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AN EXTENSION OF SION'S MINIMAX THEOREM WITH AN APPLICATION TO A METHOD FOR CONSTRAINED GAMES

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Sion's minimax theorem is extended for noncompact sets, and for certain two-person zero-sum games on constrained sets a sequential unconstrained solution method is given.

I. Introduction. It is an important question in two-person zero-sum games, whether there exists a saddle point strategy, and if so, how it is to be computed. Existence theorems are known almost only for the case that the sets of strategies are compact. Often these sets are given by numerically complicated conditions and because of the necessity to consider the boundary of the constraint region you cannot apply analytical methods.

First we extend Sion's minimax theorem [7] for noncompact sets. With it we then give a solution method for a frequently occuring type of games over constrained sets. This method approximates a solution from the interior of the admissible sets and makes it possible to apply analytical methods like those for the whole spaces. It can be regarded as an extension of the widely used Interior Penalty Method of Nonlinear Programming to saddle point problems.

II. A minimax theorem for noncompact sets. Let X and Y be not empty subsets of real linear topological Hausdorff spaces $\mathscr{X}$ and $\mathscr{Y}$, respectively, and let $\mathbf{R}$ denote the real numbers.

DEFINITION 1.
(a) A function $f: X \to \boldsymbol{R}$ is called
(i) *inf-compact* if $\{x \mid x \in X,\ f(x) \leqq a\}$, $a \in \boldsymbol{R}$, is compact,
(ii) *sup-compact* if $\{x \mid x \in X,\ f(x) \geqq a\}$, $a \in \boldsymbol{R}$, is compact.
(b) A function $f: X \times Y \to \boldsymbol{R}$ is called (x_1, y_1)-*sup inf-compact*, for a fixed $(x_1, y_1) \in X \times Y$, if $f(x_1, \cdot)$ is inf-compact and $f(\cdot, y_1)$ is sup-compact.

If $f: X \times Y \to \boldsymbol{R}$ is u.s.c.-l.s.c., i.e., $f(x, y)$ is upper semi-continuous in x for each $y \in Y$ and lower semi-continuous in y for each $x \in X$, and X and Y are compact sets, then $f(x, y)$ is (x_1, y_1)-sup inf-compact for all $(x_1, y_1) \in X \times Y$. Thus the following theorem generalizes Theorem 3.4 of Sion [7].

THEOREM 1. *Let X and Y be convex sets, and $f: X \times Y \to \boldsymbol{R}$ an* u.s.c.-l.s.c. *and quasi-concave-convex function, that is* (x_1, y_1)-*sup inf-compact for a fixed* $(x_1, y_1) \in X \times Y$. *Then we have*

$$\max_{x \in X} \min_{y \in Y} f(x, y) = \min_{y \in Y} \max_{x \in X} f(x, y) .$$

Proof. Let $(x_1, y_1) \in X \times Y$ with the property given above be fixed.

(i) $L_1 := \{x \mid x \in X,\ f(x, y_1) \geqq f(x_1, y_1)\}$ is compact, $f(x, y_1)$ is u.s.c. in x and $\sup_{x \in X} f(x, y_1) = \sup_{x \in L_1} f(x, y_1)$. Since an u.s.c. function on a compact set takes its maximum, there exists $\varphi(y_1) := \max_{x \in X} f(x, y_1)$. Now, for $a \in \boldsymbol{R}$, we have

$$\begin{aligned}\{y \mid y \in Y,\ \sup\nolimits_{x \in X} f(x, y) \leqq a\} \\ = \bigcap_{x \in X} \{y \mid y \in Y,\ f(x, y) \leqq a\} \subset \{y \mid y \in Y,\ f(x_1, y) \leqq a\} .\end{aligned}$$

$\{y \mid y \in Y,\ f(x, y) \leqq a\}$, $x \in X$, is closed because $f(x, y)$ is l.s.c. in y for each $x \in X$; $\{y \mid y \in Y,\ f(x_1, y) \leqq a\}$ is compact because $f(x_1, y)$ is inf-compact in y, and thus with $\varphi(y) := \sup_{x \in X} f(x, y)$ the level sets $\{y \mid y \in Y,\ \varphi(y) \leqq a\}$ are compact. $M_1 := \{y \mid y \in Y,\ \varphi(y) \leqq \varphi(y_1)\}$ is compact and not empty because

$$\varphi(y_1) = f(x', y_1) , \quad \text{for some} \quad x' \in X .$$

$\inf_{y \in Y} \varphi(y) = \inf_{y \in M_1} \varphi(y)$, $\varphi(y)$ is l.s.c., and since a l.s.c. function on a compact set takes its minimum, there exists $\min_{y \in Y} \sup_{x \in X} f(x, y)$. Equivalently there exists $\max_{x \in X} \inf_{y \in Y} f(x, y)$.

(ii) Suppose

$$(1) \qquad \sup_{x \in X} \inf_{y \in Y} f(x, y) < k < \inf_{y \in Y} \sup_{x \in X} f(x, y) .$$

For all $x \in X$ the sets

$$B_x := \{y \mid y \in Y,\ f(x, y) \leqq k\}$$

are closed, and B_{x_1} is compact. From (1) it follows that the family of the complements $\{B_x^C\}_{x \in X}$ is an open covering of Y, for if this were wrong, we would have a $y_0 \in Y$ with $f(x, y_0) \leqq k$ for all $x \in X$ and thus $\inf_{y \in Y} \sup_{x \in X} f(x, y) \leqq \sup_{x \in X} f(x, y_0) \leqq k$, which contradicts (1). Further we have

$$(2) \qquad Y \subset B_{x_1} \cup B_{x_1}^C .$$

Since the family $\{B_x^C\}_{x \in X}$ covers Y, it covers also B_{x_1}. B_{x_1} is compact and thus covered by a finite covering

$$\{B_{x_2}^C, \cdots, B_{x_m}^C\} , \quad x_i \in X , \quad i = 2, \cdots, m .$$

With (2) this means $\{B_{x_1}^C, B_{x_2}^C, \cdots, B_{x_m}^C\}$ covers Y. We have found a finite set $X_1 = \{x_1, \cdots, x_m\} \subset X$ such that for each $y \in Y$ there exists an $x \in X_1$ with $f(x, y) > k$.

Similarly there exists a finite set $Y_1 \subset Y$ such that for each $x \in X$ there exists a $y \in Y_1$ with $f(x, y) < k$.

Following now the second part of Sion's proof of Theorem 3.4 in [7], we come to a contradiction to (1), and we have

$$\sup_{x \in X} \inf_{y \in Y} f(x, y) = \inf_{y \in Y} \sup_{x \in X} f(x, y) . \tag{3}$$

(iii) Let $\varphi(y) := \sup_{x \in X} f(x, y)$ and $\psi(x) := \inf_{y \in Y} f(x, y)$, then by (i) there exist $x_0 \in X$, $y_0 \in Y$ such that

$$\varphi(y_0) = \inf_{y \in Y} \varphi(y) , \qquad \psi(x_0) = \sup_{x \in X} \psi(x) .$$

By (3) we then get, for $x \in X$, $y \in Y$,

$$f(x, y_0) \leqq \sup_{x \in X} f(x, y_0) = \varphi(y_0) = \psi(x_0) = \inf_{y \in Y} f(x_0, y) \leqq f(x_0, y) . \tag{4}$$

Putting in (4) $x = x_0$, $y = y_0$, we get

$$f(x_0, y_0) = \max_{x \in X} f(x, y_0) = \min_{y \in Y} f(x_0, y) ,$$

and thus

$$\min_{y \in Y} \max_{x \in X} f(x, y) = \max_{x \in X} \min_{y \in Y} f(x, y) .$$

III. A sequential unconstrained solution method. Let A and B be closed convex and not empty subsets of real linear topological Hausdorff spaces $\mathscr{X}$ and $\mathscr{Y}$, respectively, and $f: A \times B \to \boldsymbol{R}$ may be a payoff function. Then we consider the two-person zero-sum game (A, B, f). A strategy may be called optimal if it is a saddle point component. Let $\operatorname{int} A$ denote the interior, $\operatorname{cl} A$ the closure and $\operatorname{bd} A$ the boundary of A. We assume that $\operatorname{int} A \neq \emptyset$, $\operatorname{int} B \neq \emptyset$, such that all points of A and B may be reached from the interior:

$$\operatorname{cl} \operatorname{int} A = A , \qquad \operatorname{cl} \operatorname{int} B = B .$$

DEFINITION 2. A function g: $\operatorname{int} A \times \operatorname{int} B \to \boldsymbol{R}$ is called a *barrier function* of $A \times B$, if

(i) $g(x, y)$ is bounded above in x for each $y \in \operatorname{int} B$, and bounded below in y for each $x \in \operatorname{int} A$.

(ii) $g(x, y)$ is u.s.c.-l.s.c. on $(\operatorname{int} A \times \operatorname{int} B)$, i.e., the level sets

$$\{y \mid y \in \operatorname{int} B,\ g(x, y) \leqq a\} , \quad (x \in \operatorname{int} A,\ a \in \boldsymbol{R}) ,$$

and

$$\{x \mid x \in \operatorname{int} A,\ g(x, y) \geqq a\}\ , \quad (y \in \operatorname{int} B,\ a \in \boldsymbol{R})\ ,$$

are closed.

REMARK. If A and B are compact, and if $g(x, y)$ is u.s.c.-l.s.c. on $(\operatorname{int} A \times \operatorname{int} B)$, the boundedness condition is fulfilled: Let $x' \in \operatorname{int} A$ be fixed, then the level sets $L_y := \{x \mid x \in \operatorname{int} A,\ g(x, y) \geqq g(x', y)\}$, $(y \in \operatorname{int} B)$, are not empty, closed and contained in $\operatorname{int} A$, and thus compact, when A is compact. Since

$$\sup_{x \in \operatorname{int} A} g(x, y) = \sup_{x \in L_y} g(x, y) = \max_{x \in L_y} g(x, y)\ ,$$

there exists $\max_{x \in \operatorname{int} A} g(x, y)$, for each $y \in \operatorname{int} B$.

Now let $g(x, y)$ be a barrier function of $A \times B$, then for a positive real sequence $\{r_n\}_{n \in N} \subset \boldsymbol{R}$, with $r_n \to +0$ for $n \to \infty$, we define on $(\operatorname{int} A \times \operatorname{int} B)$ the family of payoffs

$$p_n(x, y) := f(x, y) + r_n g(x, y)\ , \quad (n \in \boldsymbol{N})\ .$$

THEOREM 2. *Let* $p_n(x, y)$ *be quasi-concave-convex on* $(\operatorname{int} A \times \operatorname{int} B)$, *and* $f(x, y)$ *be continuous in each variable, quasi-concave-convex, bounded above in* x *for each* $y \in B$, *bounded below in* y *for each* $x \in A$, *and* (x_0, y_0)*-sup inf-compact on* $(A \times B)$, *for a fixed* $(x_0, y_0) \in (\operatorname{int} A \times \operatorname{int} B)$. *Then we have*:

(i) *The game* $(\operatorname{int} A, \operatorname{int} B, p_n(x, y))$, $n \in \boldsymbol{N}$, *has optimal strategies* x_n *and* y_n.

(ii) $\{x_n\}_{n \in N}$ *and* $\{y_n\}_{n \in N}$ *have cluster points, and these points are optimal strategies for the game* (A, B, f).

Proof. Let $(x_0, y_0) \in (\operatorname{int} A \times \operatorname{int} B)$ with the property given above be fixed.

(i) The function $p_n(x, y)$, $(n \in \boldsymbol{N})$ satisfies the conditions of Theorem 1: By our assumptions on the functions f and g we have the obvious fact that $p_n = f + r_n g$ is u.s.c.-l.s.c. on $(\operatorname{int} A \times \operatorname{int} B)$; cf. Definition 2.
For $x = x_0$ we have

$$\begin{aligned}&\{y \mid y \in \operatorname{int} B,\ p_n(x_0, y) \leqq a\} \\ &\qquad \subset \{y \mid y \in B,\ f(x_0, y) \leqq a - r_n \cdot \inf_{y \in \operatorname{int} B} g(x_0, y)\} =: P^a_{x_0}\ .\end{aligned}$$

$P^a_{x_0}$ is compact because $f(x_0, y)$ is inf-compact in y. So $p_n(x_0, \cdot)$ is inf-compact. Similarly it follows that $p_n(\cdot, y_0)$ is sup-compact.

From Theorem 1 we get the existence of a saddle point (x_n, y_n) of p_n over $(\operatorname{int} A \times \operatorname{int} B)$.

(ii) Denote

$$v_n := p_n(x_n, y_n) = \mathrm{val}\,(\mathrm{int}\, A,\ \mathrm{int}\, B,\ p_n(x, y))\,, \quad (n \in N)\,,$$
$$v_0 := \mathrm{val}\,(A, B, f) = f(x', y'), \text{ for a saddle point } (x', y') \text{ of } f \text{ on } A \times B, \text{ which exists by Theorem 1.}$$

We have

$$(1) \qquad f(x_n, y) + r_n g(x_n, y) \geqq v_n\,, \quad \text{for each} \quad y \in \mathrm{int}\, B\,,$$

$$(2) \qquad f(x, y_n) + r_n g(x, y_n) \leqq v_n\,, \quad \text{for each} \quad x \in \mathrm{int}\, A\,,$$

$$f(x', y) \geqq v_0\,, \qquad f(x, y') \leqq v_0\,, \quad \text{for each} \quad x \in A \text{ and } y \in B\,.$$

For an arbitrary but fixed real $\delta > 0$ let $x_\delta \in \mathrm{int}\, A$ and $y_\delta \in \mathrm{int}\, B$ be δ-optimal strategies in the game (A, B, f):

$$(3) \qquad f(x_\delta, y) \geqq v_0 - \delta\,, \qquad f(x, y_\delta) \leqq v_0 + \delta\,, \quad \text{for each} \quad x \in A\,, \quad y \in B\,.$$

From (1), (2) and (3) we get

$$[f(x_\delta, y_n) + r_n g(x_\delta, y_n)] - f(x_\delta, y_n) - \delta \leqq v_n - v_0 \leqq [f(x_n, y_\delta) + r_n g(x_n, y_\delta)] - f(x_n, y_\delta) + \delta\,,$$

and

$$r_n g(x_\delta, y_n) - \delta \leqq v_n - v_0 \leqq r_n g(x_n, y_\delta) + \delta\,.$$

The boundedness of g then implies

$$r_n \cdot \inf_{y \in \mathrm{int}\, B} g(x_\delta, y) - \delta \leqq v_n - v_0 \leqq r_n \cdot \sup_{x \in \mathrm{int}\, A} g(x, y_\delta) + \delta\,,$$

and for $r_n \to +0$, $(n \to \infty)$,

$$-\delta \leqq \liminf_{n\to\infty}\,(v_n - v_0) \leqq \limsup_{n\to\infty}\,(v_n - v_0) \leqq \delta\,.$$

Since $\delta > 0$ is arbitrarily chosen, that gives

$$(4) \qquad \lim_{n\to\infty} v_n = v_0\,.$$

From (1) we get for each $y \in \mathrm{int}\, B$

$$(5) \qquad \begin{aligned} f(x_n, y) &\geqq v_n - r_n g(x_n, y) \\ &\geqq v_n - r_n \cdot \sup_{x \in \mathrm{int}\, A} g(x, y)\,, \end{aligned}$$

and since $v_n \to v_0$, $r_n \to 0$, $(n \to \infty)$, there exists a constant c independent of n such that

$$f(x_n, y_0) \geqq c\,, \quad \text{for all} \quad n \in N\,.$$

$f(\cdot, y_0)$ is sup-compact, so $\{x_n\}_{n \in N}$ is contained in a compact set.

Hence there exists a subset $\{x_{n(\alpha)}\}$ of the sequence $\{x_n\}$ which converges to an $\hat{x} \in A$. With (5) and (4) we get for each $y \in \operatorname{int} B$ by $x_{n(\alpha)} \to \hat{x}$

$$\begin{aligned} f(\hat{x}, y) &= \lim f(x_{n(\alpha)}, y) \\ &\geqq \lim (v_{n(\alpha)} - r_{n(\alpha)} \cdot \sup_{x \in \operatorname{int} A} g(x, y)) \\ &\geqq \lim v_{n(\alpha)} = v_0 \, . \end{aligned} \tag{6}$$

$f(\hat{x}, y)$ is continuous in y and thus (6) provides

$$f(\hat{x}, y) \geqq v_0 \, , \quad \text{for all} \quad y \in B \, ;$$

i.e., $\hat{x}$ is an optimal strategy for (A, B, f).

COROLLARY. *For every sequence* $\{r_n\}_{n \in N} \subset \boldsymbol{R}$, $r_n \to +0$, $(n \to \infty)$, *the values* $v_n = \operatorname{val}(\operatorname{int} A, \operatorname{int} B, f + r_n g)$ *converge to the value* $v_0 = \operatorname{val}(A, B, f)$, *and if* (A, B, f) *has a unique solution, the whole sequences of corresponding optimal strategies* $\{x_n\}_{n \in N}$, $\{y_n\}_{n \in N}$ *converge to the solution of* (A, B, f).

The *solution method* is now: Construct a barrier function g of $A \times B$, choose a positive nullsequence $\{r_n\}_{n \in N} \subset \boldsymbol{R}$, build $p_n = f + r_n g$, find an optimal strategy x_n of $(\operatorname{int} A, \operatorname{int} B, p_n(x, y))$ and take a cluster point of $\{x_n\}_{n \in N}$ as an optimal strategy for (A, B, f), on the premises that the conditions of Theorem 2 are satisfied.

An example: Let A and B be given by

$$\begin{aligned} A &= \{x \in \mathscr{X} \mid G_i(x) \leqq 0, \ i = 1, \cdots, m\} \, , \\ B &= \{y \in \mathscr{Y} \mid H_j(y) \leqq 0, \ j = 1, \cdots, n\} \, , \end{aligned}$$

with some continuous convex functions

$$\begin{aligned} G_i &\colon \mathscr{X} \longrightarrow \boldsymbol{R} \, , \quad (i = 1, \cdots, m) \, , \\ H_j &\colon \mathscr{Y} \longrightarrow \boldsymbol{R} \, , \quad (j = 1, \cdots, n) \, . \end{aligned}$$

Under the hypothesis that

$$\begin{aligned} \operatorname{int} A &= \{x \in \mathscr{X} \mid G_i(x) < 0, \ i = 1, \cdots, m\} \neq \varnothing \\ \operatorname{int} B &= \{y \in \mathscr{Y} \mid H_j(y) < 0, \ j = 1, \cdots, n\} \neq \varnothing \, , \end{aligned}$$

we can take as barrier functions of $A \times B$ for example:

$$\begin{aligned} g_1(x, y) &:= \sum_{i=1}^{m} \lg(-\max[G_i(x), -1]) - \sum_{j=1}^{n} \lg(-\max[H_j(y), -1]) \, , \\ g_2(x, y) &:= \sum_{i=1}^{m} \frac{1}{G_i(x)} - \sum_{j=1}^{n} \frac{1}{H_j(y)} \, . \end{aligned}$$

Both are well defined on $\operatorname{int} A \times \operatorname{int} B$.

$$h_{1j}(y) := -\lg(-\max[H_j(y), -1]) \quad \text{and} \quad h_{2j}(y) := -\frac{1}{H_j(y)}$$

are convex, bounded below by 0 and l.s.c. on $\operatorname{int} B$; so $\sum_{j=1}^{n} h_{kj}(y)$ has these properties, too, $(k = 1, 2)$. If B is compact, we can take also $h_{1'j}(y) := -\lg(-H_j(y))$, which then gives the mostly used barrier functions in the Interior Penalty Method of Nonlinear Programming, tracing back to Frisch [5] and Caroll [2], respectively.

IV. Computational aspects. In the differentiable case, a necessary and for a (strictly quasi-) concave-convex function $h(x, y)$ sufficient condition in order that $(\bar{x}, \bar{y})$ is a saddle point of $h(x, y)$ over open (convex) sets, is

$$(*) \qquad \frac{\partial h(\bar{x}, \bar{y})}{\partial x} = 0 , \qquad \frac{\partial h(\bar{x}, \bar{y})}{\partial y} = 0 .$$

(If the sets include their boundaries, the condition is much more complicated (cf. [3]).) In our method at each stage (r_n) we have to solve such a system (*) for $p_n(x, y)$.

This can be done by fixpoint methods, like the Newton Method or its modifications.

We can also take the gradient methods of [1] or [3] to solve $(\operatorname{int} A, \operatorname{int} B, p_n)$ directly.

All the methods need a starting point in the interior of the regions. Mostly such a point is known in advance, but if not, it can be computed by a method given in [4], p. 195. Then none of the algorithms mentioned above leaves the interior of the sets A and B because of the boundary properties of the barrier function. Thus the algorithms work as on the whole spaces so that it is justified to call our method a (sequential) unconstrained solution method.

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