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KRULL RINGS

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We extend the notion of a Krull domain to commutative rings with identity which may contain zero divisors. In order to do this we present a definition of the divisors of an arbitrary ring, and show that the collection of divisors is a commutative semigroup with identity and is a group if and only if the ring is completely integrally closed. In addition, an extension of unique factorization domains to arbitrary commutative rings is used to investigate the relationship between Krull rings and unique factorization rings. In particular, it is shown that a unique factorization ring is a Krull ring with trivial class group.

1. Divisors and complete integral closure. The divisors of an integral domain together with the concept of complete integral closure is important in the study of unique factorization domains and Krull domains. To extend the notion of divisors to rings which contain zero divisors, let R be a ring with total quotient ring K. If A is a subset of K such that A is an R-module and there exists a regular element $d \in R$ where dA is a subset of R, then A is called a fractional ideal of R. If, in addition, A contains a regular element of R, then A is called a *regular fractional ideal* of R. In what follows, $F(R)$ will denote the collection of regular fractional ideals of R.

For $A, B \in F(R)$, set $[A: B] = \{x \in K \mid xB \subset A\}$. Then $[A: B] \in F(R)$. In particular, $[R: A] \in F(R)$ for each $A \in F(R)$. Define $\sim$ on $F(R)$ by: $A \sim B$ if and only if $[R: A] = [R: B]$. Note that $\sim$ is an equivalence relation on $F(R)$. $D(R)$ will denote the collection of equivalence classes induced on $F(R)$ by $\sim$. If $A \in F(R)$, then $\operatorname{div} A$ represents the equivalence class containing A. As in the domain case, a well-defined operation may be defined on $D(R)$ by: $\operatorname{div} A + \operatorname{div} B = \operatorname{div} AB$ for $A, B \in F(R)$. With this operation, $D(R)$ is a commutative semigroup with identity $\operatorname{div} R$. If $D(R)$ is a group, then $D(R)$ is called the *group of divisors* of R.

For $A \in F(R)$, set $\bar{A} = [R: [R: A]]$. Then $\bar{A} \in F(R)$ and $A \sim \bar{A}$. In fact, for $A, B \in F(R)$, $A \sim B$ if and only if $\bar{A} = \bar{B}$. A partial order may be defined on $D(R)$ by: $\operatorname{div} A \leqq \operatorname{div} B$ if and only if $\bar{B} \subset \bar{A}$. With this partial order, $D(R)$ is a partially ordered semigroup with identity.

The following proposition is of importance since it gives a necessary and sufficient condition for a ring to be completely integrally closed.

PROPOSITION 1.1. *Let R be a ring with total quotient ring K.*

Then R is completely integrally closed if and only if $D(R)$ is group.

Proof. Suppose that R is completely integrally closed. Let $A \in F(R)$. It only needs to be shown that div A has an additive inverse in $D(R)$.

Let $x \in [R: A[R: A]]$. Then $x[R: A] \subset [R: A]$. Therefore $x^n A[R: A] \subset A[R: A] \subset R$ for each positive integer n. Let d and d' be regular elements of R which are contained in A and $[R: A]$ respectively. Hence dd' is regular and $dd'x^n \in R$ for each positive integer n. Thus by complete integral closure, $[R: A[R: A]] \subset R$. But then it follows that $[R: A[R: A]] = R$ and so $\operatorname{div} R = \operatorname{div} A + \operatorname{div} [R: A]$, and $\operatorname{div} [R: A]$ is the additive inverse of $\operatorname{div} A$ in $D(R)$.

The proof of the converse is similar to the analogous theorem in the domain case.

An immediate corollary to the above proposition is that if R is completely integrally closed, then $D(R)$ is a lattice ordered abelian group. In fact if $\operatorname{div} A$ and $\operatorname{div} B$ are elements of $D(R)$, it follows that $\operatorname{glb} \{\operatorname{div} A, \operatorname{div} B\} = \operatorname{div} (A + B)$ and $\operatorname{lub} \{\operatorname{div} A, \operatorname{div} B\} = \operatorname{div} (A \cap B)$.

A regular fractional ideal of a ring R is called a *principal fractional ideal* if it is of the form Ra where a is a regular element of the total quotient ring of R. It can be shown that $\bar{A}$ is equal to the intersection of all principal fractional ideals which contain A for each regular fractional ideal A of a ring R. If $A = \bar{A}$, then A is said to be a *divisorial ideal* of R.

2. Krull rings. Let R be a ring with total quotient ring K such that $R \neq K$. Then R is called a *Krull ring* if there exist a family $\{(V_\alpha, P_\alpha) \mid \alpha \in I\}$ of discrete rank one valuation pairs of K with associated valuations $\{V_\alpha \mid \alpha \in I\}$ such that

(I) $R = \bigcap \{V_\alpha \mid \alpha \in I\}$.

(II) $v_\alpha(a) = 0$ almost everywhere on I for each regular $a \in K$, and each P_α is regular ideal of V_α.

Note that for P_α to be a regular ideal of V_α, it is necessary and sufficient that P_α contain a regular element of R. The last condition also means that for each regular element $a \in R$, a is an element of $V_\alpha - P_\alpha$ for all except finitely many $\alpha \in I$.

For $A \in F(R)$, $\inf \{v_\alpha(a) \mid a \in A\}$ exists for each $\alpha \in I$ since the value of a regular element is finite. For $\alpha \in I$, $v_\alpha(A)$ will denote this infinum.

PROPOSITION 2.1. *Let R be a Krull ring with a defining family, $\{v_\alpha \mid \alpha \in I\}$, of discrete rank one valuations on K where K is the total quotient ring of R.*

(i) *Let* $A, B \in F(R)$ *where* B *is divisorial. Then* $A \subset B$ *if and only if* $v_\alpha(A) \geqq v_\alpha(B)$ *for each* $\alpha \in I$.

(ii) *For each* $A \in F(R)$, $v_\alpha(A) = 0$ *almost everywhere on* I.

Proof. The proof is similar to the analogous theorem in the domain case.

PROPOSITION 2.2. *Let* R *be a Krull ring with total quotient ring* K. *Then* R *is completely integrally closed and every nonempty set of divisorial ideals of* R *has a maximal element.*

Proof. Let $\{v_\alpha \mid \alpha \in I\}$ be a defining family of discrete rank one valuations on K. Let d be a regular element in R and $x \in K$ such that $dx^n \in R$ for each positive integer n. Hence $v_\alpha(d) + nv_\alpha(x) \geqq 0$ for each positive integer n. Suppose that $x \in K - R$. Then there exists $\beta \in I$ such that $v_\beta(x) < 0$. Since d is fixed, n may be chosen large enough so that $v_\beta(d) + nv_\beta(x)$ is strictly less than zero, a contradiction. Hence $v_\alpha(x) \geqq 0$ for each $\alpha \in I$, and $x \in R$. Therefore, R is completely integrally closed.

That each nonempty set of divisorial ideals of R has a maximal element follows from Proposition 2.1.

Let R be a Krull ring. Then by Proposition 2.2, every nonempty collection of positive elements of $D(R)$ has a minimal element. Let the set of all minimal positive elements of $D(R)$ be indexed by a set I. For each $\alpha \in I$, let M_α be the divisorial ideal of R such that $\operatorname{div} M_\alpha$ is a minimal positive element of $D(R)$. Thus $\{M_\alpha \mid \alpha \in I\}$ is the collection of maximal divisorial ideals of R. The proof of the following lemma is omitted since its proof is similar to that of the domain case.

LEMMA 2.3. *Let* R *be a completely integrally closed ring with maximal divisorial ideal* M. *If* $\operatorname{div} M \leqq \operatorname{div} A_1 + \operatorname{div} A_2 + \cdots + \operatorname{div} A_n$, *where each* $\operatorname{div} A_i$ *is a positive element of* $D(R)$, *then* $\operatorname{div} M \leqq \operatorname{div} A_i$ *for some* i.

PROPOSITION 2.4. *Let* R *be a Krull ring. Using the notation preceding the lemma, each element of* $D(R)$ *is uniquely of the form*

$$\sum n_\alpha \operatorname{div} M_\alpha$$

where $n_\alpha = 0$ *almost everywhere on* I.

Proof. Let $\operatorname{div} A \in D(R)$. By the corollary to Proposition 1.1, there exists $B \in F(R)$ such that $\operatorname{div} B$ is the least upper bound of

$\{\operatorname{div} A, \operatorname{div} R\}$. Then, since $D(R)$ is an abelian group, $\operatorname{div} A = \operatorname{div} B$ $(\operatorname{div} B - \operatorname{div} A)$ and each element of $D(R)$ may be written as the difference of two nonnegative elements of $D(R)$. Thus, if it is shown that each positive element of $D(R)$ is of the desired form, then so is every element of $D(R)$. That this is indeed the case is straight forward. Uniqueness of the above representation follows from Lemma 2.3.

It can be shown that a finite complete direct sum of Krull rings is a Krull ring. However, it is not necessary that each summand of a complete direct sum be Krull in order that it be a Krull ring. The following proposition demonstrates this possibility.

PROPOSITION 2.5. *Let R be a Krull ring with total quotient ring K, and S be a ring which is its own total quotient ring. Then $R \oplus S$ is a Krull ring.*

Proof. Let $\{(V_\alpha, P_\alpha) \mid \alpha \in I\}$ be a defining family of discrete rank one valuation pairs of K for R. Note that the total quotient ring of $R \oplus S$ is $K \oplus S$. For each $\alpha \in I$, define $w_\alpha(x, s) = v_\alpha(x)$ where v_α is a determining valuation on K for (V_α, P_α). Then each w_α is a discrete rank one valuation on $K \oplus S$.

Let $W_\alpha = \{(x, s) \in K \oplus S \mid w_\alpha(x, s) \geqq 0\}$. Then $W_\alpha = W_\alpha \oplus S$ and

$$\bigcap \{W_\alpha \mid \alpha \in I\} = R \oplus S .$$

Since $w_\alpha(x, s) = 0$ almost everywhere on I for each regular element $(x, s) \in K \oplus S$, it follows that $R \oplus S$ is a Krull ring.

3. Unique factorization rings. Let R be a ring with total quotient ring K such that $R \neq K$. $H(R)$ will denote the subgroup of $D(R)$ whose elements are of the form $\operatorname{div} Ra$ where a is a regular element of K. The factor group $C(R) = D(R)/H(R)$ is called the *divisor class group* of R.

In the domain case, R is a unique factorization domain if and only if R is a Krull ring and $C(R)$ is trivial. To investigate what happens when R is an arbitrary ring with identity, a characterization of a unique factorization ring will be used. This characterization states that every unique factorization ring is a finite complete direct sum of unique factorization domains and of special principal ideal rings (1).

In connection with the following proposition, a ring R is said to have *Property* (M) if each nonempty collection of regular principal ideals of R contains a maximal element.

PROPOSITION 3.1. *Let R be a ring which is not its own total* *quotient ring. Consider the following statements:*

(1) *R is a unique factorization ring.*

(2) *R is a Krull ring and $C(R)$ is trivial.*

(3) *R is a Krull ring and each maximal divisorial ideal of R is principal.*

(4) *R has property (M) and the intersection of two regular principal ideals is principal.*

(5) *R is a Krull ring and the intersection of two regular principal ideals is principal.*

Then $(1) \Rightarrow (3) \Leftrightarrow (5) \Leftrightarrow (2) \Rightarrow (4)$.

Proof. $(2) \Leftrightarrow (3)$. It is clear that $(2) \Rightarrow (3)$. To see that $(3) \Rightarrow (2)$, let A be a divisorial ideal of R. Then by Proposition 2.4

$$\operatorname{div} A = \sum n_\alpha \operatorname{div} M_\alpha$$

where $\{M_\alpha \mid \alpha \in I\}$ is the collection of maximal divisorial ideals of R and $n_\alpha = 0$ almost everywhere on I. By (3), each M_α is principal and it follows that

$$\operatorname{div} A = \sum n_\alpha \operatorname{div} M_\alpha = \operatorname{div} \prod M_\alpha^{n_\alpha}$$

is a principal divisor of R. Therefore, $C(R)$ is trivial and $(3) \Rightarrow (2)$.

$(2) \Rightarrow (4)$. Since each regular principal ideal of a ring is divisorial, it follows from Proposition 2.2 that every collection of regular principal ideals of a Krull ring has a maximal element. Hence, R has property (M). If a and b are regular elements of R, then $\overline{(a) \cap (b)} \subset (a)$ and $\overline{(a) \cap (b)} \subset (b)$. Thus $(a) \cap (b)$ is divisorial since $(a) \cap (b) \subset \overline{(a) \cap (b)}$, and it follows from the assumption that $C(R)$ is trivial that $(a) \cap (b)$ is principal. Therefore $(2) \Rightarrow (4)$.

$(5) \Leftrightarrow (3)$. Let K be the total quotient ring of R and let M be a maximal divisorial ideal of R. Thus, if $S = \{a \in K \mid M \subset Ra\}$, then $M = \bigcap \{Ra \mid a \in S\}$. Therefore $M \subset Ra \cap R \subset R$ for each $a \in S$. If $Ra \cap R = R$ for each $a \in S$, then $R \subset \bigcap \{Ra \mid a \in S\} = M$, a contradiction. Hence there exists $a \in S$ such that $M \subset Ra \cap R$ which is properly contained in R. But $Ra \cap R$ is a divisorial ideal of R, and by the maximality M, $M = Ra \cap R$. Let d be a regular element of R such that $ad \in R$. Then $dM = Rad \cap Rd$, and since ad is regular, it follows that dM is principal. Hence M is principal and $(5) \Rightarrow (3)$. Conversely, since the intersection of two regular principal ideals is divisorial, from Proposition 2.4 it follows that the intersection of two regular principal ideals of R is principal. Therefore $(5) \Leftrightarrow (3)$.

$(1) \Rightarrow (2)$. Since R is a unique factorization ring, R is a finite complete direct sum of unique factorization contains and special principal ideal rings. But each unique factorization domain is a Krull domain with trivial class group and a complete direct sum of special principal ideal rings is its own total quotient ring. Hence R may be written as $R_1 \oplus S$ where R_1 is a Krull ring and S is its own total quotient ring. Therefore by Proposition 2.5, R is a Krull ring.

Noting that the divisor class group of a finite complete direct sum is the complete direct sum of the divisor class groups of each summand, and that the divisor class group of S is trivial, it follows that $C(R)$ is trivial. Therefore $(1) \Rightarrow (2)$, and the implications and equivalences in the statement of the proposition hold.

Of particular interest in Proposition 3.1 is the statement that (1) implies (2). However, it has not been shown whether the converse holds. If (2) did imply (1), then it would follow that each Krull ring with trivial class group is a finite complete direct sum of unique factorization domains and special ideal rings. This, in the author's opinion, is too strong a result. Accordingly, the proof or disproof that (2) implies (1) is left as an open problem.

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