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THE SUPPORT OF AN EXTREMAL DILATATION

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We introduce a density condition which applies to subsets, E, of a bounded region Ω in the complex plane. If E satisfies this condition, then it is possible to construct a quasiconformal mapping F, of Ω, subject to the following conditions: F is extremal for its boundary values; F is conformal throughout $\Omega - E$; F is *not* conformal on E. The construction makes essential use of the Hamilton-Reich-Strebel characterization of extremal quasiconformal maps.

0. Introduction. In all that follows, Ω denotes a bounded domain in the complex plane. Let κ denote an element of $\mathscr{L}^\infty(\Omega)$. We say that κ is an *extremal dilatation* (on Ω) if $\|\kappa\|_\infty \neq 0$ and κ is the complex dilatation of a quasiconformal mapping of Ω which is extremal for its boundary values.

R. Hamilton, E. Reich and K. Strebel have given an incisive characterization of extremal dilatations. Their result follows ([1], [3], [4], [5]):

Let $B(\Omega)$ denote the space of functions, f, analytic on Ω, for which

$$\|f\| = \int |f(z)|\, dA(z) < \infty \qquad \text{(area measure)}\,.$$

Then κ is an extremal dilatation if and only if $(0 < \|\kappa\|_\infty < 1)$ *and*

$$(*) \qquad \sup_{\substack{\|f\|=1 \\ f \in B(\Omega)}} \left| \int_\Omega f(z)\kappa(z)\, dA(z) \right| = \|\kappa\|_\infty\,.$$

It is well known that a bounded measurable function κ may be supported on a small subset of Ω and still satisfy condition (*). In this paper we attempt to quantify this feature. We show that subsets of Ω which satisfy a certain density condition will always support extremal dilatations.

Density conditions which are *necessarily* satisfied by the support of an extremal dilatation are known in the case that Ω is the unit disk or the upper half plane. Some of these are discussed in [2]. They have features in common with the present sufficient condition, but in no case is there a complete characterization.

1. A sufficient condition. If E is a subset of Ω, χ_E denotes the indicator function of E:

$$\chi_E(z) = \begin{cases} 1, & z \in E \\ 0, & z \notin E . \end{cases}$$

Let E denote a subset of Ω. *We say that E is analytically thick in Ω if there is a bounded analytic function, h, defined on Ω for which* $\|h\|_\infty = 1$ *and*

$$\int_{H(x)} \chi_E(z)|h(z)|\,dA(z) = (1 + o(1))\int_{H(x)} |h(z)|\,dA(z) , \tag{1.1}$$

as $x \to 1$; *in* (1.1), $H(x) = \{z \in \Omega\colon\ |h(z)| > x\}$ *for* $0 \leqq x < 1$; *also,* $dA(z)$ *denotes Lebesgue planar measure.*

THEOREM 1. *Suppose E is analytically thick in Ω. Then there is an extremal dilatation, κ, defined on Ω for which*

$$\{z \in \Omega\colon\ \kappa(z) \neq 0\} \subset E .$$

Proof. The proof of Theorem 1 depends on Lemma 2 of § 4 and on the theorem of Hamilton-Reich-Strebel.

Let h be given as in the definition. By Lemma 2, condition (1.1) implies

$$\int_\Omega \chi_E(z)|h(z)|^n dA(z) = (1 + o(1))\int_\Omega |h(z)|^n dA(z),\ n \longrightarrow \infty . \tag{1.2}$$

Let $N = \{1, 2, 3, \cdots\}$ and let $\|\cdot\|_1$ denote the norm in $\mathscr{L}^1(\Omega)$. For each $n \in N$, set $k_n(z) = h^n(z)/\|h^n\|_1$. So, $\|k_n\|_1 = 1$ and, by (1.2)

$$\lim_{n\to\infty} \int_{\Omega - E} |k_n(z)|\,dA(z) = 0 . \tag{1.3}$$

It can be shown that $\{k_n\colon\ n \in N\}$ is a normal family; there are two possibilities:

(1) at least one subsequence of $\langle k_n \rangle_{n \in N}$ converges, uniformly on compact subsets of Ω, to a function $K(z)$ which is analytic and not identically zero on Ω.

(2) $\langle k_n \rangle_{n \in N}$ converges to zero uniformly on compact subsets of Ω.

In Case 1, apply Fatou's theorem to the given subsequence: we see, by (1.3)

$$\int_{\Omega - E} |K(z)|\,dA(z) \leqq \varlimsup_{n\to\infty} \int_{\Omega - E} |k_n(z)|\,dA(z) = 0 .$$

Therefore measure $(\Omega - E) = 0$ since K is analytic and not identically zero.

In Case 2, we construct a sequence, $\langle A_n \rangle$, of mutually disjoint compact subsets of Ω, and a subsequence, $\langle K_n\rangle$, of $\langle k_n\rangle$, such that

$$\int_{\Omega - A_n} |K_n(z)|\,dA(z) \leqq \frac{1}{n}, \quad n \in \boldsymbol{N}. \tag{1.4}$$

First, A_1 and K_1 are chosen arbitrarily. Suppose $K_1, K_2, \cdots, K_n$ and $A_1, A_2, \cdots, A_n$ have been chosen; we take K_{n+1}, from the vanishing sequence $\langle k_n\rangle$, so that

$$\int_{\bigcup_{j=1}^{n} A_j} |K_{n+1}(z)|\,dA(z) \leqq \frac{1}{2(n+1)}.$$

Since $\|K_{n+1}\|_1 = 1$, we may choose A_{n+1}, disjoint from $A_1, A_2, \cdots, A_n$, so that

$$\int_{A_{n+1}} |K_{n+1}(z)|\,dA(z) \geqq 1 - \frac{1}{n+1};$$

this is the same as (1.4).

Now we set

$$\kappa(z) = \begin{cases} \overline{K_n(z)}/|K_n(z)|, & z \in E \cap A_n,\ n \in \boldsymbol{N} \\ 0 & , \text{otherwise}. \end{cases}$$

Take $n \in \boldsymbol{N}$; (1.4) implies

$$\begin{aligned} \left|\int_\Omega K_n(z)\kappa(z)dA(z)\right| &\geqq \left|\int_{A_n} K_n(z)\kappa(z)dA(z)\right| - 1/n \\ &= \int_{E\cap A_n} |K_n(z)|\,dA(z) - 1/n \geqq \int_E |K_n(z)|\,dA(z) - 2/n. \end{aligned}$$

Combine this with (1.3); since $\langle K_n\rangle$ is a subsequence of $\langle k_n\rangle$, we have

$$\lim_{n\to\infty} \left|\int_\Omega K_n(z)\kappa(z)dA(z)\right| = \|\kappa\|_\infty.$$

It now follows, from the theorem of Hamilton, Reich and Strebel, that $\kappa/2$ is an *extremal dilatation*. As κ is supported within E, we are through.

2. An example. Set Ω: $|z-1|<1$ and $h(z)=e^{-1}\exp\{-(2i/\pi)\log z)$. Then $|h(re^{i\theta})|=e^{-1}\exp\{(2/\pi)\theta\}$ and, if $e^{-2}<x<1$ and $\theta(x)=\pi/2(1+\log x)$, we have

$$H(x) = \{re^{i\theta}\colon\ \theta(x) < \theta < \pi/2 \quad \text{and} \quad 0 < r < 2\cos\theta\}.$$

For $-\pi/2 < \theta < \pi/2$, we set $l(\theta) = \{re^{i\theta}\colon\ 0 < r < 2\cos\theta\}$.

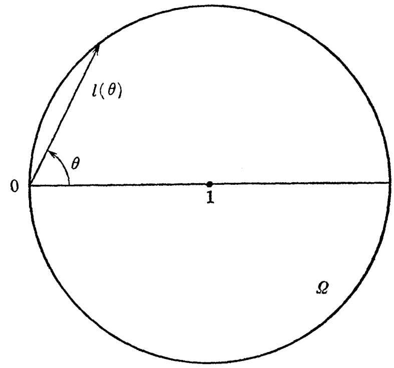

FIGURE 1

Now, let $E \subset \Omega$. We assume that the *linear density of E on $l(\theta)$ approaches one as θ approaches $\pi/2$*: that is, we assume

$$(2.1) \qquad \int_0^{2\cos\theta} \chi_E(re^{i\theta})\, dr = (1 + o(1))2\cos\theta,\ \theta \longrightarrow \pi/2 .$$

It is a consequence of (2.1) that

$$(2.2) \qquad \int_0^{2\cos\theta} \chi_E(re^{i\theta}) r dr = (1 + o(1)) \int_0^{2\cos\theta} r dr , \qquad \theta \longrightarrow \pi/2 .$$

This can be seen in a few lines; we integrate $\int_0^{2\cos\theta} \chi_E(re^{i\theta}) r dr$ by parts, then use (2.1) and the estimate

$$\int_0^{2\cos\theta} \int_0^t \chi_E(re^{i\theta}) dr dt \leqq \int_0^{2\cos\theta} t dt .$$

In turn, from (2.2), we see

$$\int_{\theta(x)}^{\pi/2} e^{-1} \exp\left\{\frac{2\theta}{\pi}\right\} \int_0^{2\cos\theta} \chi_E(re^{i\theta}) r dr d\theta$$
$$= (1 + o(1)) \int_{\theta/(x)}^{\pi/2} e^{-1} \exp\left\{\frac{2\theta}{\pi}\right\} \int_0^{2\cos\theta} r dr d\theta, \qquad \text{as } x \longrightarrow 1 ;$$

and this is the same as

$$\int_{H(x)} \chi_E(z) |h(z)| dA(z) = (1 + o(1)) \int_{H(x)} |h(z)| dA(z) , \qquad x \longrightarrow 1 .$$

By Theorem 1, E is an extremal support. So, if E satisfies condition (2.1), there is an extremal quasiconformal mapping of Ω which is conformal outside of E but not conformal throughout Ω.

3. LEMMA 1. *Let f and g denote integrable functions defined on $(0, 1)$. We assume:* $0 \leqq f(r) \leqq g(r)$ *for all* r, $0 < r < 1$; $\int_x^1 g(r) dr >$

0 *for all* x, $0 \leqq x < 1$; *and*

$$\int_x^1 rf(r)dr = (1 + o(1))\int_x^1 rg(r)dr\,, \qquad x \longrightarrow 1\,. \tag{3.1}$$

Then

$$\int_0^1 r^n f(r)dr = (1 + o(1))\int_0^1 r^n g(r)dr\,, \qquad n \longrightarrow \infty\,. \tag{3.2}$$

Proof. Let $\varepsilon > 0$ be fixed. By condition (3.1), we may choose $x(\varepsilon)$, in $(0, 1)$, so that,

$$\int_x^1 r(g(r) - f(r))dr \leqq \varepsilon/2\int_x^1 rg(r)dr \tag{3.3}$$

if $x(\varepsilon) \leqq x < 1$. This implies that

$$\int_y^1\int_x^1 r(g(r) - f(r))drdx \leqq \varepsilon/2\int_y^1\int_x^1 rg(r)drdx$$

holds as long as $x(\varepsilon) \leqq y < 1$. We interchange the order of integration and obtain

$$\int_{r=y}^1 r(g(r) - f(r))(r - y)dr \leqq \varepsilon/2\int_{r=y}^1 rg(r)(r - y)dr\,;$$

then, by (3.3), we see that

$$\int_{r=y}^1 r^2(g(r) - f(r))dr \leqq \varepsilon/2\int_{r=y}^1 r^2 g(r)dr$$

for any y, $x(\varepsilon) \leqq y < 1$.

Repeat this argument with the same $x(\varepsilon)$. We see that (3.3) is valid with r replaced by r^n. Thus,

$$\int_{x(\varepsilon)}^1 r^n(g(r) - f(r))dr \leqq \varepsilon/2\int_{x(\varepsilon)}^1 r^n g(r)dr \tag{3.4}$$

holds for all $n \in \boldsymbol{N}$.

Set $M = \int_0^1 g(t) - f(t)dt$. Then, by (3.4), if $n \in \boldsymbol{N}$, we have

$$\int_0^1 r^n(g(r) - f(r))dr \leqq Mx(\varepsilon)^n + \varepsilon/2\int_{x(\varepsilon)}^1 r^n g(r)dr\,. \tag{3.5}$$

Now, set $x_1(\varepsilon) = (x(\varepsilon) + 1)/2$. Since $\int_{x_1(\varepsilon)}^1 g(t)dt > 0$, we may choose $N(\varepsilon, f, g) \in \boldsymbol{N}$ so that, if $n \geqq N(\varepsilon, f, g)$, we have

$$Mx(\varepsilon)^n \leqq \varepsilon/2(x_1(\varepsilon))^n\int_{x_1(\varepsilon)}^1 g(r)dr \leqq \varepsilon/2\int_0^1 r^n g(r)dr$$

(just note that $x(\varepsilon)<x_1(\varepsilon)$). Combine this with (3.5); if $n \geqq N(\varepsilon, f, g)$, we have

$$\int_0^1 r^n(g(r) - f(r))dr \leqq \varepsilon \int_0^1 r^n g(r)dr .$$

We proved that

$$\int_0^1 r^n(g(r) - f(r))dr = o(1)\int_0^1 r^n g(r)dr , \qquad n \longrightarrow \infty ,$$

and (3.2) now follows.

4. LEMMA 2. The technique here is to perform an iterated integration over the level curves of $|h|$. For the sake of completeness, we establish the existence of an appropriate induced measure on these curves. So, the proof is a little longer than is perhaps necessary.

LEMMA 2. *Let h denote a bounded analytic function on Ω with $\|h\|_\infty = 1$. For $0 \leqq x < 1$, we set $H(x) = \{z \in \Omega: |h(z)| > x\}$. Then, if $E \subset \Omega$ and*

$$\int_{H(x)} X_E(z)|h(z)|dA(z) = (1 + o(1))\int_{H(x)} |h(z)|dA(z) \tag{4.1}$$

as $x \to 1$, it follows that

$$\int_\Omega \chi_E(z)|h(z)|^n dA(z) = (1 + o(1))\int_\Omega |h(z)|^n dA(z) , \tag{4.2}$$

as $n \to \infty$.

Proof. Set $\Omega' = \{z \in \Omega: |h(z)| \neq 0$ and $|h'(z)| \neq 0\}$. The lemma is trivial when h is a constant function. If h is not constant (as we assume from now on), the set $\Omega - \Omega'$ is negligible with regard to integration.

We construct an open cover of Ω'. For each $z \in \Omega'$, $U(z)$ will denote an open subset of Ω' which contains z; moreover, *we assume h is one-to-one in each $U(z)$.*

Now, let $\{P_n: n \in N\}$ be a C^∞ partition of unity, on Ω', subordinate to the cover $\{U(z): z \in \Omega'\}$. So, for each $n \in N$, there is a set $U(n) \in \{U(z): z \in \Omega'\}$ which contains the support of P_n. Set $h[U(n)] = S(n)$ and let $S(n) \xrightarrow{z_n} U(n)$ $(w \to z_n(w))$ denote the inverse of h defined in $S(n)$. For $0 < r < 1$, $n \in N$ we set

$$\Theta_n(r) = \{\theta: 0 \leqq \theta < 2\pi,\ re^{i\theta} \in S(n)\}$$

and we define

$$f_n(r) = \int_{\Theta_n(r)} P_n(z_n(re^{i\theta}))\chi_E(z_n(re^{i\theta}))\,|z_n'(re^{i\theta})|^2 r d\theta$$

$$g_n(r) = \int_{\Theta_n(r)} P_n(z_n(re^{i\theta}))\,|z_n'(re^{i\theta})|^2 r d\theta$$

and

$$f(r) = \sum_{n\in N} f_n(r), \quad g(r) = \sum_{n\in N} g_n(r) .$$

It is clear that $0 \leqq f(r) \leqq g(r)$, $0 < r < 1$.

If $n \in N$ is arbitrary and $0 \leqq x < 1$ and $0 \leqq \theta < 2\pi$, note that

$$\chi_{H(x)}(z_n(re^{i\theta})) \equiv \Phi_x(r) = \begin{cases} 1, & x < r < 1 \\ 0, & 0 < r \leqq x . \end{cases}$$

Take $N \in \boldsymbol{N}$ and suppose $0 \leqq x < 1$: then, with $w = re^{i\theta}$,

$$\begin{aligned} \int_{H(x)} \chi_E(z)\,|h(z)|^N dA(z) &= \sum_{n\in N}\int_{\Omega'} P_n(z)\chi_{H(x)}(z)\chi_E(z)\,|h(z)|^N dA(z) \\ &= \sum_{n\in N}\int_{S(n)} P_n(z_n(w))\chi_{H(x)}(z_n(w))\chi_E(z_n(w))\,|w|^N\,|z_n'(w)|^2 dA(w) \\ &= \sum_{n\in N}\int_{r=0}^{1}\int_{\Theta_n(r)} P_n(z_n(\cdot))\chi_{H(x)}(z_n(\cdot))\chi_E(z_n(\cdot))r^N\,|z_n'(\cdot)|^2 r d\theta dr \\ &= \sum_{n\in N}\int_{r=0}^{1} \Phi_x(r) f_n(r) r^N dr = \int_0^1 \Phi_x(r) f(r) r^N dr . \end{aligned}$$

We conclude from the Monotone Convergence Theorem that f is integrable on $(0, 1)$. In summary, if $0 \leqq x < 1$ and $N \in \boldsymbol{N}$ we have

$$\int_{H(x)} |h(z)|^N \chi_E(z) dA(z) = \int_x^1 f(r) r^N dr \tag{4.3}$$

and, by the same reasoning,

$$\int_{H(x)} |h(z)|^N dA(z) = \int_x^1 g(r) r^N dr . \tag{4.4}$$

By (4.4), $\int_x^1 g(r)dr > 0$ if $0 < x < 1$. By hypothesis (4.1) and equations (4.3) and (4.4), in the case $N = 1$, we see

$$\int_x^1 f(r) r dr = (1 + o(1)) \int_x^1 g(r) r dr , \qquad x \longrightarrow 1 .$$

Thus, by Lemma 1,

$$\int_0^1 f(r) r^N dr = (1 + o(1)) \int_0^1 g(r) r^N dr , \qquad N \longrightarrow \infty .$$

So, by (4.3) and (4.4), in the case $x = 0$,

$$\int_{H(o)} |h(z)|^N \chi_E(z) dA(z) = (1 + o(1)) \int_{H(o)} |h(z)|^N dA(z) ,$$

as $N \to \infty$. Since $\Omega - H(o)$ is countable, we are through.

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