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COUNTABLE DECOMPOSITIONS OF E^n

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We investigate the nature of upper semicontinuous decompositions of E^n that contain only countably many cellular nondegenerate elements. Such decompositions do not in general yield decomposition spaces which are homeomorphic to E^n. However, we consider certain additional restrictions on the decomposition which insure that the decomposition space is homeomorphic with $E^n (n \geqq 5)$. It is interesting to note that these same restrictions do not guarantee that such a decomposition of E^3 yields a decomposition space homeomorphic with E^3.

1. Introduction. R. H. Bing has exhibited [3] a very simple decomposition of E^3 for which the decomposition space fails to be a topological manifold. The decomposition consists of points and a null sequence (that is, only finitely many elements of the sequence have diameter greater than an arbitrary $\varepsilon > 0$) of cellular sets each of which is contained in one of two fixed affine planes. Furthermore, an arc contained in the straight line which is the intersection of these two planes meets every nondegenerate element of the decomposition.

In contrast with the 3-dimensional example of Bing, countable, cellular, upper semicontinuous decompositions of $E^n (n \geqq 5)$ yield decomposition spaces which are homeomorphic with E^n if there exists a tame arc or even a tame polyhedron of codimension at least two that meets all of the nondegenerate elements in the decomposition. We shall prove this fact and set forth other conditions which imply that such decompositions yield spaces homeomorphic with E^n.

2. Preliminaries. All decompositions in this paper are upper semicontinuous; that is, if G is a decomposition of X, then each element of G is compact and the natural projection $\pi: X \to X/G$ is closed. If G is a decomposition of X we let N_G denote the union of the nondegenerate elements of G; hence, N_G is a subset of X.

We use B^n and E^n to denote the n-cell and Euclidean n-space, respectively. A subset K of E^n is said to be *cellular* if there exists a sequence $B_1, B_2, B_3, \cdots$, of topological n-cells in E^n such that for each i, B_{i+1} is contained in the interior of B_i and $K = \cap B_i$. A compact subset of E^n is said to be *cell-like* if it admits an embedding in some Euclidean space as a cellular subset. We say that a decomposition of E^n is cellular (resp. cell-like) if each element of the decomposition is cellular (resp. cell-like). A decomposition with only countably many nondegenerate elements is called a countable decomposition.

A continuous function is called a map. Let f and g be maps from X into a metric space S. We say that g is an *ε-approximation* of f if for each x in X, the distance between $f(x)$ and $g(x)$ is less than or equal to ε. We say that a closed subset X of E^n (or a space homeomorphic with E^n) has *embedding codimension* greater than k, if each closed tame polyhedron L of dimension less than or equal to k can be moved off X by a small ambient isotopy of E^n with support arbitrarily close to $X \cap L$ [9]. A metric space S satisfies the *disjoint disk property* [5] if any two maps of the standard 2-cell B^2 into S can be approximated by maps having disjoint images.

We will have need of the following known facts which we state without proof.

THEOREM 2.1 (*Lifting to within ε theorem*) [1], [11], [12], [14]. *Let G be a cell-like upper semicontinuous decomposition of E^n, $\pi: E^n \to E^n/G$ the natural projection, f a map from a finite polyhedron K into E^n/G (which is a metric space). Then for each $\varepsilon > 0$ there is a map $\tilde{f}: K \to E^n$ such that $\pi \circ \tilde{f}$ is an ε-approximation of f.*

The following theorem due to R. H. Bing [4] was stated only for dimension three and in slightly different terms. Nevertheless, Bing's proof works equally well in the following setting.

THEOREM 2.2. *Let H be a cellular upper semicontinuous decomposition of E^n. If the closure of N_H contains only countably many elements of H, then E^n/H is homeomorphic with E^n.*

The following remarkable theorem is due to R. D. Edwards [8]. We state it only for decompositions of E^n.

THEOREM 2.3. *Let G be a cell-like decomposition of E^n. If E^n/G is finite dimensional and satisfies the disjoint disk property, then E^n/G is homeomorphic with E^n.*

3. Countable cellular decompositions of E^n. When contrasted with the Bing decomposition of §1, our first theorem shows that countable decompositions of $E^n (n \geqq 5)$ behave differently than countable decompositions of E^3.

THEOREM 3.1. *Let G be a cellular upper semicontinuous decom-*

position of $E^n(n \geqq 5)$ such that G contains only countably many nondegenerate elements. Suppose P is a polyhedron of dimension at most $(n-2)$ which is embedded in E^n as a tame closed subset. If P meets every nondegenerate element of G, then E^n/G is homeomorphic to E^n.

Proof. The diagram in Figure 1 will aid the reader in keeping track of various maps. We begin by letting $\pi: E^n \to E^n/G$ be the natural projection. Since $\pi|(E^n - N_G)$ is one-to-one and $\pi(N_G)$ is countable, it is easily seen [10, page 44] that E^n/G is finite dimensional. Hence, we will complete the proof by showing that E^n/G has the disjoint disk property.

Let f, g be maps of B^2 into E^n/G. By Theorem 2.1 there is a map $\tilde{f}: B^2 \to E^n$ so that $\pi \circ \tilde{f}$ is an ε-approximation of f. By general position [13] we may also assume that $\tilde{f}$ is a tame embedding of B^2 into E^n which meets P in only finitely many points.

Let H be the upper semicontinuous decomposition of E^n consisting of points and the nondegenerate elements of G which meet $\tilde{f}(B^2)$. Let $\pi_1: E^n \to E^n/H$ be the projection map and $\pi_2: E^n/H \to E^n/G$ be the map given by $\pi_2 = \pi \circ \pi_1^{-1}$.

LEMMA 3.2. *The decomposition space E^n/H is homeomorphic with E^n.*

Proof of Lemma 3.2. By Theorem 2.2 we need only show that $\bar{N}_H$, the closure of N_H, contains only countably many elements of H. Clearly N_H contains only countably many elements of H. If the point p is a limit point of N_H, that is not itself a point of N_H, then there exists a sequence of nondegenerate elements of H which converge to p. But each nondegenerate element of H meets both $f(B^2)$ and P. Hence p must be one of the finite number of points in the intersection of $f(B^2)$ and P. Therefore, the closure of N_H contains only countably many elements of H.

LEMMA 3.3. *Let G' be then decomposition of E^n/H consisting of elements $\pi_2^{-1}(x)$, $x \in E^n/G$. Then G' is a cell-like upper semicontinuous decomposition of E^n/H (Recall that E^n/H is homeomorphic with E^n). Furthermore, the map π_2 is essentially the natural projection of E^n/H onto $(E^n/H)/G'$.*

Proof of Lemma 3.3. The proof is straight-forward. Nondegenerate elements of G' are homeomorphic with nondegenerate elements of G. The map π_2 is easily seen to be closed and onto.

We now finish the proof of Theorem 3.1. By Theorem 2.1 there is a map $\tilde{g}: B^2 \to E^n/H$ so that $\pi_2 \circ \tilde{g}$ is an ε-approximation to g. We may also assume that $\tilde{g}(B^2) \cap \pi_1(\bar{N}_H) = \varnothing$ since $\pi_1(\bar{N}_H)$ contains only countably many points. Hence, $\pi_1^{-1} \circ \tilde{g}$ is a map from B^2 to E^n whose image misses $\bar{N}_H$. By a slight general position adjustment if necessary we may assume in addition that $\pi_1^{-1} \circ \tilde{g}(B^2)$ does not meet $\tilde{f}(B^2)$. It is now an easy matter to check that $\pi \circ \tilde{f}$ and $\pi_2 \circ \tilde{g}$ have disjoint images and are the desired approximations to f and g, respectively.

The proof of Theorem 3.1 can easily be adjusted to give a proof of the following theorem.

THEOREM 3.4. *Let G be a cellular upper semicontinuous decomposition of $E^n (n \geqq 5)$ such that G contains only countably many nondegenerate elements. Suppose K is a closed subset of E^n which has embedding codimension greater than two and which meets every nondegenerate element of G. Then E^n/G is homeomorphic to E^n.*

4. Generalizations. R. H. Bing [2] and W. T. Eaton [7] have constructed decompositions of E^n consisting of points and a Cantor set worth of tame arcs. The resulting decomposition spaces have been called dogbone spaces. The reader who is familiar with these decompositions will realize that there is a tame arc in each E^n (which lies in a hyperplane which slices the dogbones in half) which meets every nondegenerate element. Consequently, Theorems 3.1 and 3.4 cannot be strengthened by dropping the hypothesis that the decomposition has only countably many nondegenerate elements.

Daverman has given an example [6] of a nonmanifold decomposition of $E^n (n \geqq 5)$ consisting of points and a null sequence of cellular sets. Daverman's example can be obtained by starting with a modified dogbone decomposition of E^n and then "tubing" together [15] certain of the nondegenerate elements of H to obtain the nondegenerate elements of the Daverman decomposition. It turns out that the closure of the nondegenerate elements of the modified dogbone decomposition H has embedding codimension equal to two and meets all of the nondegenerate elements of Daverman's decomposition. Therefore, the dimension restriction in Theorem 3.4 is the best possible.

We now consider the dimension restrictions in Theorem 3.1. Clearly we may not assume that the polyhedron has dimension n since E^n itself could serve for the polyhedron. In many cases a polyhedron of dimension $(n-1)$ is sufficient to insure that the decomposition space is homeomorphic with E^n.

THEOREM 4.1. *Let G be a cellular upper semicontinuous decomposition of $E^n (n \geqq 5)$ such that G contains only countably many nondegenerate elements. Suppose P is a polyhedron of dimension* *at most* $(n-1)$ *which is embedded in* E^n *as a tame closed subset and which meets every nondegenerate element of* G. *If for each element* g *of* G *and for each open* $(n-1)$*-simplex* σ *of* P $g \cap \sigma$ *has embedding codimension greater than one* σ, *then* E^n/G *is homeomorphic with* E^n.

COROLLARY 4.2. *Let* G *be a cellular upper semicontinuous decomposition of* $E^n (n \geqq 5)$ *such that* G *contains only countably many nondegenerate elements each of which has dimension at most* $(n-3)$. *Suppose* P *is a polyhedron of dimension at most* $(n-1)$ *which is embedded in* E^n *as a tame closed subset. If* P *meets every nondegenerate element of* G, *then* E^n/G *is homeomorphic with* E^n.

Proof of Corollary 4.2. If g is an element of G and σ an open $(n-1)$-simplex of P, then dimension $(g \cap \sigma) \leqq n-3$. Hence, $g \cap \sigma$ is nowhere dense in σ and $g \cap \sigma$ cannot separate any open subset of σ [10, p. 48]. Hence, $g \cap \sigma$ has embedding codimension greater than one in σ.

Proof of Theorem 4.1. Once again we make reference to Figure 1; however, the proof will be slightly different. We let $\pi\colon E^n \to E^n/G$ be the natural projection. As before we need only show that E^n/G has the disjoint disk property.

Let f, g be maps of B^2 into E^n/G. By Theorem 2.1 there is a map $\tilde{f}\colon B^2 \to E^n$ so that $\pi \circ \tilde{f}$ is an ε-approximation of f. By general position arguments, we may also suppose that $K = \tilde{f}(B^2) \cap P$ is a tame 1-dimensional polyhedron which meets the $(n-2)$-skeleton of P in a finite number of points and that for each open $(n-1)$-simplex σ of p and each nondegenerate element g of G we have $\tilde{f}(B^2) \cap \sigma \cap g = \varnothing$. Furthermore, we assume that $\tilde{f}$ is a tame embedding. Let H be the upper semicontinuous decomposition consisting of points and the elements of G that meet $\tilde{f}(B^2)$. By Theorem 3.1 E^n/H is homeomorphic with E^n. We define the natural maps π_1 and π_2 as in the proof of Theorem 3.1. (See Figure 1.)

$$\begin{tikzcd}
 & E^n \arrow[dd, "\pi"] \arrow[dr, "\pi_1"] & & \\
 & & E^n/H \arrow[dl, "\pi_2"'] & \\
B^2 \arrow[uur, "\tilde{f}"] \arrow[r, "f"] & E^n/G & & B^2 \arrow[ll, "g"'] \arrow[ul, "\tilde{g}"']
\end{tikzcd}$$

FIGURE 1

LEMMA 4.3. *Let $h: B^2 \to E^n/H$. The map h can be approximated by a map $h_1: B^2 \to E^n/H$ such that $h_1(B^2) \cap \pi_1(K) = \varnothing$.*

Proof of Lemma 4.3. Let $h: B^2 \to E^n/H$ be a map. Let h' be a close approximation of h so that $h'(B^2)$ misses the image under π_1 of the finite number of nondegenerate elements of H that meet K. By Theorem 2.1 we find a map $h''\, B^2 \to E^n$ so that $\pi_1 \circ h''$ is a close approximation of h' and such that $h''(B^2)$ misses the finite number of nondegenerate elements of H that meet K. By general position we may also assume that $h''(B^2)$ misses K. Hence, $\pi_1 \circ h''$ is a close approximation of h that misses $\pi_1(K)$.

We now conclude the proof of Theorem 4.1. By Theorem 2.1 we find a map $\tilde{g}: B^2 \to E^n/H$ so that $\pi_2 \circ \tilde{g}$ is an ε-approximation to g. By Lemma 4.3 we may assume that $\tilde{g}(B^2)$ misses $\pi_1(K)$. Since H has only countably many nondegenerate elements, we may also assume that $\tilde{g}(B^2)$ misses the image of these elements under π_1. The closure $\bar{N}_H$ of N_H is easily seen to be contained in $N_H \cup K$. Hence $\pi_1^{-1} \circ \tilde{g}$ is a map from B^2 to E^n whose image misses $\bar{N}_H$. By a slight general position adjustment we may assume in addition that $\pi_1^{-1} \circ \tilde{g}(B^2)$ does not meet $\tilde{f}(B^2)$. The maps $\pi \circ \tilde{f}$ and $\pi_2 \circ \tilde{g}$ have disjoint images and are the desired approximations to f and g, respectively.

REFERENCES

1. S. Armentrout and T. M. Price, *Decompositions into compact sets with UV properties*, Trans. Amer. Math. Soc., **141** (1969), 433–442.
2. R. H. Bing, *A decomposition of E^3 into points and tame arcs such that the decomposition space is topologically different from E^3*, Ann. of Math., (2) **65** (1957), 484–500.
3. ———, *Point-like decompositions of E^3*, Fund. Math., **50** (1962), 431–453.
4. ———, *Upper semicontinuous decompositions of E^3*, Ann. of Math., (2) **65** (1957), 363–374.
5. J. W. Cannon, *Shrinking cell-like decompositions of manifolds*, Codimension three, Ann. of Math., (2) **110** (1979), 83–112.
6. R. J. Daverman, *A nonshrinkable decomposition of S^n determined by a null sequence of cellular sets*, Proc. Amer. Math. Soc., **75** (1979), 171–176.
7. W. T. Eaton, *A generalization of the dogbone space to E^n*, Proc. Amer. Math. Soc., **39** (1973), 379–387.
8. R. D. Edwards, *Approximating certain cell-like maps by homeomorphisms*, (manuscript). See Notices Amer. Math. Soc., **24** (1977), A-649, #751-G5.
9. ———, *Demension Theory, I, Geometric Topology*, Proceedings of the Geometric Topology Conference held at Park City, Utah, 1974 (edited by L. C. Glaser and T. B. Rushing), Springer-Verlag (New York), 194–211.
10. W. Hurewicz and H. Wallman, *Dimension Theory*, Princeton, 1941.
11. G. Kozlowski, *Factorization of certain maps up to homotopy*, Proc. Amer. Math. Soc., **21** (1969), 88–92.
12. R. C. Lacher, *Cell-like mappings* 1, Pacific J. Math., **30** (1969), 717–731.
13. C. P. Rourke and B. J. Sanderson, *Introduction to Piecewise-linear Topology*, Springer-Verlag, 1972.

14. S. Smale, *A Vietoris mapping theorem for homotopy*, Proc. Amer. Math. Soc., **8** (1957), 604–610.
15. D. G. Wright, *A decomposition of $E^n (n \geqq 3)$ into points and a null sequence of cellular sets*, General Topology and its Applications, **10** (1979), 297–304.

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