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A FIXED POINT THEOREM FOR PRODUCT SPACES

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We prove the following result in this paper: Let (X, d) be a complete metric space and Y be a space having the fixed point property. Let $f: X \times Y \to X \times Y$ be a continuous map. If f is a contraction mapping in the first variable, then f has a fixed point.

This result is a generalization to the result obtained in Nadler [5].

Other results are proved concerning the fixed point theorem for product spaces.

The concept "continuous height-selection" is discussed and its relation to the existence of fixed points for a function is also discussed.

1. **Introduction.** Given a function $f: X \to X$, fixed point theory is concerned with the questions of the existence, nature and number of points $x \in X$ such that $f(x) = x$.

The problem of whether the fixed-point property (f.p.p. for short) is, or is not necessarily invariant under cartesian products is an old one (see [2] and [3] for its history). As is well-known, examples due to Connell in the category of metric spaces, to Lopez in the category of polyhedra, and to Bredon in the category of polyhedra with the Shih condition show that the answer is negative in these categories, ([2] and [3]). In a certain sense the reason for the failure in the latter examples of the invariance of the f.p.p. under products is due to special local properties of the space under consideration. The first indication that the invariance could fail for homotopy-theoretic reasons came to light when Bredon discovered cell complexes X_α of the form $S^k \bigcup_\alpha D^{2m}$ with the f.p.p. whose product $X_\alpha \times X_\beta$ fails to have the f.p.p. ([1]). A nice article by Husseini [4] shows that there are also manifolds with the f.p.p. whose product fails to have f.p.p.

Although the f.p.p. is not preserved under cartesian products in general, it is preserved when the maps $f: X \times Y \to X \times Y$ have special contraction properties. We prove the following theorem.

THEOREM 1.1. *Let (X, d) be a complete metric space, let Y be a topological space with the* f.p.p. *and let f be a continuous function from $X \times Y$ into $X \times Y$. If f is a locally contraction mapping in the first variable, then f has a fixed point.*

(For the definition of locally contraction mapping see §2.)

The above theorem is a generalization of the following two theorems obtained by Nadler (see [5]).

THEOREM 1.2 (Nadler). *Let (X, d_X) be a complete metric space, let (Y, d_Y) be a metric space with the* f.p.p., *and let f be a function from $X \times Y$ into $X \times Y$. If f is uniformly continuous on $X \times Y$ and a contraction mapping in the first variable, then f has a fixed point.*

THEOREM 1.3 (Nadler). *Let (X, d_X) be a complete metric space,* let *(Y, d_Y) be a metric space with the* f.p.p., *and let f be a function from $X \times Y$ into $X \times Y$. If (X, d_X) is locally compact and f is continuous on $X \times Y$ and a contraction mapping in the first variable, then f has a fixed point.*

In Theorem 1.1 we don't need f to be uniformly continuous on $X \times Y$; neither we need the metrizability of Y nor the locally compactness of X. For these reasons, Theorem 1.1 is a generalization of each of Theorem 1.2 and Theorem 1.3.

In what follows $pr_1: X \times Y \to X$ will denote the first projection mapping defined by $pr_1(x, y) = x$, while $pr_2: X \times Y \to Y$ will denote the second projection mapping defined by $pr_2(x, y) = y$.

2. Fixed point theorem and product spaces. Let us start this section with the following definition.

DEFINITION 2.1. Let (X, d) be a metric space and Y be any space. Let f be a mapping from $X \times Y$ into $X \times Y$. Then f is said to be a *locally contraction mapping in the first variable* if and only if for any $y \in Y$ there exist an open set $V(y)$ containing y and a real number $\lambda(y) \in [0, 1)$ such that

$$d(pr_1 f(x, v), pr_1 f(x_*, v)) \leqq \lambda(y) d(x, x_*) \text{ for all } x, x_* \in X \text{ and all } v \in V(y) .$$

A mapping $f: X \times Y \to X \times Y$ is called a *contraction mapping in the first variable* if and only if there exists a real number $\lambda \in [0, 1)$ such that for any $y \in Y$ we have

$$d(pr_1 f(x_1, y), pr_1 f(x_2, y)) \leqq \lambda d(x_1, x_2) \quad \text{for all} \quad x_1, x_2 \in X .$$

It is clear that every contraction mapping in the first variable must be locally contraction in the first variable but the converse is not true in general. For example, let $X = Y = R$, $d(x, y) = |x - y|$ for all $x, y \in R$, and let $f: R \times R \to R \times R$ be defined by $f(x, y) = (xy^2/(1 + y^2), xy^2/(1 + y^2))$ (notice that both X and Y have the usual

topology on the real line). For $y \in Y$ let

$$r = |y| + 1 , \qquad V(y) = (-r, r) \quad \text{and} \quad \lambda(y) = \frac{r^2}{1 + r^2} .$$

Then for any $x, x' \in X$ and any $v \in V(y)$, we have

$$|pr_1 f(x, v) - pr_1 f(x', v)| = \frac{v^2}{1 + v^2}|x - x'| \leqq \lambda(y)|x - x'| .$$

Hence f is a locally contraction mapping in the first variable. Since

$$|pr_1 f(0, y) - pr_1 f(1, y)| = \frac{y^2}{1 + y^2} \quad \text{and} \quad \operatorname{Lim}_{y \to \infty} \frac{y^2}{1 + y^2} = 1 ,$$

therefore the mapping f is not contraction in the first variable.

Let us prove the first of our main results.

Proof of Theorem 1.1. Let x_0 be a fixed point in X. Define the mapping $g: Y \to Y$ as follows:

Let $y \in Y$. In order to define $g(y)$ we first define a sequence $\{t_n(y)\}$ as follows:

$$t_0(y) = x_0 , \qquad t_n(y) = pr_1 f(t_{n-1}(y), y) , \qquad n = 1, 2, 3, \cdots .$$

For simplicity we shall write t_n to stand for $t_n(y)$.

We shall prove below, that as a consequence of local contractibility, it follows that $\{t_n\}$ is a Cauchy sequence in X (see Lemma 1 below). But (X, d) is complete so there is a point t_y in X such that $t_n \to t_y$.

Now define $g(y) = pr_2 f(t_y, y)$. We are going to prove in Lemma 2 that $pr_1 f(t_y, y) = t_y$.

We are also going to prove in Lemma 3 that g is continuous.

Since Y has the f.p.p., there is a point $y_0 \in Y$ such that $g(y_0) = y_0$. By Lemma 2 we have $pr_1 f(t_{y_0}, y_0) = t_{y_0}$. But $y_0 = g(y_0) = pr_2 f(t_{y_0}, y_0)$. Henceforth $f(t_{y_0}, y_0) = (t_{y_0}, y_0)$ which completes the proof of the theorem.

LEMMA 1. *The sequence $\{t_n\}$ is a Cauchy sequence in X.*

Proof. Since f is a locally contraction mapping in the first variable, there exists a real number $\lambda \in [0, 1)$ such that

$$d(pr_1 f(t_{n-1}, y), pr_1 f(t_n, y)) \leqq \lambda d(t_{n-1}, t_n) , \qquad n \geqq 1 .$$

By induction, we find

$$d(t_n, t_{n+1}) \leqq \lambda^n d(t_0, t_1) .$$

Using the triangle inequality, we find, for $m > n$,

$$\begin{aligned} d(t_n, t_m) &\leqq d(t_n, t_{n+1}) + \cdots + d(t_{m-1}, t_m) \\ &\leqq (\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1}) d(t_0, t_1) \\ &= \lambda^n \frac{1 - \lambda^{m-n}}{1 - \lambda} d(t_0, t_1) \\ &< \frac{\lambda^n}{1 - \lambda} d(t_0, t_1) \ . \end{aligned}$$

Since $\lambda^n \to 0$ as $n \to \infty$, this inequality shows that $\{t_n\}$ is a Cauchy sequence.

LEMMA 2. $pr_1 f(t_y, y) = t_y$.

Let $u = pr_1 f(t_y, y)$. By way of contradiction, suppose that $u \neq t_y$. Then $d(u, t_y) = \varepsilon > 0$. Since f is continuous, there exists an open set $U \times V$ in $X \times Y$ such that

$$U \subset S_{\varepsilon/4}(t_y),\ (t_y, y) \in U \times V \quad \text{and} \quad f(U \times V) \subset S_{\varepsilon/4}(u) \times Y \ .$$

Since $\lim_{n\to\infty} t_n = t_y$, there is a natural number $K \geqq 1$ such that $t_n \in U$ for all $n \geqq K$. But $pr_1 f(t_k, y) = t_{k+1} \in U$. Therefore $f(t_k, y) \notin S_{\varepsilon/4}(u) \times Y$ which contradicts the fact that $f(U \times V) \subset S_{\varepsilon/4}(u) \times Y$.

Therefore our assumption is incorrect and consequently we have the required conclusion.

LEMMA 3. *The function $g: Y \to Y$ defined as in the proof of Theorem* 1.1 *is a continuous function.*

Proof. Let $y \in Y$ and U be any open set containing $g(y)$. Then $f(t_y, y) \in X \times U$. Since f is a continuous function at (t_y, y), there is an open set G in Y and a positive real number $\varepsilon > 0$ such that

$$(t_y, y) \in S_\varepsilon(t_y) \times G \quad \text{and} \quad f(S_\varepsilon(t_y) \times G) \subset X \times U \ .$$

Let W be an open set in Y and let λ be a real number with $\lambda \in [0, 1)$ such that $y \in W$ and

$$d(pr_1 f(x, v), pr_1 f(x_*, v)) \leqq \lambda d(x, x_*) \text{ for all } x, x_* \in X \text{ and all } v \in W \ .$$

Since $\lambda^m \to 0$ as $m \to \infty$, we can choose a natural number $n \geqq 1$ such that

$$\lambda^n < \frac{\varepsilon}{8} \frac{1 - \lambda}{d(t_0, t_1) + (\varepsilon/8)} \ ,$$

and

$$d(t_y, t_m) < \frac{\varepsilon}{8} \quad \text{for all} \quad m \geqq n .$$

Since $f(t_n, y) \in X \times U$ and f is a continuous function at (t_n, y), there exists a basic open set $U_n \times V_n$ in $X \times Y$ such that

$$(t_n, y) \in U_n \times V_n , \qquad U_n \subset S_{\varepsilon/8}(t_y) , \qquad V_n \subset G \cap W \quad \text{and}$$
$$f(U_n \times V_n) \subset X \times U .$$

Since f is continuous at (t_{n-1}, y) and $f(t_{n-1}, y) \in U_n \times Y$, there exists a basic open set $U_{n-1} \times V_{n-1}$ in $X \times Y$ such that

$$(t_{n-1}, y) \in U_{n-1} \times V_{n-1} , \qquad U_{n-1} \subset S_{\varepsilon/8}(t_{n-1}) , \qquad V_{n-1} \subset V_n \quad \text{and}$$
$$f(U_{n-1} \times V_{n-1}) \subset U_n \times Y .$$

Inductively, suppose that $U_n, U_{n-1}, \cdots, U_i, V_n, V_{n-1}, \cdots, V_i (n-1 \geqq i \geqq 1)$ are chosen such that

$$(t_i, y) \in U_i \times V_i , \qquad U_i \subset S_{\varepsilon/8}(t_i) , \qquad V_i \subset V_{i+1} \quad \text{and}$$
$$f(U_i \times V_i) \subset U_{i+1} \times Y .$$

Since f is continuous at (t_{i-1}, y) and $f(t_{i-1}, y) \in U_i \times Y$, there exists a basic open set $U_{i-1} \times V_{i-1}$ in $X \times Y$ such that

$$(t_{i-1}, y) \in U_{i-1} \times V_{i-1} , \qquad U_{i-1} \subset S_{\varepsilon/8}(t_{i-1}) , \qquad V_{i-1} \subset V_i \quad \text{and}$$
$$f(U_{i-1} \times V_{i-1}) \subset U_i \times Y .$$

In this way, the sets $U_n, U_{n-1}, \cdots, U_0, V_n, V_{n-1}, \cdots, V_0$ are defined with the above mentioned properties. We claim that $g(V_0) \subset U$.

To prove the claim, let $z \in V_0$. Then $(t'_0, z) \in U_0 \times V_0$, where $t'_0 = x_0$. Thus $f(t'_0, z) \in U_1 \times Y$, i.e., $t'_1 = pr_1 f(t'_0, z) \in U_1$. Consequently $d(t'_1, t_1) < \varepsilon/8$.

Using the triangle inequality we have $d(t'_0, t'_1) = d(t_0, t'_1) \leqq d(t_0, t_1) + d(t_1, t'_1) < d(t_0, t_1) + \varepsilon/8$. Since $f(U_1 \times V_1) \subset U_2 \times Y$ and $(t'_1, z) \in U_1 \times V_1$, therefore $f(t'_1, z) \in U_2 \times Y$, i.e., $t'_2 = pr_1 f(t'_1, z) \in U_2$ (For brevity we have used t'_n to stand for $t_n(z)$).

In the same way, we conclude that $t'_i = pr_1 f(t'_{i-1}, z) \in U_i (i = 1, 2, \cdots, n)$.

Since $t'_n \in U_n$ and $U_n \subset S_{\varepsilon/8}(t_y)$, therefore $d(t'_n, t_y) < \varepsilon/8$. Using the triangle inequality we find, for $m \geqq n$,

$$d(t'_m, t_y) \leqq d(t_y, t'_n) + d(t'_n, t'_{n+1}) + \cdots + d(t'_{m-1}, t'_m)$$
$$< \frac{\varepsilon}{8} + \lambda^n d(t'_0, t'_1) + \lambda^{n+1} d(t'_0, t'_1) + \cdots + \lambda^{m-1} d(t'_0, t'_1)$$

$$= \frac{\varepsilon}{8} + \frac{\lambda^n}{1-\lambda}(1-\lambda^{m-n})d(t_0', t_1')$$

$$< \frac{\varepsilon}{8} + \frac{\lambda^n}{1-\lambda}\left(d(t_0, t_1) + \frac{\varepsilon}{8}\right) < \frac{\varepsilon}{8} + \frac{\varepsilon}{8} = \frac{\varepsilon}{4}.$$

If $t_z = \lim_{n\to\infty} t_n'$, then the above inequality shows that $d(t_z, t_y) \leqq \varepsilon/4$.

Therefore $(t_z, z) \in S_\varepsilon(t_y) \times G$ and consequently $f(t_z, z) \in X \times U$, i.e., $g(z) = pr_2 f(t_z, z) \in U$.

Therefore our claim is proved and the proof of the lemma is completed.

The following result is an immediate corollary to Theorem 1.1.

COROLLARY 2.2. *Let (X, d) be a complete metric space, let Y be a topological space with the* f.p.p. *and let f be a continuous function from $X \times Y$ into $X \times Y$. If f is a contraction mapping in the first variable, then f has a fixed point.*

A mapping $f: X \times Y \to X \times Y$ is called a *strong contraction mapping in the first variable* provided the following condition is satisfied: "There exists a positive constant K such that for all $x_1, x_2 \in X$ and all $y_1, y_2 \in Y$, we have $d(pr_1 f(x_1, y_1), pr_1 f(x_2, y_2)) \leqq K d(x_1, x_2)$ whenever $x_1 \neq x_2$".

It is clear that there exists a contraction mapping in the first variable which is not strong contraction in the first variable. It is also clear that there exists a strong contraction mapping in the first variable which is not contraction mapping in the first variable. For examples, let $X = Y = R$, $d(x, y) = |x - y|$ for all $x, y \in R$ and let f, g be two functions from $R \times R$ into $R \times R$ defined by the rules $f(x, y) = (2x, y)$ and $g(x, y) = ((x + y)/2, y)$. Then it is clear that f is a strong contraction mapping in the first variable with $k = 2$ but not a contraction mapping in the first variable. It is also clear that g is a contraction mapping in the first variable with $\lambda = 1/2$ but not a strong contraction mapping in the first variable because

$$|pr_1 g(2, y) - pr_1 g(0, 0)| = \left|\frac{y}{2} + 1\right| \quad \text{and} \quad \lim_{y\to\infty}\left(\frac{y}{2} + 1\right) = +\infty.$$

As the following proof shows we have the following result.

PROPOSITION 2.3. *Let (X, d) be a metric space which has no isolated point. If $f: X \times Y \to X \times Y$ is a continuous mapping and a strong contraction mapping in the first variable, then $pr_1 f(\{x\} \times Y)$ is a set consisting of a single element for all $x \in X$.*

Proof. We shall prove the proposition by contradiction. Assume

that there exist $x \in X$ and $y, y_* \in Y$ such that $pr_1 f(x, y) \neq pr_1 f(x, y_*)$. For simplicity, we shall write $t_1 = pr_1 f(x, y)$ and $t_2 = pr_1 f(x, y_*)$. Let $\varepsilon = d(t_1, t_2)$. Then $\varepsilon > 0$. Let $S_r(p)$ denote the open sphere in (X, d) with center at $p \in X$ and radius $r > 0$.

Since f is continuous, therefore there exist an open set U in X and open sets V_1 and V_2 in Y such that

$$(x, y) \in U \times V_1 , \qquad (x, y_*) \in U \times V_2 , \qquad U \subset S_{\varepsilon/6k}(x) ,$$

$f(U \times V_1) \subset S_{\varepsilon/6}(t_1) \times Y$ and $f(U \times V_2) \subset S_{\varepsilon/6}(t_2) \times Y$.

Since (X, d) has no isolated point, therefore there exists a point $u \in U - \{x\}$.

Therefore $d(u, x) < \varepsilon/6K$. But f is a strong contraction mapping in the first variable, therefore we have

$$d(pr_1 f(u, y_*), pr_1 f(x, y)) \leqq Kd(u, x) < \frac{\varepsilon}{6} .$$

Consequently $pr_1 f(u, y_*) \in S_{\varepsilon/6}(t_1)$ which is impossible because $pr_1 f(u, y_*) \in S_{\varepsilon/6}(t_2)$ and $S_{\varepsilon/6}(t_1) \cap S_{\varepsilon/6}(t_2) = \varnothing$.

Therefore we have the required contradiction and the proof of the proposition is completed.

3. Fixed point theorem and continuous height-selection. Let $f: X \times Y \to X \times Y$ be a continuous mapping. A point $y_x \in Y$ for which $pr_2 f(x, y_x) = y_x$ is called a *fixed f-height* of x. The set

$$\{y_x: y_x \text{ is a fixed } f\text{-height of } x\}$$

is called the *fixed f-height* of x and is denoted by $F(f, x)$. The set

$$\cup \{F(f, x): x \in X\}$$

is called the *fixed f-height* of X and is denoted by $F(f, X)$.

THEOREM 3.1. *If Y has the* f.p.p., *then $F(f, x) \neq \varnothing$ for all spaces X and all $x \in X$. Conversely if $F(f, X) \neq \varnothing$ for some space X and all continuous maps $f: X \times Y \to X \times Y$, then Y has the* f.p.p.

Proof. Let Y have the f.p.p. and let X be any topological space. Let $x \in X$ and $f: X \times Y \to X \times Y$ be any continuous map. Define the function $g: Y \to Y$ by the rule $g(y) = pr_2 f(x, y)$. Then g is continuous because f is continuous. But Y has the f.p.p., therefore there exists an element $y_x \in Y$ such that $g(y_x) = y_x$. Thus $pr_2 f(x, y_x) = y_x$, i.e., $F(f, x) \neq \varnothing$.

Conversely, suppose that $F(f, X) \neq \varnothing$ for some space X and all continuous maps $f: X \times Y \to X \times Y$. Let $g: Y \to Y$ be any continuous

map. Define $h: X \times Y \to X \times Y$ by the rule $h(x, y) = (x, g(y))$. Then h is continuous. Since $F(h, X) \neq \emptyset$, there exists an element $a \in X$ such that $F(h, a) \neq \emptyset$. Let $y \in F(h, a)$.

Then $pr_2 h(a, y) = y$. It follows from the definition of h that $g(y) = y$ which completes the proof of the theorem.

A *continuous height-selection* of f is a continuous mapping $g: X \to F(f, X)$ such that $g(x) \in F(f, x)$ ($F(f, X)$ has the subspace topology from Y).

Let us state our next result in this section.

THEOREM 3.2. *Let X and Y be topological spaces with the* f.p.p., *and let $f: X \times Y \to X \times Y$ be a continuous map. If f has a continuous height-selection, then f has a fixed point.*

Proof. Since Y has the f.p.p., by Theorem 3.1 the set $F(f, X) \neq \emptyset$. Let $g: X \to F(f, X)$ be a continuous height-selection of f. Define the map $h: X \to X$ by the rule $h(x) = pr_1 f(x, g(x))$. We claim that h is continuous. To prove our claim we let $x \in X$ and G be any open set containing $h(x)$. Then $G \times Y$ is an open set containing the point $f(x, g(x))$. Since f is continuous, therefore there exists an open set $U \times V$ containing $(x, g(x))$ such that $f(U \times V) \subset G \times Y$. But g is continuous and V is an open set containing $g(x)$. Therefore there exists an open set W containing x such that $g(W) \subset V$. Now, it is clear that $h(W) \subset G$. Hence h is continuous.

Let $t \in X$ be a fixed point of h. Then $h(t) = t$, i.e., $pr_1 f(t, g(t)) = t$. Therefore $f(t, g(t)) = (t, v)$ for some $v \in Y$. But $g(t) \in F(f, t)$. Therefore $pr_2 f(t, g(t)) = g(t)$, i.e., $f(t, g(t)) = (u, g(t))$ for some $u \in X$. Henceforth $f(t, g(t)) = (t, g(t))$ which completes the proof of the theorem.

THEOREM 3.3. *Let X and Y be spaces with* f.p.p. *Let f be a continuous function from $X \times Y$ into $X \times Y$. If $pr_1 f(\{x\} \times Y)$ is a singleton set for any $x \in X$, then f has a fixed point.*

Proof. Choose a point y_0 in Y. By Theorem 3.1 there exists an element $x_0 \in X$ such that $f(x_0, y_0) = (x_0, v)$ for some $v \in Y$. Apply Theorem 3.1 again, there exists an element $y_{x_0} \in Y$ such that $f(x_0, y_{x_0}) = (u, y_{x_0})$ for some $u \in X$. But $pr_1 f(\{x_0\} \times Y)$ is a singleton set, therefore $u = x_0$. Henceforth $f(x_0, y_{x_0}) = (x_0, y_{x_0})$ which completes the proof of the theorem.

If we combine Theorem 3.3 and Proposition 2.3 then we can get the following result by observing that if X has the f.p.p. then X must be connected.

COROLLARY 3.4. *Let (X, d) be a metric space with the* f.p.p., *let Y be a space with the* f.p.p. *and let f be a continuous mapping from $X \times Y$ into $X \times Y$. If f is a strong contraction mapping in the first variable, then f has a fixed point.*

Using the proof of Theorem 1.1 and the technique used in the proof of Theorem 3.2 one can obtain the following theorem.

THEOREM 3.5. *Let (X, d) be a complete metric space, let Y be a topological space with the* f.p.p. *and let f be a continuous function from $Y \times X$ into $Y \times X$. If f is a locally contraction mapping in the second variable, then f has a continuous height-selection and, hence, f has a fixed point.*

Actually the proof of Lemma 3 (§2) gives us a way to construct such continuous height-selection. More explicitely if we define $h: Y \to F(f, Y)$ ($F(f, Y)$ is a subspace of X) by the rule $h(y) = t_y$, then h will be a continuous height-selection of f.

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