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ESTIMATES OF MEROMORPHIC FUNCTIONS AND SUMMABILITY THEOREMS

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The main goal of this paper is to prove the following theorem.

THEOREM 1. **Let L be an unbounded operator in a Hilbert space $\mathfrak{H}$, having a discrete spectrum $\{\lambda_j\}\subset G = B_R \cup P_{q,h}$, where $B_R = \{\lambda: |\lambda| \leqq R\}$, $P_{q,h} = \{\lambda: \operatorname{Re}\lambda \geqq 0,\ |\lambda|>1,\ |\operatorname{Im}\lambda| \leqq h(\operatorname{Re}\lambda)^q,\ h>0,\ -\infty < q < 1\}$, and for some $\gamma < \infty$, $L^{-1} \in \sigma_\gamma$. Also let the estimate**

$$\|(I\lambda - L)^{-1}\| \leqq Cd^{-1}(\lambda, G),\ \lambda \bar{\in} G$$

hold outside the domain $G' = B_R \cup P_{q,2h}$, and for some $a > 0$, $p>0$

$$\sum_{|\lambda_j|\leqq t} 1 = n(t) \leqq dt^p$$

provided t is sufficiently large.

Then $L \in A(\alpha, \mathfrak{H})$ for any $\alpha > \max 0,\ p-(1-q)$.

Besides, if the numbers a or h can be chosen arbitrarily small and $p-(1-q)>0$, then $\alpha = p-(1-q)$ is admissible.

Introduction. Let L be an unbounded linear operator in a separable complex Hilbert space $\mathfrak{H}$ with domain of definition $\mathscr{D}(L)$ which is dense in $\mathfrak{H}$, having a discrete spectrum $\sigma(L)$. Let $\{e_j\}_{j=1}^{\infty}$ be a sequence consisting of bases in the root subspaces of L, where e_j is a root vector corresponding to the eigenvalue λ_j. To each vector $x \in \mathfrak{H}$ we associate its Fourier series $\sum (x, e_j^*)e_j$ with respect to this system (not necessarily convergent), where $\{e_j^*\}$ is a system which is biorthogonal to $\{e_j\}$.

We write $L \in \mathscr{A}(\alpha, \mathfrak{M}, \mathfrak{H})$ if for an arbitrary vector x in $\mathfrak{M}$, where $\mathfrak{M}$ is some linear manifold in $\mathfrak{H}$, the Fourier series $\sum (x, e_j^*)e_j$ is summable in $\mathfrak{H}$ to x by the Abel method of order α with parenthesis.

If we suppose that L has no associated vectors and all its eigenvalues $\{\lambda_j\}$ lie in the sector $\Lambda_\theta = \{\lambda: |\arg\lambda| \leqq \pi/2\theta,\ 1/2 \leqq \theta < \infty\}$ then the Abel method of summability of order $\alpha(\alpha \leqq \theta)$ consists in replacing the series $\sum (x, e_j^*)e_j$ by series

$$(1)\qquad u_x(t) = \sum_{j=1}^{\infty} e^{-\lambda_j^{\alpha} t}(x, e_j^*)e_j\,;$$

it is required that for any $t>0$ after possible recombination of its terms and appropriate use of parenthesis (not depending on $x \in \mathfrak{M}$, or $t>0$) this series converges in $\mathfrak{H}$ and its sum $u_x(t)$ converges to x in $\mathfrak{H}$ as $t \to +0$. The branch of the function λ^α in (1) is selected so that $\lambda^\alpha > 0$ if $\lambda > 0$. In the general case, when there do exist associated vectors, the factors for the vectors e_j in the series (1) are defined by calculating the integral

$$\frac{1}{2\pi i}\int e^{-\lambda^\alpha t}(\lambda I - L)^{-1} x \, d\lambda$$

along a contour which surrounds a corresponding eigenvalue (see [9], where the Abel method was first introduced).

By σ_p we denote the collection of all compact operators A, for which $\sum s_j^p(A) < \infty$, where $s_j(A)$ are eigenvalues of operator $(AA^*)^{1/2}$, and by σ_∞ the collection of all compact operators.

The following result combines those of many authors.

THEOREM. *Let L be an unbounded operator in a Hilbert space $\mathfrak{H}$ having a discrete spectrum $\{\lambda_j\} \subset G = B_R \cup \Lambda_\theta$, where $B_R = \{\lambda: |\lambda| \leqq R\}$, $\Lambda_\theta = \{\lambda: |\arg \lambda| \leqq \pi/2\theta\}$, and its inverse operator $L^{-1} \in \sigma_\gamma$ for some $\gamma < \theta$. If the estimate*

$$\|(\lambda I - L)^{-1}\| \leqq C d^{-1}(\lambda, G), \quad \lambda \,\overline{\in}\, G$$

holds outside the domain G, where $d(\lambda, G)$ is the distance between λ and G, then

(1) *the system of root vectors of operator L is complete in the space $\mathfrak{H}$.*

(2) *$L \in \mathscr{A}(\alpha, \mathfrak{H})$, if $\alpha \in (\gamma, \theta)$.*

M. V. Keldysh [6], [7] proved the first assertion in the case $L = (I + V)H$, where $H = H^* > 0$, $V \in \sigma_\infty$. Subsequently, the Keldysh method was generalized by many authors, in particular, in a similar form the first assertion was proved by S. Agmon [1], by I. C. Gohberg and M. G. Krein [3]. The second assertion is much stronger. In [9] V. B. Lidskii proved, that $L \in \mathscr{A}(\alpha, \mathscr{D}(L), \mathfrak{H})$, if $\alpha \in (\gamma, \theta)$. Recently V. I. Macaev noticed that, indeed, the second assertion holds.[1]

In many cases the spectrum of operator L lies asymptotically in an arbitrarily small sector Λ_θ, i.e., the number θ may be chosen arbitrarily large. Such cases occur for some differential operators and are valid for operators which can be represented in the form $L = (I + V)H$, where $H > 0$ and $V \in \sigma_\infty$. In this situation the interval for α is equal to (γ, ∞). For applications (see [9]) the most important case is when the order of summability $\alpha = 1$. In

[1] This note is reported in the appendix to the book [5]. The appendix is written by M. S. Agranovich.

this connection it is highly important to clear up the general conditions under which the interval for α can be extended. Indeed, it can be extended if the spectrum of operator L lies asymptotically not only in an arbitrarily small sector but in some domain which is bounded by parabolas, lines, or hyperbolas.

The following theorem, which can be considered as the continuation of the previous theorem, formulates the exact result.

THEOREM 1. *Let L be an unbounded operator in a Hilbert space $\mathfrak{H}$, having a discrete spectrum $\{\lambda_j\} \subset G = B_R \cup P_{q,h}$, where $B_R = \{\lambda: |\lambda| \leqq R\}$, $P_{q,h} = \{\lambda: \operatorname{Re}\lambda \geqq 0,\ |\lambda| > 1,\ |\operatorname{Im}\lambda| \leqq h(\operatorname{Re}\lambda)^q,\ h > 0,\ -\infty < q < 1\}$, and for some $\gamma < \infty$, $L^{-1} \in \sigma_\gamma$. Also let the estimate*

$$\|(I\lambda - L)^{-1}\| \leqq Cd^{-1}(\lambda, G),\ \lambda \overline{\in} G$$

hold outside the domain $G' = B_R \cup P_{q,2h}$, and for some $a > 0$, $p > 0$

$$\sum_{|\lambda_j| \leqq t} 1 = n(t) \leqq at^p$$

provided t is sufficiently large.

Then $L \in \mathscr{A}(\alpha, \mathfrak{H})$ for any $\alpha > \max 0, p - (1 - q)$.

Besides, if the numbers a or h can be chosen arbitrarily small and $p - (1 - q) > 0$, then $\alpha = p - (1 - q)$ is admissible.

Some results about extension of the interval for α were obtained by V. B. Lidskii [10], by V. E. Katznelson and M. S. Agranovich (see [2]). All these results dealt with operators which can be represented in the form of a weakly perturbed self-adjoint positive operator, and the proofs of the appropriate statements used the specific properties of those operators.

Theorem 1 includes and generalizes these results. For its proof we use another more general method, where new estimates for meromorphic functions play the basic role. These estimates have independent significance; the following theorem formulates the relevant result.

THEOREM 2. *Let $F(\lambda)$ be a meromorphic function of finite order γ in the sector Λ_θ, and its poles $\{\lambda_j\}$ lie in the domain $P_{q,h} = \{\lambda: \operatorname{Re}\lambda > 0,\ |\lambda| > 1,\ |\operatorname{Im}\lambda| < h(\operatorname{Re}\lambda)^q,\ h > 0,\ -\infty < q < 1\}$. Also let the estimate $|F(\lambda)| \leqq C$ hold on the boundary of the domain $P_{q,2h}$ and for some $a > 0$, $p > 0$*

$$\sum_{|\lambda_j| \leqq t} 1 = n(t) \leqq at^p$$

provided t is sufficiently large.

Then there exists a sequence $r_1 < r_2 < \cdots < r_k \to \infty$, *such that the estimate*

$$|F(\lambda)| \leqq C \exp(\sigma a h |\lambda|^{p-(1-q)})$$

holds for all $|\lambda| = r_k$, $\lambda \in P_{q,2h}$, *where the constants* C, σ *do not depend on* λ, a, h, *if* $0 < h < h_0$ *and* h_0 *is any fixed number.*

In the case when the function $F(\lambda)$ is meromorphic in the whole complex plane and has the finite order γ, and when its poles $\{\lambda_j\}$ are scattered in the sector Λ_θ, $\theta > \gamma$, $n(t) \leqq at^p$ and $|F(\lambda)| \leqq C$ on $\partial\Lambda_\theta$ one can obtain, using the well-known theorem of Titchmarsh (see for example [1], p. 278), the following estimate

$$|F(\lambda)| \leqq C \exp |\lambda|^{p+\varepsilon}, \ \varepsilon > 0, \ |\lambda| = r_k, \ \lambda \in \Lambda_\theta ,$$

where $r_1 < r_2 < \cdots < r_k \to \infty$. This estimate was used by V. B. Lidskii [9] for his summability theorem.

We note that in the case when $\{\lambda_j\}$ are concentrated close to the real axis, namely, in some domain $P_{q,h}$, $q < 1$, Theorem 2 gives a much sharper estimate.

Proof of Theorems 1 *and* 2. In this section we will denote

1. $\Lambda_\theta = \{\lambda\colon \arg|\lambda| \leqq \pi/2\theta\}$,
2. $P_{q,h} = \{\lambda\colon \operatorname{Re}\lambda > 0, \ |\lambda| > 1, \ |\operatorname{Im}\lambda| < h(\operatorname{Re}\lambda)^q, \ h > 0, \ -\infty < q < 1\}$,
3. $P_{q,h}^{+} = \{\lambda\colon \operatorname{Re}\lambda > 0, |\lambda| > 1, 0 < |\operatorname{Im}\lambda| < h(\operatorname{Re}\lambda)^q, h > 0, -\infty < q < 1\}$,
4. $B(z, r) = \{\lambda\colon |\lambda - z| \leqq r\}$, $B(0, r) = B_r$.

If the sequence $\{\lambda_n\}$ lies in the upper half-plane ($\operatorname{Im}\lambda_n > 0$) and

$$\sum_{n=1}^{\infty} \frac{\operatorname{Im}\lambda_n}{1 + |\lambda_n|^2} < \infty ,$$

then the product

$$B(\lambda) = \prod_{n=1}^{\infty} \frac{\lambda - \lambda_n}{\lambda - \bar{\lambda}_n} \frac{|1 + \lambda_n^2|}{1 + \lambda_n^2}$$

converges and is called Blaschke product for the sequence $\{\lambda_n\}$.

We will start with the proof of the Theorem 2. The estimates of Blaschke product will play the main role in proving this theorem. First we will establish several lemmas.

LEMMA 1. *Given any number* $\varepsilon > 0$ *and complex numbers* a_1, $a_2, \cdots, a_N$, *there is a system of circles in the complex plane, with the sum of the radii not greater than* $2\varepsilon N$, *such that for each* *point z lying outside these circles one has the inequalities*

$$|z - a_k| \geqq k\varepsilon, \ n = 1, \cdots, N,$$

if the numbers a_k have been enumerated in increasing order of $|z - a_k|$.

Proof. This lemma is essentially equivalent to H. Cartan's well-known theorem about estimating from below the modulus of a polynomial, and its proof can be obtained by following the proof of Cartan's theorem (see [8], Chap. 1, § 7).

Let E be a set in the complex plane and suppose, that for any r sufficiently large the set $E \cap B_r$ may be covered by a system of circles, such that the total sum of their radii is not greater than δr, where the number δ does not depend on r and $0 < \delta < 1$. The number δ_0 which is the minimum of such δ, we will define as the linear density of the set E.

LEMMA 2. *Let $\{\lambda_k\}$ be a sequence in the complex plane, such that for all t sufficiently large*

$$\sum_{|\lambda_k| \leqq t} 1 = n(t) \leqq at .$$

Given any number $0 < \delta < 1$, there exists the set E of linear density $\leqq \delta$, such that for all $z \in D$ one has the inequalities

$$|z - \lambda_k| \geqq \frac{\delta k}{4a}, \ k = 1, 2, \cdots, \tag{3}$$

if the numbers λ_k have been enumerated in order of increasing $|z - \lambda_k|$.

Proof. Let $\lambda_1, \cdots, \lambda_{k_0}$ be the points of the sequence $\{\lambda_k\}$ lying in the circle B_{2r}. From the inequality $n(t) \leqq at$, $(t > t_0)$ it follows that $k_0 \leqq 2ar$. Fix any δ, $0 < \delta < 1$. According to Lemma 1 (in this lemma we put $\varepsilon = \delta/4a$) there exists a set E_r which consists of the circles with total sum of the radii $\leqq 2\varepsilon k_0 \leqq \delta r$, such that for all $z \bar{\in} E_r$ one has the inequalities

$$|z - \lambda_k| \geqq \frac{\delta k}{4a}, \ k = 1, 2, \cdots, k_0, \tag{4}$$

if the numbers $\lambda_1, \cdots, \lambda_{k_0}$ have been enumerated in order of increasing $|z - \lambda_k|$.

By the inequality $n(t) \leqq at$, we conclude $|\lambda_k| \geqq k/a$ for all $\lambda_k \bar{\in}$

B_{2r}. Consequently, if $|z| \leqq r$, then

$$|z - \lambda_k| \geqq \frac{|\lambda_k|}{2} \geqq \frac{k}{2a}, \ k = k_0 + 1, \ k_0 + 2, \cdots . \tag{5}$$

Let $G_r = B_r \backslash E_r$, $G = \bigcup_{r > t_0} G_r$, $E = C \backslash G$, where by C we denote the complex plane. According to (4), (5), the inequalities (3) hold, if $z \in G_r$ for all $r > t_0$, consequently, they hold for $z \in G$, i.e., for $z \overline{\in} E$.

Evidently, $B_r \cap E \subset E_r$; therefore the set $B_r \cap E$ may be covered by a system of circles with sum of the radii $\leqq \delta r$. Hence the linear density of the set E is not greater than δ. Thus Lemma 2 is proved.

LEMMA 3. *Let the sequence $\{\lambda_k\}$ lie in the domain $P^+_{q,h}$ and*

$$\sum_{|\lambda_k| \leqq t} 1 = n(t) \leqq at$$

provided t is sufficiently large.

Given any number $0 < \delta < 1$ there exists a set E, such that its linear density $\leqq \delta$ and for all $\lambda \in P^+_{q,h} \backslash E$ one has the inequality

$$|B(\lambda)| \geqq \exp(-\sigma a h \delta^{-1} |\lambda|^q), \ \lambda \in P^+_{q,h} \backslash E ,$$

where the function $B(\lambda)$ is defined by (2), and the constant σ does not depend on λ, a, h, δ, if $0 < h < h_0$ and h_0 is any fixed number.

Proof. Denote $\lambda = \mu + i\nu$, $\lambda_n = \mu_n + i\nu_n$. Then

$$\begin{aligned} |B(\lambda)|^{-2} &= \prod_{k=1}^{\infty} \left| \frac{\lambda - \bar{\lambda}_k}{\lambda - \lambda_k} \right| = \prod_{k=1}^{\infty} \frac{(\mu - \mu_k)^2 + (\nu + \nu_k)^2}{|\lambda - \lambda_k|^2} \\ &= \prod_{k=1}^{\infty} \left(1 + \frac{4\nu\nu_k}{|\lambda - \lambda_k|^2} \right) . \end{aligned}$$

Taking into account that $\nu \leqq h\mu^q$, $\nu_k \leqq h\mu_k^q$, $|\lambda - \lambda_k| \geqq |\mu - \mu_k|$, $q < 1$, we obtain

$$\begin{aligned} |B(\lambda)|^{-2} &\leqq \prod_{k=1}^{\infty} \left(1 + \frac{4h^2\mu^q\mu_k^q}{|\lambda - \lambda_k|^2} \right) \leqq \prod_{k=1}^{\infty} \left(1 + \frac{4h^2\mu^q[(\mu_k - \mu) + \mu]^q}{|\lambda - \lambda_k|^2} \right) \\ &\leqq \prod_{k=1}^{\infty} \left(1 + \frac{4h^2\mu^{2q}}{|\lambda - \lambda_k|^2} \right) \prod_{k=1}^{\infty} \left(1 + \frac{4h^2\mu^q}{|\lambda - \lambda_k|^{2-q}} \right) . \end{aligned}$$

According to Lemma 2, there exists a set E such that its linear density $\leqq \delta$ and for all $\lambda \overline{\in} E$ one has the inequalities (3) after enumerating the sequence $\{\lambda_k\}$ properly. Hence, if $\lambda \in P^+_{q,h} \backslash E$, then

$$
\begin{aligned}
&-2ln|B(\lambda)| \\
&\quad \leqq \sum_{k=1}^{\infty} ln\Big(1 + \frac{64a^2h^2\mu^{2q}}{\delta^2k^2}\Big) + ln\Big(1 + \frac{4(4a)^{2-q}h^2\mu^q}{\delta^{2-q}k^{2-q}}\Big) \\
&\quad < \int_{1/2}^{\infty}\Big[ln\Big(1 + \frac{64a^2h^2\mu^{2q}}{\delta^2x^2}\Big) + ln\Big(1 + \frac{4(4a)^{2-q}h^2\mu^q}{\delta^{2-q}x^{2-q}}\Big)\Big]dx \\
(6) &\quad \leqq xln\Big(1 + \frac{64a^2h^2\mu^{2q}}{\delta^2x^2}\Big)\Big|_{1/2}^{\infty} + xln\Big(1 + \frac{4(4a)^{2-q}h^2\mu^q}{\delta^{2-q}x^{2-q}}\Big)\Big|_{1/2}^{\infty} \\
&\qquad + \int_{1/2}^{\infty}\Big[\frac{128a^2h^2\mu^{2q}}{\delta^2x^2 + 64a^2h^2\mu^{2q}} + \frac{4(2-q)(4a)^{2-q}h^2\mu^q}{\delta^{2-q}x^{2-q} + 4(4a)^{2-q}h^2\mu^q}\Big]dx \\
&\quad \leqq ln(1 + C_1a^2h^2\delta^{-2}\mu^{2q}) + ln(1 + C_2a^{2-q}\delta^{q-2}h^2\mu^q) + C_3ah\delta^{-1}\mu^q \\
&\qquad + C_4ah^{2/2-q}\delta^{-1}\mu^{q/2-q} \leqq Cah\delta^{-1}\mu^q \leqq Cah\delta^{-1}|\lambda|^q ,
\end{aligned}
$$

where the constant C does not depend on λ, a, h, δ, if $0 < h < h_0$ and h_0 is any fixed number. We note that in (6) the following estimates were used:

$$
\int_{1/2}^{\infty}\frac{\omega^2dx}{\delta^2x^2 + \omega^2} = \omega\delta^{-1}\,\mathrm{arc}\,tg\Big|_{1/2}^{\infty} < \frac{\pi}{2}\omega\delta^{-1} ;
$$

$$
\begin{aligned}
\int_{1/2}^{\infty}\frac{\omega dx}{\delta^{2-q}x^{2-q} + \omega} &\leqq \frac{\omega}{\delta^{2-q}}\Big[\int_{1/2}^{\omega^{1/2-q}\delta^{-1}}\frac{dx}{x^{2-q} + \omega\delta^{q-2}} + \int_{\omega^{1/2-q}\delta^{-1}}^{\infty}\frac{dx}{x^{2-q} + \omega\delta^{q-2}}\Big] \\
&< \frac{\omega}{\delta^{2-q}}\Big[\int_{1/2}^{\omega^{1/2-q}\delta^{-1}}\frac{dx}{\omega\delta^{q-2}} + \int_{\omega^{1/2-q}\delta^{-1}}^{\infty}\frac{dx}{x^{2-q}}\Big] \\
&< \omega^{1/2-q}\delta^{-1} + \frac{1}{1-q}\omega^{1/2-q}\delta^{-1} = \frac{2-q}{1-q}\omega^{1/2-q}\delta^{-1} .
\end{aligned}
$$

The estimates (6) prove Lemma 3.

LEMMA 4. *Let the sequence* $\{\lambda_k\}$ *lie in the domain* $P_{q,h}$ *and*

$$
(7) \qquad \sum_{|\lambda_k|\leqq t} 1 = n(t) \leqq at^p .
$$

Then there exists a holomorphic function $\Delta(\lambda)$ *in the domain* $P_{q,2h}$, *such that*

(a) $\Delta(\lambda_k) = 0$ *for all points of the sequence* $\{\lambda_k\}$, *and* λ_k *is an s-multiple root of* $\Delta(\lambda)$, *if it is repeated in the sequence s times.*

(b) $|\Delta(\lambda)| \leqq 1$, *if* $\lambda \in P_{q,2h}$.

(c) *given* $\delta > 0$ *there exists a set* E, *such that its linear density* $\leqq \delta$ *and for* $\lambda \in P_{q,2h}\backslash E$ *one has the estimate*

$$
(8) \qquad |\Delta(\lambda)| \geqq \exp(-\sigma ah\delta^{-\beta}|\lambda|^{p-(1-q)}) ,
$$

where the constants $\sigma, \beta > 0$ *do not depend on* λ, a, h, δ, *if* $0 < h < h_0$ *and* h_0 *is any fixed number.*

Proof. Let us consider the function $\rho(\lambda) = [\lambda^{1-q} + 3h(1-q)i + \tau]^{p/1-q}$. It is easy to verify, that provided τ is sufficiently large, the function $\rho(\lambda)$ maps the domain $P_{q,2h}$ inside the domain $P^{+}_{q',h'}$, i.e., $\rho(P_{q,h}) \subset P^{+}_{q',h'}$, where $q' = 1 - (1-q)/p$, $h' = 6ph$. Hence, the sequence $\{\rho_n\} = \{\rho(\lambda_n)\}$ lies in the domain $P^{+}_{q',h'}$ and furthermore, according to (7), we have

$$\sum_{|\rho_n| \leqq t} 1 = \sum_{|\lambda_n^{1-q} + 3h(q-1)i+\tau|^{p/1-q} \leqq t} 1$$
$$\leqq \sum_{|\lambda_n|^p \leqq 2t} 1 = \sum_{|\lambda_n| \leqq 2^{1/p} t^{1/p}} 1 \leqq 2at\ .$$

According to Lemma 3, given any number $\delta_1 > 0$, there exists a set E, such that its linear density $\leqq \delta_1$ and for $\lambda \in P^{+}_{q,h} \backslash E$ one has the estimate

$$|B(\rho)| \geqq \exp[-\sigma a h \delta_1^{-1} |\rho|^{q'}]\ , \tag{9}$$

where $B(\rho) = \Pi(\rho - \rho_k)(\rho - \bar{\rho}_k)^{-1}$. Evidently, $|B(\rho)| < 1$ if $\operatorname{Im} \rho > 0$. Taking into account that $\rho^{-1}(p^{+}_{q',h'}) \supset P_{q,2h}$, we find that the function $\Delta(\lambda) = B(\rho(\lambda))$ is holomorphic in the domain $P_{q,2h}$ and satisfies conditions (a) and (b). By virtue of (9), we have

$$|\Delta(\lambda)| \geqq \exp(-\sigma a h \delta_1^{-1} |\lambda^{1-q} + 3h(1-q)i + \tau|^{pq'/1-q})$$
$$\geqq \exp(-\sigma_1 a h \delta_1^{-1} |\lambda|^{p-(1-q)})\ ,$$

if $\lambda \in \rho^{-1}(P^{+}_{q',h'}) \backslash \rho^{-1}(E) \supset P_{q,2h} \backslash \rho^{-1}(E)$. Thus, the proof of Lemma 4 will be complete if we show that the set $\rho^{-1}(E)$ has the linear density $\leqq C\delta_1^{1/\beta}$, where the constants $C > 0$, $\beta > 0$ do not depend on a, h, δ_1. Then, supposing $\delta = C\delta_1^{1/\beta}$, we will obtain the estimate (8).

It is sufficient to show that if the set $\mathfrak{M}$ has linear density $\leqq \varepsilon < 1$, then its image $\xi(\mathfrak{M})$ under the map $\xi(\lambda) = \lambda^{\kappa}$ has the linear density $\leqq C\varepsilon^{\kappa'}$, where $\kappa' = \min(\kappa, 1)$ and the constant C does not depend on ε.

For any r sufficiently large, there exists circles $B(z_i, \varepsilon_i)$, such that they cover the set $\mathfrak{M} \cap B_r$ and $\sum \varepsilon_i \leqq \varepsilon r$. If $\varepsilon_i \leqq |z_i|$, then

$$\begin{aligned} |(z_i + \varepsilon_i e^{i\phi})^{\kappa} - z_i^{\kappa}| &\leqq |z_i|^{\kappa} |(1 + \varepsilon_i e^{i\varphi} z_i^{-1})^{\kappa} - 1| \\ &\leqq C_1 \varepsilon_i |z_i|^{\kappa-1} \end{aligned} \tag{10}$$

by virtue of the simple inequality $|(1+z)^{\kappa} - 1| \leqq C_1(\kappa)|z|$, which holds for $|z| \leqq 1$. If $|z_i| < \varepsilon_i$, then $B(z_i, \varepsilon_i) \subset B(0, 2\varepsilon_i) \subset B(0, 2r\varepsilon)$. Taking into account (10), we find that the set $\xi(\mathfrak{M}) \cap B(0, r^{\kappa})$ may be covered by circles with the sum of the radii not greater than $(2r\varepsilon)^{\kappa} + C_1 \sum \varepsilon_i |z_i|^{\kappa-1} \leqq (2r\varepsilon)^{\kappa} + C_1 r^{\kappa-1} \sum \varepsilon_i \leqq C_2 r^{\kappa}(\varepsilon^{\kappa} + \varepsilon)$.

This means that the set $\xi(\mathfrak{M})$ has linear density $\leqq 2C_2\varepsilon^{\kappa'}$, where $\kappa' = \min(\kappa, 1)$. Hence Lemma 4 is established.

LEMMA 5. *Let the sequence* $\{\lambda_k\}$ *lie in the domain* $P_{q,h}$ *and*

$$\sum_{|\lambda_k|\leq t} 1 = n(t) \leqq at^p, \tag{11}$$

provided t is sufficiently large.

Then outside the domain $P_{q,2h}$ *one has the estimate*

$$|V(\lambda)| \geqq C\exp(-|\lambda|^{p+\varepsilon}),\ \varepsilon > 0,\ \lambda \in C\backslash P_{q,2h}, \tag{12}$$

where the function $V(\lambda)$ *is the canonical product for the sequence* $\{\lambda_k\}$

$$V(\lambda) = \prod_{k=1}^{\infty}\left(1 - \frac{\lambda}{\lambda_k}\right)\exp\left(\frac{\lambda}{\lambda_k} + \cdots + \frac{\lambda^\nu}{\nu\lambda_k^\nu}\right),\ \nu = [p]. \tag{13}$$

Proof. If $\pm\lambda \,\overline{\in}\, P_{q,2h}$, $\lambda_n \in P_{q,h}$, then there exist the constants C_1, C_2 depending only on h, q, such that

$$|\lambda - \lambda_k| \geqq C_1|\operatorname{Im}\lambda| \geqq C_1C_2|\lambda|^q. \tag{14}$$

If $-\lambda \in P_{q,2h}$, then the estimate (14) holds for $q = 1$, and consequently (14) holds for all $\lambda \,\overline{\in}\, P_{q,2h}$.

It follows from condition (11), that $|\lambda_{k+1}| \geqq a^{-1/p}k^{1/p}$. Taking into account that

$$\sum_{k=1}^{N} k^\kappa \leqq \begin{cases} 2N^{\kappa+1} & \text{if } \kappa \neq -1 \\ 2lnN, & \text{if } \kappa = -1, \end{cases}$$

we obtain the estimate

$$\begin{aligned} \sum_{|\lambda_k|\leq 2|\lambda|} ln\left|\exp\left(\frac{\lambda}{\lambda_k} + \cdots + \frac{\lambda^\nu}{\nu\lambda_k^\nu}\right)\right| &\geqq -\sum_{k=1}^{n(2|\lambda|)}\left|\frac{\lambda}{\lambda_k}\right| + \cdots \\ &+ \frac{1}{\nu}\left|\frac{\lambda}{\lambda_k}\right|^\nu \geqq -C_3\sum_{s=1}^{\nu}\sum_{k=1}^{n(2|\lambda|)}|\lambda|^s k^{-s/p} \\ &\geqq -2C_3\sum_{j=1}^{\nu}|\lambda|^s n(2|\lambda|)^{1-s/p} ln n(2|\lambda|) \\ &\geqq -C_4\sum_{s=1}^{\nu}|\lambda|^s|\lambda|^{p-s}ln|\lambda| = -\nu C_4|\lambda|^p ln|\lambda|. \end{aligned} \tag{15}$$

Further, using (14), we have

$$\begin{aligned} \sum_{|\lambda_k|<2|\lambda|} ln\left|1 - \frac{\lambda}{\lambda_k}\right| &\geqq \sum_{|\lambda_k|<2|\lambda|} ln\frac{C_1C_2|\lambda|^q}{|\lambda_k|} \\ &\geqq n(2|\lambda|)ln\frac{1}{2}C_1C_2|\lambda|^{q-1} \geqq -C_5|\lambda|^p ln|\lambda|. \end{aligned}$$

Since

$$\sum_{|\lambda_k|>2|\lambda|} ln\Big(1 - \frac{\lambda}{\lambda_k}\Big) + \frac{\lambda}{\lambda_k} + \cdots + \frac{\lambda^\nu}{\nu\lambda_k^\nu} = - \sum_{|\lambda_k|>2|\lambda|} \sum_{s=\nu+1}^{\infty} \frac{\lambda^s}{s\lambda_k^s},$$

we finally get the estimate

$$\begin{aligned}\prod_{|\lambda_k|>2|\lambda|} &\left|\Big(1 - \frac{\lambda}{\lambda_k}\Big)\exp\Big(\frac{\lambda}{\lambda_k} + \cdots + \frac{\lambda^\nu}{\nu\lambda_k^\nu}\Big)\right| \\ &\geqq \prod_{|\lambda_k|>2|\lambda|} \exp\Big(- \sum_{s=\nu+1}^{\infty} \frac{1}{\nu}\Big|\frac{\lambda}{\lambda_k}\Big|\Big)^s \\ &\geqq \prod_{|\lambda_k|>2|\lambda|} \exp\Big(-\Big|\frac{\lambda}{\lambda_k}\Big|^{\nu+1}\Big(1 - \Big|\frac{\lambda}{\lambda_k}\Big|\Big)^{-1}\Big) \\ &\geqq \prod_{|\lambda_k|>2|\lambda|} \exp\Big(-2\Big|\frac{\lambda}{\lambda_k}\Big|^{\nu+1}\Big) \qquad (17) \\ &= \exp\Big(-2 \sum_{|\lambda_k|>2|\lambda|} \Big|\frac{\lambda}{\lambda_k}\Big|^{\nu+1}\Big) \\ &\geqq \exp\Big(- \sum_{|\lambda_k|>2|\lambda|} \Big|\frac{\lambda}{\lambda_k}\Big|^{\rho+\varepsilon/2}\Big) \\ &\geqq C_6 \exp(-|\lambda|^{\rho+\varepsilon}),\end{aligned}$$

if $\varepsilon > 0$ is chosen, such that $\rho + \varepsilon/2 \leqq \nu + 1$, and $|\lambda|$ is sufficiently large. The estimates (15)-(17) prove Lemma 5.

REMARK. According to Titchmarsh's theorem, there exists a sequence $r_1 < r_2 < \cdots < r_k \to \infty$, such that the estimate (12) holds not only for $\lambda \in C\backslash P_{q,2h}$, but also for $|\lambda| = r_k$.

LEMMA 6.[2] *Let the function $f(\lambda)$ be holomorphic in the sector $\Lambda_\theta = \{\lambda: |\arg\lambda| < \pi/2\theta,\ \theta \geqq 1/2\}$, let f have no zeros inside this sector and let the order of its growth be $\rho < \infty$. Then for any given $\delta > 0$ inside the sector $\Lambda_{\theta+\delta}$ one has the estimate*

$$|f(\lambda)| \geqq \exp(-\sigma|\lambda|^{\max(\rho+\delta,\theta)}),\ \lambda \in \Lambda_{\theta+\delta},\ |\lambda| > 1, \qquad (18)$$

where the constant σ does not depend on λ.

Proof. If the function $\psi(z)$ is holomorphic in the circle $|z| \leqq 1$ and has no zeros in this circle, then its modulus satisfies the inequality ([8], Chap. 1, § 6):

$$ln|\psi(z)| \geqq -\frac{c|z|}{1 - |z|},\ |z| < 1,$$

where $c = 2\psi^{-1}(0) ln \max_{|z|\leqq 1}|\psi(z)|$. The function $z(\mu) = (\mu-1)(\mu+1)^{-1}$

[2] A similar lemma (unpublished) was obtained by another method by G. V. Radzievskii.

maps the right half-plane into the unit circle, and as $|\mu| \to \infty$ we have asymptotically ($\mu = |\mu| e^{i\phi}$)

$$\frac{\left|\frac{\mu-1}{\mu+1}\right|}{1-\left|\frac{\mu-1}{\mu+1}\right|} = \frac{\sqrt{|\mu|^2+1-2|\mu|\cos\phi}}{\sqrt{|\mu|^2+1+2|\mu|\cos\phi}-\sqrt{|\mu|^2+1-2|\mu|\cos\phi}}$$

$$= \frac{|\mu| - \cos\phi + 0(|\mu|^{-1})}{2\cos\phi + 0(|\mu|^{-1})} .$$

Hence, the function $g(\mu)$, which is holomorphic, bounded and has no zeros in the right half-plane, satisfies the inequality

$$ln|g(\mu)| \geqq \frac{-C|\mu|}{2\cos(\arg\mu)}, \quad \text{if } \mu \in \varLambda_{1+\delta},\ |\mu| > 1 . \tag{19}$$

Suppose that $\theta > \rho$. Then $\theta - \tau > \rho$ for some $\tau > 0$. In this case the function $f(\lambda)\exp(-\lambda^{\theta-\tau})$ is holomorphic and bounded in the sector $\varLambda_\theta$. If $\lambda = \mu^{1/\theta}$, then the function $g(\mu) = f(\mu^{1/\theta})\exp(-\mu^{\theta-\tau/\theta})$ satisfies (19) and one has the estimate

$$ln|f(\lambda)| \geqq -\sigma|\lambda|^\theta, \ \lambda \in \varLambda_{\theta+\delta},\ |\lambda| > 1 . \tag{20}$$

In case $\theta \leqq \rho$ we consider the function $f_\phi(\lambda) = f(\lambda)\exp(-e^{i\phi\pi/2}\lambda^{\rho+\delta})$, which is holomorphic and bounded in the sector $\varLambda^\phi_{\rho+2\delta} = \{\lambda: -\pi/2((1/\rho + 2\delta) + \phi) < \arg\lambda < \pi/2(1/(\rho+2\delta) - \phi)\} \subset \varLambda_\theta$, if $1/(\rho+2\delta) - 1/\theta < \phi < 1/\theta - 1/(\rho+2\delta)$. All sectors $\varLambda^\phi_{\rho+2}$ cover the sector $\varLambda_\theta$, when ϕ changes in the indicated limits. Therefore, it is sufficient to show that the function $f(\lambda)$ satisfies (18) in every sector $\varLambda^\phi_{\rho+3\delta}$.

The function $\mu(\lambda) = (e^{+i\pi/2}\lambda)^{\rho+2\delta}$ maps the sector $\varDelta^\phi_{\rho+2}$ into the right half-plane. Then the function $g(\mu) = f_\phi(e^{-i\phi\pi/2}\mu^{1/\rho+2\delta})$ is holomorphic and bounded in the right half-plane, and hence its modulus satisfies (19). From this inequality we obtain

$$lnf(\lambda) \geqq -\sigma|\lambda|^{\rho+2\delta}, \ \lambda \in \varLambda^\phi_{\rho+3\delta},\ |\lambda| > 1 . \tag{21}$$

The inequalities (20), (21) prove Lemma 6.

Now let us go on to the proof of Theorem 2.

Proof of Theorem 2. Let $F(\lambda) = F_1(\lambda)(F_2(\lambda))^{-1}$, where $F_1(\lambda)$, $F_2(\lambda)$ are holomorphic functions of finite order $\leqq \gamma$ in the sector $\varLambda_\theta$, and $\{\lambda_n\}$ are the zeros of the function $F_2(\lambda)$. According to Lemma 4, there exists a function $\varDelta(\lambda)$, which satisfies the condition (a)–(c) of this lemma. By $V(\lambda)$ we denote the canonical product (13) for the sequence $\{\lambda_n\}$. Let us consider the function

$$G(\lambda) = F(\lambda)\Delta(\lambda) = \frac{F_1(\lambda)}{\phi(\lambda)}\psi(\lambda),$$

where

$$\phi(\lambda) = \frac{F_2(\lambda)}{V(\lambda)}, \ \psi(\lambda) = \frac{\Delta(\lambda)}{V(\lambda)} .$$

The function $G(\lambda)$ is holomorphic in the domain $P_{q,2h}$; we want to show that it has growth of finite order in that domain. Let $\mathscr{D}_k = P_{q,2_h} \cap B_{r_k}$ where r_k are the numbers that were mentioned in the remark after Lemma 5. According to Lemma 5 and Titchmarsh's theorem, we have

$$|\phi(\lambda) \exp(-\lambda^{\gamma'})|_{\partial\mathscr{D}_k} \leqq C, \ |\psi(\lambda) \exp(-\lambda^{\gamma'})|_{\partial\mathscr{D}_k} \leqq C ,$$

if $\gamma' > \max(\gamma, \theta)$, and the constant C does not depend on k. By virtue of the maximum principle, we have the functions $\phi(\lambda)\exp(-\lambda^{\gamma'})$ and $\psi(\lambda)\exp(-\lambda^{\gamma'})$ are bounded in the domain $P_{q,2h}$, i.e., the functions $\phi(\lambda)$, $\psi(\lambda)$ have finite order $\leqq \gamma'$ in the domain $P_{q,2h}$. By inequality (12) we find that the function $\phi(\lambda)$ has order $\leqq \gamma'$ in the domain Λ_θ. As soon as $\phi(\lambda)$ has no zeros in the sector Λ_θ, we conclude from Lemma 6 that the function $\phi^{-1}(\lambda)$ has order $\leqq\gamma'$ in sector $\Lambda_{\theta+\delta}$, $\delta > 0$. Hence, the function $G(\lambda)$ has order $\leqq \gamma'$ in domain $P_{q,2h}$.

For $\lambda \in \partial P_{q,2h}$ we have the inequality $|G(\lambda)| \leqq C$, insofar as

$$|F(\lambda)|_{\partial P_{q,2h}} \leqq C, \ |\Delta(\lambda)|_{\partial P_{q,2h}} \leqq 1 .$$

Using the Phragmen-Lindelöf principle (see, for example [8], chap. 1, § 14), we have $|G(\lambda)| \leqq C$ for all $\lambda \in P_{q,2h}$. Hence, according to Lemma 6,

$$|F(\lambda)| \leqq C|\Delta(\lambda)|^{-1} \leqq C\exp(-\sigma a h \delta^{-\beta}|\lambda|^{p-(1-q)}) , \tag{22}$$

if $\lambda \in P_{q,2h}\backslash E$, where the set E has linear density $\leqq \delta$; the constants C, σ, β do not depend on λ, a, h, δ, and one can choose δ arbitrarily small.

Obviously, if the set E has linear density $< 1/2$, then there exist numbers $0 < r_1 < r_2 < 1 \cdots < r_k \to \infty$, such that the circles $|\lambda| = r_k$ do not interesect the set E. Then, the assertion of theorem 2 follows from the estimate (22).

Essentially, Theorem 1 may be considered as a corollary of the Theorem 2.

Proof of Theorem 1. Fix the number α, such that $p-(1-q)<\alpha$. All eigenvalues of the operator L, except for a finite number, lie in the sector $\Lambda_{\alpha+\varepsilon}$, $\varepsilon > 0$. Without loss of generality, we suppose that there are no eigenvalues of the operator L outside the sector $\Lambda_{\alpha+\varepsilon}$.

Let $x \in \mathfrak{H}$. Consider the integral

$$u_x(t) = \frac{1}{2\pi i}\int_\Gamma e^{-\lambda^\alpha t}(\lambda I - L)^{-1}x d\lambda,\ t > 0 ,$$

where the contour Γ is the boundary of the sector $\Lambda_{\alpha+\varepsilon}$. Since $\|(\lambda I - L)^{-1}\| \leqq C|\lambda|^{-1}$ for $\lambda \in \Gamma$, the function $u_x(t)$ is correctly defined for all $t > 0$. For the proof of the theorem we have to show that

(a) the function $u_x(t)$ can be represented in the form (1) and the series (1) converges in $\mathfrak{H}$ after some rearrangement of parentheses not depending on t and x.

(b) $\lim_{t\to+0} u_x(t) = x$.

It follows from $L \in \sigma_\gamma$, that $(\lambda I - L)^{-1}$ is a meromorphic function of order $\leqq \gamma$ (see, for example, [9]). According to Theorem 2, there exists a sequence $r_1 < r_2 < \cdots < r_k \to \infty$, such that for $|\lambda| = r_k$ one has the estimate

$$\|(\lambda I - L)^{-1}\| \geqq \exp(-\sigma a h|\lambda|^{p-(1-q)}) , \tag{23}$$

where the constant σ does not depend on a, h, $0 < h < h_0$.

Let $K_n = \{\lambda: r_n \leqq |\lambda| \leqq r_{n+1}\}$, $\mathscr{D}_n = \Lambda_{\alpha+\varepsilon} \cap K_n$. It follows from estimate (23), that for any $t > 0$ $(\alpha > p - (1-q))$

$$u_x(t) = \sum_{n=1}^{\infty}\int_{\partial\mathscr{D}_n} e^{-\lambda^\alpha t}(I\lambda - L)^{-1}x d\lambda , \tag{24}$$

and the series converges in $\mathfrak{H}$.

Calculating the integrals in (24), we obtain the assertion (a). We note also that in the case when either a, or h can be chosen arbitrarily small and $p - (1-q) > 0$, the assertion (a) is valid for $\alpha = p - (1-q)$.

As was mentioned before, under assumption $\|(I\lambda - L)^{-1}\| \leqq Cd^{-1}(\lambda, \Lambda_{\alpha+\varepsilon})$, the assertion (b) was proved by V. B. Lidskii [9] for any $x \in \mathscr{D}(L)$, and by V. I. Macaev (see [5]) for any $x \in \mathfrak{H}$.

We note only that (b) is valid for x, which can be represented as a finite linear combination of eigenvectors of the operator L, and all such x are closed in $\mathfrak{H}$. Consequently, the assertion (b) is valid for all x, if $\|u_x(t)\| \leqq C\|x\|$, where the constant C does not depend on t, $0 < t \leqq 1$. But this fact may be proved also by using the ideas from a theorem of E. Hille about the generation of holomorphic semi-groups (see [4], § 12.8).

As was shown in [9], the summability theorems have an important role in solving some nonstationary differential equations. The applications of Theorem 1 to such problems will be considered by the author in a subsequent paper.

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