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COMPUTATION OF THE SURGERY OBSTRUCTION GROUPS $L_{4k}(1; \boldsymbol{Z}_P)$

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The $4k$-dimensional simply connected surgery obstruction group with coefficients $\boldsymbol{Z}_P$ (i.e., the group of nonsingular even quadratic forms over $\boldsymbol{Z}_P$) is computed in terms of the classical Witt group and a Gauss sum invariant.

1. Introduction. Let $L_{4k}(1; \boldsymbol{Z}_P)$ be the simply connected surgery obstruction group, with coefficient $\boldsymbol{Z}_P = \boldsymbol{Z}[1/p: p \in P]$, in dimension $4k$, of [1]. By definition, this is the Witt group of even, nonsingular quadratic forms over the ring $\boldsymbol{Z}_P$. We compute $L_{4k}(1; \boldsymbol{Z}_P)$ in terms of the classical Witt group $W(\boldsymbol{Z}_P)$ ([4]).

Let $\gamma_p: W(\boldsymbol{Q}_p) \rightarrow \mathscr{U}$ denote the "p-primary Gauss sum" character of [4], Appendix 4, where $\mathscr{U} \subset \boldsymbol{C}^{\cdot}$ is the multiplicative group of roots of unity. Define $\Phi_P: W(\boldsymbol{Z}_P) \rightarrow \boldsymbol{Z}/8\boldsymbol{Z}$ by

$$\exp(2\pi i \Phi_P(q)/8) = \exp(2\pi i \sigma(q)/8) \cdot \prod_{p \in P} (\gamma_p(q \otimes \boldsymbol{Q}_p))^{-1} ,$$

where σ is the signature.

THEOREM 1.1. (i) *If $2 \in P$, then $L_{4k}(1; \boldsymbol{Z}_P) = W(\boldsymbol{Z}_P)$.*
(ii) *If $2 \notin P$, then $L_{4k}(1; \boldsymbol{Z}_P) \cong \ker(\Phi_P)$.*

(i) is obvious and the proof of (ii) occupies §2. An explicit description of $\ker(\Phi_P)$, necessary to obtain the ring structure, is given in §3.

The author would like to thank the referee for suggesting the brief statement and proof of Theorem 1.1 found here.

2. The proof of Theorem 1.1. For p an odd prime, let $\beta_p: W(\boldsymbol{Q}) \rightarrow W(\boldsymbol{F}_p)$ be the second residue homomorphism (called ∂_p in [4]), and $\beta_2: W(\boldsymbol{Q}) \rightarrow W(\boldsymbol{F}_2)$ the 2-adic value of the determinant. Let $\beta = \bigoplus_p \beta_p$. According to [4], $\sigma \oplus \beta: W(\boldsymbol{Q}) \rightarrow \boldsymbol{Z} \oplus \bigoplus_p W(\boldsymbol{F}_p)$ is an isomorphism.

Recall that $W(\boldsymbol{F}_2) \cong \boldsymbol{Z}/2\boldsymbol{Z}$, $W(\boldsymbol{F}_p) \cong \boldsymbol{Z}/4\boldsymbol{Z}$ if $p \equiv 3 \bmod (4)$, generated by $\langle 1 \rangle$, and $W(\boldsymbol{F}_p) = \boldsymbol{Z}/2\boldsymbol{Z} \oplus \boldsymbol{Z}/2\boldsymbol{Z}$ if $p \equiv 1 \bmod (4)$, generated by $\langle 1 \rangle$ and $\langle s_p \rangle$, where s_p is some quadratic nonresidue $\bmod(p)$. Let π_1, π_2: $W(\boldsymbol{F}_p) \rightarrow \boldsymbol{Z}/2\boldsymbol{Z}$ be the projections, $p \equiv 1 \bmod (4)$. The invariants β_p and γ_p are related by the following lemma.

LEMMA 2.1. *Let $[q] \in W(\boldsymbol{Q})$. Then:*
(i) $\gamma_p(q \otimes \boldsymbol{Q}_p) = (i\varepsilon)^{\beta_p(q)}$, *where* $\varepsilon = (-1)^{(p+1)/4}$, *if* $p \equiv 3 \bmod (4)$.

$$\text{(ii)}\quad \gamma_p(q \otimes \boldsymbol{Q}_p) = \begin{cases} (-1)^{\pi_1\beta(q)} & \textit{if} \quad p \equiv 5 \bmod (8) \\ (-1)^{\pi_2\beta(q)} & \textit{if} \quad p \equiv 1 \bmod (8) . \end{cases}$$

Proof. (i) We have $q \otimes \boldsymbol{Q}_p = n\langle p\rangle + m\langle 1\rangle$ in $W(\boldsymbol{Q}_p)$ and $\beta_p(q) = n \bmod (4)$. Therefore $\gamma_p(q \otimes \boldsymbol{Q}_p) = \gamma_p(\langle p\rangle)^{\beta_p(q)}$. By [4], $\gamma_p(\langle 4p\rangle) = \exp(\pi i(1-p)/4) = i\varepsilon$. (ii) is similar.

Let $\beta_P = \bigoplus_{p\in P}\beta_p\colon W(\boldsymbol{Z}_P) \to \bigoplus_{p\in P} W(\boldsymbol{F}_p)$. Then we have the following well-known result:

LEMMA 2.2. $\sigma \oplus \beta_P\colon W(\boldsymbol{Z}_P) \cong \boldsymbol{Z} \bigoplus_{p\in P} W(\boldsymbol{F}_p)$.

The proof is immediate from the localization sequence

$$0 \longrightarrow W(\boldsymbol{Z}_P) \longrightarrow W(\boldsymbol{Q}) \longrightarrow \bigoplus_{p\notin P} W(\boldsymbol{F}_p) \longrightarrow 0$$

of [4], Corollary IV. 3.3.

Proof of Theorem 1.1.(ii). Using the notation of [3], $L_{4k}(1; \boldsymbol{Z}_P) = \bar{W}(\boldsymbol{Z}_P)$ and we have the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \bar{W}(\boldsymbol{Z}) & \longrightarrow & \bar{W}(\boldsymbol{Z}_P) & \xrightarrow{\bar{\beta}_P} & \bar{W}(\boldsymbol{Z}_P, \boldsymbol{Z}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle i_*} & & \\
0 & \longrightarrow & W(\boldsymbol{Z}) & \longrightarrow & W(\boldsymbol{Z}_P) & \xrightarrow{\beta_P} & \bigoplus_{p\in P} W(\boldsymbol{F}_p) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \sigma_*} & & \downarrow{\scriptstyle \Phi_P} & & & & \\
 & & \boldsymbol{Z}/8\boldsymbol{Z} & \xrightarrow{=} & \boldsymbol{Z}/8\boldsymbol{Z} . & & & &
\end{array}$$

Here σ_* is the signature mod (8). The left vertical sequence is exact by [4], the top horizontal sequence by [3] or [5], and the middle horizontal sequence by Lemma 2.2. Furthermore, by [3], i_* is an isomorphism.

We claim that $\bar{W}(\boldsymbol{Z}_P) = \ker(\Phi_P)$. Clearly $\bar{W}(\boldsymbol{Z}_P) \subset \ker(\Phi_P)$ by the reciprocity formula of [4]. Suppose $\Phi_P(x) = 0$. Choose $y \in \bar{\beta}_P^{-1} i_*^{-1} \beta_P(x)$. By a diagram chase, $x - y \in W(\boldsymbol{Z})$ and $\sigma_*(x - y) = 0$. Since $\bar{W}(\boldsymbol{Z}) = \ker(\sigma_*)$, $x \in \bar{W}(\boldsymbol{Z}_P)$.

3. The ring structure. The tensor product of even quadratic forms is again even, so $L_{4k}(1; \boldsymbol{Z}_P)$ has the structure of a commutative ring. Since $\sigma \oplus \beta_P\colon L_{4k}(1; \boldsymbol{Z}_P) \to \boldsymbol{Z} \oplus \bigoplus_{p\in P} W(\boldsymbol{F}_p)$ is injective, and $\sigma(q \otimes q') = \sigma(q)\sigma(q')$, it sufficies to consider $\beta_p(q \otimes q')$.

Let $\alpha_p: W(\boldsymbol{Q}) \to W(\boldsymbol{F}_p)$ be the first residue homomorphism if $p \neq 2$, and the signature mod(2) if $p = 2$. We have:

PROPOSITION 3.1. $\beta_p(q \otimes q') = \alpha_p(q)\beta_p(q') + \alpha_p(q')\beta_p(q)$.

Proof. First assume $p \neq 2$. Diagonalize q over $\boldsymbol{Q}$ as $q_0 \otimes \langle p\rangle + q_1$, where q_0, q_1 are diagonal forms with entries prime to p. Similarly write $q' \cong q'_0 \otimes \langle p\rangle + q'_1$. Then $\beta_p(q) = \bar{q}_0$, $\alpha_p(q) = \bar{q}_1$, $\beta_p(q') = \bar{q}'_0$, $\alpha_p(q') = \bar{q}'_1$, where "$-$" denotes passing to the residue class field of $\boldsymbol{Q}_p$, and

$$\begin{aligned}\beta_p(q \otimes q') &= \beta_p(q_0 \otimes q'_0 \otimes \langle p^2\rangle + q_0 \otimes q'_1 \otimes \langle p\rangle \\ &\quad + q_1 \otimes q'_0 \otimes \langle p\rangle + q_1 \otimes q'_1) \\ &= \bar{q}_0 \otimes \bar{q}'_1 + \bar{q}_1 \otimes \bar{q}'_0 .\end{aligned}$$

The case $p = 2$ is an easy determinant argument and left to the reader.

The ring $L_{4k}(1; \boldsymbol{Z}_P)$ can now be completely determined by the values of the first residues of a set of generators, which we now describe.

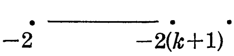
Let $(n; x_1(p_1), \cdots, x_k(p_k))$ denote the element $y \in W(\boldsymbol{Z}_P)$ with $\sigma(y) = n$, $\beta_{p_i}(y) = x_i$, $i = 1, \cdots, k$, and $\beta_p(y) = 0$ otherwise. By Theorem 1.1 and Lemma 2.1, we have

LEMMA 3.2. *Let* $2 \notin P$. *Then*: $(n; x_1(p_1), \cdots, x_k(p_k)) \in L_{4k}(1; \boldsymbol{Z}_P)$ *if and only if*

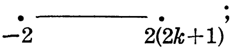
$$n + \sum_{p_i \equiv 3(4)} (-1)^{(p_i-3)/4} 2x_i + \sum_{p_i \equiv 5(8)} 4\pi_1(x_i) + \sum_{p_i \equiv 1(8)} 4\pi_2(x_i) \equiv 0 \bmod (8) .$$

Generators of $L_{4k}(1; \boldsymbol{Z}_P)$ are given by the following matrices:

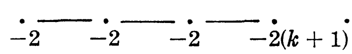
(1) $p = 4k + 3$: $(2; (-1)^{k+1}(p))$ is obtained from the weighted graph



(2) $p = 8k + 5$: $(0; s(p))$ is obtained



$(4; 1(p))$ is obtained from



(3) $p = 8k + 1$: $(0; 1(p))$ is obtained from

In general, it is hard to write down an explicit matrix realizing

$(4; s(p))$. However, by the proof of Theorem IV. 2.1 of [4], a diagonalization can be obtained in a specific case. For example, $(4; s(17))$ is represented by $\langle 51, 3, 1, 1 \rangle$.

Finally, we include the following result on signatures of even forms over $\boldsymbol{Z}_P$. Let $a_P = \text{g.c.d.}\{|\sigma(x)|: x \in L_{4k}(1; \boldsymbol{Z}_P)\}$

COROLLARY 3.3. *$a_P = 1$ (resp. 8) if and only if $2 \in P$ (resp. $P = \phi$). Otherwise, $a_P = 2$ if some $p \in P$ is 3 mod (4), and $a_P = 4$ if not.*

The proof is immediate from Lemma 3.2. This shows that Proposition 2.2. of [6] is incorrect.

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