2)(a + \lambda a^2)^2 = \gamma_1'(a + \lambda a^2) + \gamma_2'(a + \lambda a^2)^2 \quad \text{for some } \gamma_1', \gamma_2' \in K\,.$$

Simplifying and comparing coefficients with respect to the basis a, a^2, a^2a^2 we get the following system

$$\begin{aligned} \gamma_1' + \gamma_2'(2\lambda\gamma_1) &= \gamma_1 + \lambda(2\gamma_1\gamma_2) + \lambda^2(\alpha_1 + 2\gamma_1^2) + \lambda^3\beta_1 \\ \gamma_1'\lambda + \lambda_2'(1 + 2\lambda\gamma_2) &= \gamma_2 + \lambda(2\gamma_1 + 2\gamma_2^2) + \lambda^2(\alpha_2 + 2\gamma_1\gamma_2) + \lambda^3\beta_2 \\ \gamma_2'\lambda^2 &= \lambda + \lambda^2(-\gamma_2 + 2\gamma_2) + \lambda^3(-\gamma_1)\,. \end{aligned}$$

Now solving the first two equations for γ_2' and comparing to the third equation gives us

$$\lambda + \lambda^2(2\gamma_2) + \lambda^3(-4\gamma_1) + \lambda^4(-4\gamma_1\gamma_2 - \alpha_2) + \gamma^5(4\gamma_1^2 - \beta_2 + \alpha_1) - \lambda^6\beta_1 = 0 \quad \text{for all } \lambda \in K\,.$$

This implies that the field is finite and so we know that $K = \mathrm{GF}(2)$ and A is a commutative quasi division algebra. Now with respect to the basis a, a^2, a^2a^2

$$L_a = \begin{bmatrix} 0 & 1 & \alpha_1 \\ 1 & 1 & \alpha_2 \\ 0 & 0 & 1 \end{bmatrix} \quad L_{a^2} = \begin{bmatrix} 1 & 0 & \beta_1 \\ 1 & 0 & \beta_2 \\ 0 & 1 & 1 \end{bmatrix} \quad L_{a^2a^2} = \begin{bmatrix} \alpha_1 & \beta_1 & \xi_1 \\ \alpha_2 & \beta_2 & \xi_2 \\ 1 & 1 & \alpha_1 + \beta_2 \end{bmatrix}.$$

Now the fact that L_a and L_{a^2} are similar implies that $\beta_2 = 1$. Also $\det L_{a^2} = \det(L_a + L_{a^2}) = \det(L_{a^2} + L_{a^2a^2}) = 1$ implies that $\beta_1 = 0$, $\alpha_2 = 0$ and $\xi_1 = 1$ respectively. Finally $\xi_2 = \alpha_1 = 0$ because $L_{a^2a^2}$ and $L_a + L_{a^2a^2}$ are similar to L_a. It follows that A has a basis a, a^2, a^2a^2 with the following multiplication table:

	a	a^2	a^2a^2
a	a^2	$a + a^2$	a^2a^2
a^2	$a + a^2$	a^2a^2	$a^2 + a^2a^2$
a^2a^2	a^2a^2	$a^2 + a^2a^2$	$a + a^2a^2$

It is easily checked that A is indeed a homogeneous algebra. In fact $\mathrm{Aut}(A)$ is a cyclic group of order seven generated by

$$\sigma = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

(b) Suppose a, a^2, aa^2 do form a basis. If K is finite then $K = \mathrm{GF}(2)$ and A must be a quasi division algebra. In fact it follows from the papers of Gross [1] and Shult [2] that the characteristic polynomial of L_a must be x^3+1. Then with respect to our basis a, a^2, aa^2 we have

$$L_a = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad L_{a^2} = \begin{bmatrix} 0 & \beta_1 & \gamma_1 \\ 0 & \beta_2 & \gamma_2 \\ 1 & \beta_3 & \gamma_3 \end{bmatrix} \quad L_{aa^2} = \begin{bmatrix} 1 & \gamma_1 & \xi_1 \\ 0 & \gamma_2 & \xi_2 \\ 0 & \beta_2 & 1+\gamma_2 \end{bmatrix}.$$

Now the equations $\det(\gamma_1 L_a + L_{a^2}) = \det(\beta_2 L_a + L_{aa^2}) = \det L_{a^2} = \det L_{aa^2} = \det(L_{a^2} + L_{aa^2}) = 1$ imply that $\beta_1 = 1 + \gamma_2 = (\beta_2 = \gamma_1) = (\beta_2 = \xi_2) = (\beta_3 + \xi_1) = 1$ respectively. Since A is a homogeneous quasi division algebra generated by each of its elements we know that each automorphism (except the identity) is fixed point free. Consider the automorphism σ for which $\sigma(a) = a^2$. It is easily checked that this automorphism has an eigenvalue if $\beta_3 = 0$. So we must have $\beta_3 = 1$ and $\xi_1 = 0$. It follows that A has a basis a, a^2, aa^2 with the multiplication table:

	a	a^2	aa^2
a	a^2	aa^2	a
a^2	aa^2	$a + a^2 + aa^2$	$a + aa^2$
aa^2	a	$a + aa^2$	$a^2 + aa^2$

It is easily checked that this is a homogeneous algebra. In fact $\mathrm{Aut}(A)$ is a cyclic group of order 7 generated by

$$\sigma = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Now assume that K is infinite. Then with respect to the basis a, a^2, aa^2 we have

$$L_a = \begin{bmatrix} 0 & 0 & \alpha_1 \\ 1 & 0 & \alpha_2 \\ 0 & 1 & 0 \end{bmatrix} \quad L_{a^2} = \begin{bmatrix} 0 & \beta_1 & \gamma_1 \\ 0 & \beta_2 & \gamma_2 \\ 1 & \beta_3 & -\beta_2 \end{bmatrix} \quad L_{aa^2} = \begin{bmatrix} \alpha_1 & \gamma_1 & \xi_1 \\ \alpha_2 & \gamma_2 & \xi_2 \\ 0 & -\beta_2 & -\alpha_1 - \gamma_2 \end{bmatrix}.$$

Suppose $\alpha_1 = 0$. Then $\det(\lambda_1 L_a + \lambda_2 L_{a^2} + \lambda_3 L_{aa^2}) = 0$ for all $\lambda_1, \lambda_2, \lambda_3 \in K$. But

$$\begin{aligned} \det(\lambda_1 L_a + \lambda_2 L_{a^2} + \lambda_3 L_{aa^2}) &= \lambda_1^2\lambda_2\gamma_1 + \lambda_1^2\lambda_3\xi_1 + \lambda_1\lambda_2^2(\beta_3\gamma_1 + \beta_1\beta_2 + \beta_1\alpha_2) \\ &+ \lambda_1\lambda_2\lambda_3(2\alpha_2\gamma_1 + \beta_1\alpha_2 + \beta_3\xi_1) + \lambda_1\lambda_3^2(\gamma_1\gamma_2 - \beta_2\xi_1 + \alpha_2\xi_1) \\ &+ \lambda_2^2\lambda_3(\alpha_2\beta_3\gamma_1 + \alpha_2\beta_1\beta_2 + \beta_1\xi_2 - \beta_2\xi_1) + \lambda_2\lambda_3^2(\alpha_2\beta_1\gamma_2 + \alpha_2\beta_3\xi_1 + \xi_2\gamma_1 - \gamma_2\xi_1) \\ &+ \lambda_3^3\alpha_2(\gamma_1\gamma_2 - \beta_2\xi_1) + \lambda_2^3(\beta_1\gamma_2 - \beta_2\gamma_1) = 0\,. \end{aligned}$$

Since K is infinite this implies that all the coefficients must be zero. It follows that $\gamma_1 = \xi_1 = \beta_1(\beta_2 + \alpha_2) = \beta_1\gamma_2 = 0$. If $\beta_1 = 0$ the equation $xy = a$ has no solution which is impossible in a nonzero homogeneous algebra. Hence $\beta_1 \neq 0$ and $\beta_2 = -\alpha_2$. If $\alpha_1 = 0$, then rank $L_a = 2$ but rank $(L_{aa^2} - \alpha_2 L_a) < 2$ which is impossible. We conclude that $\alpha_1 \neq 0$ and so A must be a quasi division algebra. Since $\langle a \rangle = A$ we know that no automorphism of A (except the identity) can have and eigenvalue.

Now consider the automorphism σ for which $\sigma(a) = \mu(a + \lambda a^2)$ where λ is arbitrary and μ may depend on λ. Then it can be checked that

$$\sigma = \begin{bmatrix} \mu & \mu^2\lambda^2\beta_1 & \mu^3(\lambda(\beta_1+2\alpha_1)+\lambda^2(\beta_3\alpha_1+2\gamma_1)+\lambda^3(\beta_1\beta_2+\beta_3\gamma_1)) \\ \mu\lambda & \mu^2(1+\lambda^2\beta_2) & \mu^3(\lambda(\beta_2+2\alpha_2)+\lambda^2(\beta_1+\beta_3\alpha_2+2\gamma_2)+\lambda^3(\beta_2^2+\beta_3\gamma_2)) \\ 0 & \mu^2(2\lambda+\lambda^2\beta_2) & \mu^3(1+\lambda\beta_3-\lambda^2\beta_2+\lambda^3\beta_1) \end{bmatrix} .$$

Suppose char $K \neq 2$. If $\beta_3 \neq 0$ then letting $\lambda = -2/\beta_3$ gives σ an eigenvalue which is impossible. Hence we must have $\beta_3 = 0$. But then consider the automorphism τ for which $\tau(a) = \nu a^2$. It is easily checked that τ has an eigenvalue. Hence it follows that char $K = 2$.

We now consider $\sigma(a^2a^2) = \sigma(a^2)\sigma(a^2)$. This gives us a system of 3 equations which can be solved to get

$$\begin{aligned} 1 + \lambda\beta_3 &+ \lambda^4(\beta_1\beta_3 + \beta_3^2\alpha_2 + \beta_2^2) + \lambda^5\beta_3(\beta_2^2 + \beta_3\gamma_2 + \beta_3\alpha_1) \\ &+ \lambda^6(\beta_1^2 + \beta_1\beta_2\beta_3 + \beta_3^2\gamma_1) \\ &= \mu[1 + \lambda^4(\beta_2^2 + \beta_3\alpha_1 + \beta_3\gamma_2) + \lambda^6(\beta_1^2 + \beta_3^2\xi_2 + \beta_2\beta_3\alpha_1 + \beta_2\beta_3\gamma_2) \\ &+ \lambda^7(\beta_3^2\xi_1 + \beta_1\beta_3\alpha_1 + \beta_1\beta_3\gamma_2)] \end{aligned}$$

or $f = \mu g$ where f and g are polynomials λ. Squaring we have $f^2 = \mu^2 g^2$. But since $\sigma(a) = \mu(a + \lambda a^2)$ we have $\alpha_2 = \mu^2 h$ where $h = \alpha_2 + \lambda(\beta_1 + \alpha_1 + \gamma_2 + \beta_3\alpha_2) + \lambda^2(\gamma_1 + \beta_2^2 + \beta_3\gamma_2)$ and so comparing we have

$$\alpha_2 g^2 = f^2 h$$

and so

$$\begin{aligned} \alpha_2(1 + \lambda^8 * + \lambda^{12} * + \lambda^{14} *) &= (1 + \lambda^2\beta_3 + \lambda^8 * + \lambda^{10} * + \lambda^{12} *) \\ &\times (\alpha_2 + \lambda(\beta_1 + \alpha_1 + \gamma_2 + \beta_3\alpha_2) + \lambda^2(\alpha_1 + \beta_2^2 + \beta_3\gamma_2)) . \end{aligned}$$

Since the field is infinite and λ is arbitrary all the coefficients of λ must be zero. Considering the coefficients of λ, λ^4, and λ^2 in that order we conclude that

$$\begin{aligned} \beta_1 + \alpha_1 + \gamma_2 + \beta_3\alpha_2 &= 0 \\ \gamma_1 + \beta_2^2 + \beta_3\alpha_2 &= 0 \\ \alpha_2 &= 0 . \end{aligned}$$

Now $\alpha_2 = 0$ and since A is a quasi division algebra we may without loss of generality assume that L_a has an eigenvalue of 1. That is, we may assume that $\alpha_1 = 1$. But again since $\sigma(a) = \mu(a + \lambda a^2)$ we conclude that

$$1 = \mu^3(1 + \lambda(\gamma_1 + \beta_3) + \lambda^2(\beta_1\beta_2 + \beta_3\gamma_1 + \beta_2) + \lambda^3(\beta_1\gamma_2 + \beta_2\gamma_1))$$

or

$$1 = \mu^3 k .$$

From above we have $f^3 = \mu^3 g^3$ and so comparing we have

$$f^3 k = g^3 .$$

That is

$$(1 + \lambda\beta_3 + \lambda^2\beta_3^2 + \lambda^3\beta_3^3 + \cdots + \lambda^{18}*)(1 + \lambda(\gamma_1 + \beta_3) + \lambda^2(\beta_1\beta_2 + \beta_3\gamma_1 + \beta_2) + \lambda^3(\beta_1\gamma_2 + \beta_2\gamma_1)) = 1 + \lambda^4* + \cdots + \lambda^{21}* .$$

As before all the coefficients of λ must be zero. Considering the coefficients of λ and λ^2 we find that

$$\gamma_1 = \beta_2(\beta_1 + 1) = 0 .$$

Now if $\beta_2 = 0$ an above equation implies that $\beta_3\gamma_2 = 0$. If $\beta_3 = 0$ then σ has an eigenvalue which in impossible. If $\gamma_2 = 0$ then $\det L_{a^2} = 0$ which is impossible. Hence we must have $\beta_2 \neq 0$ and $\beta_1 = 1$. But then $\beta_1 + 1 + \gamma_2 = 0$ implies that $\gamma_2 = 0$ which again is impossible. Hence no such algebra exists over an infinite field.

We have determined all commutative homogeneous algebras of dimension 3 and Type 4. Now let A be any 3-dimensional homogeneous algebra of Type 4. Pass from A to A^+. Then A^+ is a commutative homogeneous algebra. Suppose $A^+ \neq 0$. A^+ cannot be Type 1 since there are no nonzero commutative homogeneous algebras of Type 1. If A^+ is of Type 2 we know that $K = \mathrm{GF}(2)$. A^+ cannot be of Type 3 since there are no nonzero homogeneous algebras of Type 3. If A^+ is of Type 4 we have just shown that $K = \mathrm{GF}(2)$. So either A^+ is a zero algebra or $K = \mathrm{GF}(2)$. If A^+ is a zero algebra then A is anti-commutative. But then we have $2a^2 = 0$ and since $a^2 \neq 0$ this implies that char $K = 2$. But then A is commutative and so again we have shown above that $K = \mathrm{GF}(2)$. Hence the only nonzero homogeneous algebras of Type 4 exist over $K = \mathrm{GF}(2)$. In such cases we know that A is a commutative quasi division algebra and so we have found all nonzero 3-dimensional homogeneous algebras of Type 4. That is, if A is a nonzero 3-dimensional homogeneous algebra of Type 4 then either A has a basis a, a^2, a^2a^2 such that

	a	a^2	a^2a^2
a	a^2	$a + a^2$	a^2a^2
a^2	$a + a^2$	a^2a^2	$a^2 + a^2a^2$
a^2a^2	a^2a^2	$a + a^2a^2$	$a + a^2a^2$

or A has a basis a, a^2, aa^2 such that

	a	a^2	aa^2
a	a^2	aa^2	a
a^2	aa^2	$a + a^2 + aa^2$	$a + aa^2$
a^2a^2	a	$a + aa^2$	$a^2 + aa^2$

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UNIVERSITY OF PRINCE EDWARD ISLAND
CHARLOTTETOWN, PRINCE EDWARD ISLAND, CANADA