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THE EVOLUTION OF BOUNDED LINEAR FUNCTIONALS WITH APPLICATION TO INVARIANT MEANS

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Let S be a topological semigroup and let X be a left translation invariant, left introverted closed subspace of $CB(S)$. Let m and $\bar{\mu}$ be elements of X^*, where $\bar{\mu}(f) = \int f d\mu$ for f in $CB(S)$ and μ is a measure on S which lives on a suitable set. It is shown that the evolution and convolution of m and $\bar{\mu}$ coincide. The same argument carries over to prove that if $X \subset W(S)$, then the evolution and convolution of m and n in X^* are the same (a known result). The topological invariance of invariant means on X^* is discussed.

1. Preliminaries. Let S be a topological semigroup with separately continuous multiplication and $CB(S)$ the Banach space, under supremum norm of bounded real continuous functions on S. For each s in S, define the left and right translation operators on $CB(S)$ by $(l_s f)(t) = f(st)$ and $(r_s f)(t) = f(ts)$ for all t in S, f in $CB(S)$. The subspace X of $CB(S)$ is called left (right) introverted, if for each m in X^* the function $s \to f * m(s) = m(l_s f)(s \to m * f(s) = m(r_s f))$ is in X. $W(S)$ denotes the subspace of $CB(S)$ consisting of weakly almost periodic functions, i.e., the functions f such that the set $\{r_s f : s \in S\}$ is conditionally weak compact. $LUC(S)$ ($WLUC(S)$) is the subspace of $CB(S)$ consisting of (weakly) left uniformly continuous functions on S, i.e., the functions f such that the map $s \to l_s f$ is norm (weak) continuous. $M_\sigma(S)(M(S))$ denotes the linear space of all real valued signed Baire (regular Borel) measures on S. The mapping $T: CB(S) \to M^*(S)$ is the natural embedding of $CB(S)$ into $M^*(S)$ defined by $(Tf)(\mu) = \int f d\mu$ for f in $CB(S)$ and μ in $M(S)$. Following Granirer [4] $\sigma(CB(S), M_\sigma(S)) = \sigma(C, M_\sigma)$ denotes the weakest topology on $CB(S)$ which makes all linear functionals on $CB(S)$ of type $\int f d\mu$ for μ in M_σ continuous.

For μ in $M_\sigma(S)$ or in $M(S)$ and f in $CB(S)$ let $\mu * f(t) = \int r_t f d\mu$, $f * \mu(t) = \int l_t f d\mu$ for any t in S.

For μ in $M_\sigma(S)$ or in $M(S)$, $\bar{\mu}$ denotes the functional in $CB^*(S)$ defined by $\bar{\mu}(f) = \int f d\mu$ for f in $CB(S)$.

2. The main theorem. Before stating the main theorem we need the following lemma.

LEMMA 2.1. *Let S be a topological semigronp. For f in $CB(S)$*

and $M \geqq 0$ *let*

$$B_M(f) = \{f*m: m \text{ in } X^* \text{ and } \|m\| \leqq M\}.$$

(i) $B_M(f)$ *is pointwise compact.*

(ii) *If* S *is locally compact and* f *is in* $WLUC(S)$ *then* $T(B_MF)$ *is* $\sigma(M^*(S), M(S))$*-compact.*

(iii) *If* S *is a completely regular D-space (for definition see* [3]*), and* f *is in* $LUC(S)$*, then* $B_M(f)$ *is* $\sigma(C, M_\sigma)$*-compact.*

(iv) *If* f *is in* $W(S)$*, then* $B_M(f)$ *is weak compact.*

(v) *In each case* (i)-(iv) *the topology of pointwise convergence and the indicated topology coincide on* $B_M(f)$.

Proof. (i) By Alaoglu's theorem the set $\{m: m \text{ in } X^* \text{ and } \|m\| \leqq M\}$ is weak $*$ compact. Using this one can easily show that $B_M(f)$ is pointwise compact.

(ii) Since f is in $WLUC(S)$ and $\|f*m\| \leqq M\|f\|$, $B_M(f)$ is a norm bounded subset of $CB(S)$. Therefore this follows from [Glicksberg 3, Theorem 1.1] and preceeding part.

(iii) Since

$$\begin{aligned}|f*m(s) - f*m(s_0)| = |m(l_sf) - m(l_{s_0}f)| &\leqq \|m\|\,\|l_sf - l_{s_0}f\| \\ &\leqq M\|l_sf - l_{s_0}f\|\end{aligned}$$

for each m with $\|m\| \leqq M$ and the map $s \to l_sf$ is norm continuous, one deduce that $B_M(f)$ is an equicontinuous family of functions on S. By [4, Theorem 1 (a)] it follows that $B_M(f)$ is $\sigma(C, M_\sigma)$-conditionally compact. By part (i) $B_M(f)$ is pointwise compact and therefore $B_M(f)$ is $\sigma(C, M_\sigma)$-closed. Hence $B_M(f)$ is $\sigma(C, M_\sigma)$-compact.

(iv) This follows from [8, remark (a) after Theorem 3.3].

(v) This follows from [13, 3.8 (a), P. 61] and part (i).

Now the main theorem of the paper can be proved.

THEOREM 2.2. *Let* S *be a topological semigroup and* X *a left translation invariant, left introverted closed subspace of* $CB(S)$.

(i) *If* S *is locally compact and* $X \subset WLUC(S)$*, then for each* μ *in* $M(S)$*,* $\mu*X \subset X$ *and furthermore, for each* m *in* X^**,* $\langle \bar{\mu}, f*m\rangle = \langle \mu*f, m\rangle$ *for all* f *in* X.

(ii) *If* S *is a completely regular D-space and* $X \subset LUC(S)$*, then for each* μ *in* $M_\sigma(S)$*,* $\mu*X \subset X$ *and furthermore, for each* m *in* X^* $\langle \bar{\mu}, f*m\rangle = \langle \mu*f, m\rangle$ *for all* f *in* X.

(iii) *If* $X \subset W(S)$*, then for each* n *in* X^**,* $n*X \subset X$ *and furthermore,* $\langle n, f*m\rangle = \langle n*f, m\rangle$ *for each* m *in* X^* *and* f *in* X.

Proof. (i) Let f be in X and μ in $M(S)$. Define the functional

ψ on X^* by $\psi(m) = \int f*md\mu$ for m in X^*. It is easy to see that μ is linear. We will show that ψ is weak $*$ continuous on each ball $N_M = \{m\colon m \text{ in } X^* \text{ and } \|m\| \leqq M\}$. To see this let m_0 be a point in N_M and $\{m_\alpha\}$ a net in N_M converging weak $*$ to m_0. Then $f*m_\alpha$ converges to $f*m_0$ pointwise on S. Let $B_M(f)$ be as defined in Lemma 2.1. Hence by Lemma 2.1. (v) the pointwise topology and $\sigma(M^*(S), M(S))$ coincide on $B_M(f)$. Therefore $\int f*m_\alpha d\nu \to \int f*md\nu$ for each ν in $M(S)$. In particular

$$\psi(m_\alpha) = \int f*m_\alpha d\mu \longrightarrow \int f*md\mu = \psi(m)\ .$$

Therefore it follows from [14, Corollary A.12, p. 89] that ψ is nothing but an evaluation functional. That is, there exists g in X such that $m(g) = \int f*md\mu$ for each m in X^*. For each s in S, let $m = Q(s)$ be the evaluation functional at s in the above identity. Then $Q(s)g = g(s) = \int r_s f d\mu = \mu*f(s)$. This implies that $\mu*f$ is in X and furthermore, $m(\mu*f) = \langle \mu*f, m\rangle = \int f*md\mu = \int\langle \bar{\mu}, f*m\rangle$. This completes the proof.

(ii) The proof is similar to preceeding part.

(iii) Let f be in X. For n in X^* define the functional ψ on X^* by $\psi(m) = n(m_l f)$ for m in X^*. By an argument similar to part one and using Lemma 2.1 (v) one can show that ψ is an evaluation functional on X^*. The rest follows as in part (i).

REMARKS. (a) If in addition to hypothesis of Theorem 2.2 (i), X is also a c^*-subalgebra of $CB(S)$, then Theorem 2.2 (i) reduces to a result of Milnes [9, Lemma 3.3].

(b) It is possible to give a proof of Theorem 2.2 (ii) by a method similar to that of Granirer in [4, Lemma 3 and Theorem 4, p. 20].

(c) Let S be a topological semigroup and let X be a translation invariant, left and right introverted subspace of $CB(S)$ such that $\langle n, f*m\rangle = \langle n*f, m\rangle$ for each m and n in X^* and f in X. Let f be in X, then using Alaoglu's theorem and assumption it is easy to see that the set $\{f*m\colon m \text{ in } X^* \text{ and } \|m\| \leqq M\}$ is weak compact for each nonnegative real M. This shows that f is in $W(S)$. Hence $X \subset W(S)$.

(d) Theorem 2.2 (iii) and Preceeding remark is due to Pym [11, Theorem 4.2]. Our proof here is easier and different from that of Pym.

(e) Theorem 2.2 (iii) implies that $W(S)$ is a right introverted subspace of $CB(S)$. By an argument similar to preceeding remark (c) one can show that for each nonnegative N, the set $\{n_r f\colon n \text{ in } X^* \text{ and } \|n\| \leqq N\}$ and hence the set $\{l_s f\colon s \text{ in } S\}$ is weak compact. This in particular implies the known result that for f in $CB(S)$, $\{l_s f: s$ in $S\}$ is conditionally weak compact if $\{r_s f: s \text{ in } S\}$ is conditionally weak compact.

(f) The proof of Theorem 2.2 (i) and (ii) is independent of the topological structure of S, but it depends on the topological structure of the set on which the measure μ "lives" (see [4] for definition).

3. **Applications.** A. *Invariant means on locally compact semigroups.* Let S be a topological semigroup and X a closed subspace of $CB(S)$ containing the constant function 1. m in X^* is called a mean if $\|m\| = m(1) = 1$. If in addition X is left translation invariant, the mean m is called left invariant if $m(l_s f) = m(f)$ for all s in S and all f in X. Let S be a locally compact (resp. completely regular D-space) semigroup and $X \subset CB(S)$. X is called topological left translation invariant if $\mu * X \subset X$ for each μ is $M(S)$ (resp. $M_\sigma(S)$). The mean m on X is topological left invariant if $m(\mu * f) = m(f)$ for each probability measure μ in $M(S)$ (resp. $M_\sigma(S)$).

COROLLARY 3.1. (i) *Let S and X be as in Theorem* 2.2 (i), *then X is topological left translation invariant and the mean m on X is left translation invariant iff it is topological left invariant.*

(ii) *Let S and X be as in Theorem* 2.2 (ii), *then X is topological left translation invariant and the mean m on X is left invariant iff it is topological left invariant.*

Proof. (i) The topological left invariance of X is a part of Theorem 2.2 (i). If m is topological left invariant, then clearly it is left invariant. Suppose m is left invariant. By Theorem 2.2 (i) $\langle \mu * f, m\rangle = m(\mu * f) = \langle \bar{\mu}, f * m\rangle = \int f * m d\mu = \int m(l_s f) d\mu(s) = m(f)$ for each probability measure μ in $M(S)$ and each f in X.

(ii) Proof is similar to part (i).

REMARKS. 1. If in addition to the hypothesis of Corollary 3.1 (i), X is also a c^*-subalgebra of $CB(S)$, then Corollary 3.1 reduces to a result of Milnes [9, Corollary 3.3].

2. If S is a locally compact group Corollary 3.1 (i) reduces to a more general version of results of Namioka [10], Hulanicki [7] and Greenleaf [5, Lemma 2.2.2].

3. Corollary 3.1 (ii) is an analog of Granirer [4, Theorem 4, p. 20] for topological semigroups.

B. *Evolution and convolution of bounded linear functionals.* Let S be a topological semigroup and let X be a left (right) translation invariant, left (right) introverted closed subspace of $CB(S)$. Following Pym [11] and Day [2] for m and n in X^*, let $m \odot n$ (resp. $m*n$) be the evolution (resp. convolution) of m and n defined by $m \odot n(f) = m(n_l f)(m*n(f) = n(m_r f))$ for f in X. Notice that evolution here is the same as Arens product in Day [2]. In term of evolution and convolution Theorem 2.2 implies the following:

COROLLARY 3.2. (i) *Let S, X, $\bar{\mu}$, and m be as in Theorem* 2.2 (i) (*resp.* (ii)), *then* $\bar{\mu}*m = \bar{\mu} \odot m$ *on* X.

(ii) *Let S, X, n, and m be as in Theorem* 2.2 (iii), *then* $n*m = n \odot m$.

REMARKS. 1. Corollary 3.2 (i) implies that the bilinear mapping $(\mu, m) \in M(S) \times X^* \to \bar{\mu} \odot m \in X^*$ is separately continuous where $M(S)$ is equipped with $\sigma(M(S), TX)$ topology and X^* with weak $*$ topology. Similar assertion holds by applying part (ii). In particular in this way one gets the weakly almost periodic compactification of a topological semigroup. (See also Pym [12].)

2. Let S be a completely regular D-space semigroup and μ and ν elements of $M_\sigma(S)$. Then Corollary 3.2 (i) implies that

$$\begin{aligned}\bar{\mu} \odot \bar{\nu}(f) &= \bar{\mu}(\bar{\nu}_l f) = \int \bar{\nu}_l f \, d\mu = \iint f(ts)\,d\nu(s)\,d\mu(t)\\ &= \bar{\mu}*\bar{\nu}(f) = \bar{\nu}(\bar{\mu}_r f) = \int \bar{\mu}_r f\,d\nu = \iint f(ts)\,d\mu(t)\,d\nu(s)\end{aligned}$$

for each f in $LUC(S)$. This is an analog of Glicksberg [2, Theorem 3.1]. Note that this observation deserves more attention and may lead to a suitable way of defining the convolution of Baire measures. (See also [6, 19.23 (b)].)

REFERENCES

1. R. B. Burckel, *Weakly Almost Periodic Functions on Semigroups*, Gordon and Breach, New York, 1970.

2. M. M. Day, *Amenable semigroups*, Illinois J. Math., **1** (1957), 509-544.

3. I. Glicksberg, *Weak compactness and separate continuity*, Pacific J. Math., **11** (1961), 205-214.

4. E. Granirer, *On Baire measure on D-topological spaces*, Fund. Math., **60** (1967), 1-22.

5. F. P. Greenleaf, *Invariant means on topological groups and their applications*, Van Nostrand, Princeton, N. J., 1969.

6. E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis I*. Springer Verlag, Berlin-Gottingen-Heidelberg, 1963.

7. A. Hulanicki, *Means and Folner condition on locally compact groups*, Studia Math., **27** (1966), 87-104.

8. H. Kharaghani, *Left thick subsets of a topological semigroup*, Illinois J. Math., **22** (1978), 41-48.

9. P. Milnes, *Compactification of semitopological semigroups*, J. of Aust. Math. Soc.,

Volume XV, Part **4** (1973), 488–503.
10. I. Namioka, *On a recent theorem by Reiter*, Proc. Amer. Math. Soc., **17** (1966), 1101–1102.
11. J. S. Pym, *The convolution of linear functionals*, Proc. London Math. Soc., (3), **14** (1964), 431–444.
12. ———, *The convolution of functionals on spaces of bounded functions*, Proc. London Math. Soc., (3), **15** (1965), 84–104.
13. W. Rudin, *Functional Analysis*, Tata McGraw-Hill, New Delhi, 1974.

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