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NON-ARCHIMEDEAN GELFAND THEORY

JESUS M. DOMINGUEZ

In this paper we show that if X is a Banach algebra and X_0 is its Gelfand subalgebra, then the set X_0^* of the elements in X_0 with compact spectrum is a Gelfand algebra whose maximal ideal space is compact in the Gelfand topology. We also give a representation theorem for X_0^*, which we use to derive the Van der Put characterization of C-algebras.

Introduction. Throughout all this paper we denote by F a complete field with respect to a non-trivial rank one valuation. Also X will usually denote an algebra over F. All algebras will be understood to be commutative with identity. We shall use the notation of [3], but we shall identify the ground field F with a subset of the considered algebra. Also we shall put $C(T)$, instead of $F(T)$, to denote the algebra of all F-valued continuous functions on the topological space T.

A non-archimedean Banach algebra X is called a C-algebra if there exists a compact Hausdorff space T such that X is isometrically isomorphic to $C(T)$. In [4] N. Shilkret introduces the Gelfand subalgebra; the concept of V^*-algebra is defined in [3].

1. The subalgebras X_0 and X_0^*, and their maximal ideals.

Let X be an algebra over F and let X_0 be its Gelfand subalgebra. X_0 has the following properties:

1. If $x \in X_0$, then x is invertible in X_0 if and only if it is so in X; therefore $\sigma(x) = \sigma_{X_0}(x)$.
2. If M is a maximal ideal of X, then $M \cap X_0$ is a Gelfand ideal of X_0.
3. If F is not algebraically closed, then each maximal ideal of X_0 is of the form $M \cap X_0$, where M is a maximal ideal of X.
4. If X is a Banach algebra, then X_0 is a closed subalgebra of X.

The conditions 1, 2 and 4 are easy to check (cf. [3] or Shilkret [4]). To prove condition 3 it is enough to show that if m is a maximal ideal of X_0 and $x_1, \ldots, x_n \in m$ then there is a maximal ideal M of X containing all the x_i. Let $f(Z) = \lambda_0 + \lambda_1 Z + \cdots + \lambda_n Z^n$ be an irreducible polynomial with coefficients in F, of degree greater than one, and consider $a = \lambda_0 x_2^n + \lambda_1 x_1 x_2^{n-1} + \cdots + \lambda_n x_1^n$. Then a belongs to the subalgebra $F[x_1, x_2]$ generated by x_1, x_2 over F. Moreover the maximal ideals of X containing a are just those containing both x_1, x_2. Arguing by induction on n, we find an element $c \in F[x_1, \ldots, x_n]$ such that the maximal ideals of X containing c are just those containing all the x_i. Now, $c \in m$ hence, by condition 1, there is a maximal ideal M of X containing c and, therefore, all the x_i belong to M. (A more detailed proof can be found in Gommers [**1**].)

REMARK. The assumption of F being a valued field is necessary only in condition 4.

DEFINITION. We define the algebra

$$X_0^* = \{x \in X_0/\sigma(x) \text{ is precompact}\}.$$

We see that X_0^* is a subalgebra of X_0 containing the identity element.

THEOREM 1. *Let X be a Banach algebra. The subalgebra X_0^* has the following properties*:

1. *If $x \in X_0^*$, then x is invertible in X_0^* if and only if it is so in X; therefore $\sigma(x) = \sigma_{X_0^*}(x)$.*
2. *If M is a maximal ideal of X, then $M \cap X_0^*$ is a Gelfand ideal of X_0^*.*
3. *If F is not algebraically closed, then each maximal ideal of X_0^* is of the form $M \cap X_0^*$, where M is a maximal ideal of X.*
4. *X_0^* is a closed subalgebra of X.*

Proof. The conditions 1 and 2 are easily checked. To prove 3 we just repeat the above argument replacing X_0 by X_0^*. The proof of 4 is just the following: Since X_0 is a closed subalgebra of X it is enough to show that given a sequence (x_n) in X_0^* with $x_n \to x$, then $\sigma(x)$ is precompact. To see this pick $\varepsilon > 0$. Since $x_n \to x$ there exists n_0 such that $\|x - x_{n_0}\| < \varepsilon/2$. Now since $\sigma(x_{n_0})$ is precompact there exist $\mu_1, \ldots, \mu_r \in F$ such that $\sigma(x_{n_0}) \subset \bigcup_i B(\mu_i, \varepsilon/2)$. If $\lambda \in \sigma(x)$ then there is a maximal ideal M of X such that $\lambda = x(M)$. Hence $|\lambda - x_{n_0}(M)| \leq \|x - x_{n_0}\| < \varepsilon/2$ and therefore $\sigma(x) \subset \bigcup_i B(\mu_i, \varepsilon)$.

REMARK. If X is a Banach algebra and F is locally compact, then $\sigma(x)$ is compact for all $x \in X_0$, and thus $X_0^* = X_0$.

EXAMPLES. Assume that the valuation of F is non-archimedean, and that T is a 0-dimensional Hausdorff space.

EXAMPLE 1. $C(T)$ is a commutative algebra with an identity element. For all $f \in (C(T))_0$ one has that $f(T)$ is compact, hence $(C(T))_0^* = (C(T))_0$.

EXAMPLE 2. Let $BC(T)$ denote the algebra of all bounded continuous functions from T into F, and let $PC(T)$ denote the subalgebra of all functions $f \in BC(T)$ for which $f(T)$ is precompact. Then $BC(T)$ is a commutative Banach algebra with an identity element under the sup-norm, and $(BC(T))_0 = PC(T)$. Thus $(BC(T))_0^* = (BC(T))_0$.

EXAMPLE 3. Let $F\{Z\}$ denote the algebra of all formal power series, $\Sigma a_n Z^n$, in Z with coefficients in F for which $a_n \to 0$. Then $F\{Z\}$ is a commutative Banach algebra with an identity element under $\|\Sigma a_n Z^n\| = \max |a_n|$.

(a) If F is algebraically closed, then $(F\{Z\})_0 = F\{Z\}$.

(b) If F is not algebraically closed, then $(F\{Z\})_0 = F$.

For all F, $(F\{Z\})_0^* = F$. (See [7, Th.(6.38) p. 233].)

In the sequel $\mathfrak{M}$ will denote the set of maximal ideals M of X, $\mathfrak{M}_0^*$ the set of maximal ideals m of X_0^*, and $(\mathfrak{M}_0^*)'$ the set of Gelfand ideals m' of X_0^*. For any $x \in X_0^*$ we consider the function $\hat{x}$: $(\mathfrak{M}_0^*)' \to F$, $m' \mapsto x(m')$ and we endow $(\mathfrak{M}_0^*)'$ with the weakest topology for which each of the functions $\hat{x}$ is continuous.

THEOREM 2. *If X is a Banach algebra, then $(\mathfrak{M}_0^*)'$ is a compact Hausdorff space. Furthermore, if the valuation of F is non-archimedean then $(\mathfrak{M}_0^*)'$ is a 0-dimensional space.*

Proof. To prove the first part we just consider the map $(\mathfrak{M}_0^*)' \to \prod_{x \in X_0^*} \sigma(x)$, $m' \mapsto (x(m'))_{x \in X_0^*}$ and we argue as in the case of complex Banach algebras. The second part is trivial.

THEOREM 3. *If X is a Banach algebra, then X_0^* is a Gelfand algebra.*

Proof. If F is locally compact the result follows from the Gelfand-Mazur theorem if F is algebraically closed, and from condition 3 in Theorem 1 if F is not algebraically closed. Now assume that F is not locally compact, and let m be a maximal ideal of X_0^*. If $x \in X_0^*$ let $Z(\hat{x})$ denote the set of points of $(\mathfrak{M}_0^*)'$ where $\hat{x}$ vanishes. To see that m is a Gelfand ideal we must show that $\bigcap_{x \in m} Z(\hat{x}) \neq \varnothing$. Since $(\mathfrak{M}_0^*)'$ is compact it is enough to prove that the family $\{Z(\hat{x})/x \in m\}$ has the finite intersection property. We shall prove this in two steps:

(1) Let $x_1, x_2 \in m$ and let D_1 be the set of points in $(\mathfrak{M}_0^*)'$ where $\hat{x}_1$ does not vanish. If $\hat{x}_2/\hat{x}_1$: $D_1 \to F$ is not surjective, then there exists $x \in m$ such that $Z(\hat{x}_1) \cap Z(\hat{x}_2) = Z(\hat{x})$.

Proof. Choose $x = x_2 - \lambda x_1$, where $\lambda \notin \operatorname{Im} g(\hat{x}_2/\hat{x}_1)$.

(2) If $x_1, \ldots, x_n \in m$, then $\bigcap_i Z(\hat{x}_i) \neq \varnothing$.

Proof. By induction on n. The case $n = 1$ follows from the first two conditions of Theorem 1. Assume the result true for $n - 1$. If $\hat{x}_2/\hat{x}_1$: $D_1 \to F$ is not surjective then we have just seen in (1) that there exists $x \in m$ such that $Z(\hat{x}_1) \cap Z(\hat{x}_2) = Z(\hat{x})$. The result follows from the induction hypothesis. Now assume that $\hat{x}_2/\hat{x}_1$ is surjective and $\bigcap_i Z(\hat{x}_i) = \varnothing$. Then the set $K = \{m' \in (\mathfrak{M}_0^*)'/ |\hat{x}_j(m')| \leq |\hat{x}_1(m')| \text{ for } 2 \leq j \leq n\}$ is compact and it is contained in D_1. Since F is not locally compact, to get a contradiction it is enough to show that $\hat{x}_2/\hat{x}_1(K) = \{\lambda \in F/|\lambda| \leq 1\}$. In fact take $\lambda \in F, |\lambda| \leq 1$, and consider the $(n - 1)$ elements $x_2 - \lambda x_1$ and $x_j - x_1, 3 \leq j \leq n$. By the induction assumption there exists $m' \in Z(\hat{x}_2 - \lambda\hat{x}_1) \cap \bigcap_j Z(\hat{x}_j - \hat{x}_1)$. Since $\bigcap_i Z(\hat{x}_i) = \varnothing$, then m' must belong to D_1. So $\hat{x}_2/\hat{x}_1(m') = \lambda$ and $\hat{x}_j(m') = \hat{x}_1(m')$ for $3 \leq j \leq n$. Thus $m' \in K$ and $\hat{x}_2/\hat{x}_1(m') = \lambda$. The converse is trivial.

COROLLARY. *Let X be a Banach algebra. If the linear span of the idempotent elements is dense in X, then X is a Gelfand algebra and $\mathfrak{M}$ is a compact Hausdorff space in the Gelfand topology.*

2. Representation theorems. We assume through all this section that the valuation of F is non-archimedean and that X is a non-archimedean Banach algebra.

THEOREM 4. *If X is a V^*-algebra, then X_0^* is isometrically isomorphic to $C(\mathfrak{M}_0^*)$ under the Gelfand transformation $x \mapsto \hat{x}$.*

Proof. All we need to prove is that the Gelfand transformation is an isometry ($r_\sigma(x) = \|x\|$). In this way, we further apply the Kaplansky-Stone-Weierstrass theorem to get the desired result. Now, by condition 2 in Theorem 1, X_0^* is a V^*-algebra, and by Theorems 2 and 3 above, we are in the situation of Corollary 2, page 165 of [3]. The result now follows.

DEFINITION. A family $(x_i)_{i \in I}$ of elements in X will be called an orthogonal family if $x_i x_j = 0$ for $i \neq j$.

Let E denote the idempotent elements of X having norm one.

LEMMA. *If x belongs to the linear span of E, then $r_\sigma(x) = \|x\|$.*

Proof. (1) First suppose that there exists a finite orthogonal family $e_1, \ldots, e_n$ in E and scalars $\lambda_1, \ldots, \lambda_n$ such that $x = \sum \lambda_i e_i$. We may assume $|\lambda_1| = \max |\lambda_i|$. If we show that $\lambda_1 \in \sigma(x)$, then the result will follow from: $\max |\lambda_i| = |\lambda_1| \leq r_\sigma(x) \leq \|x\| \leq \max |\lambda_i|$.

Since e_1 is a nonzero idempotent there exists a maximal ideal M of X such that $e_1 \notin M$. But $e_1(1 - e_1) = 0$ and $e_1 e_j = 0$ implies that

$(1 - e_1) \in M$ and $e_j \in M$ for $2 \leq j \leq n$. Hence $x - \lambda_1 = -\lambda_1(1 - e_1) + \Sigma_2^n \lambda_j e_j$ belongs to M, and $\lambda_1 \in \sigma(x)$.

(2) Let $x = \Sigma_1^r \mu_j u_j$, where $u_j \in E$ and $\mu_j \in F$. Now it is enough to show that there exists a finite orthogonal family $e_1, \ldots, e_n$ in E and scalars $\lambda_1, \ldots, \lambda_n$ such that $x = \Sigma \lambda_i e_i$. The proof runs by induction on r. For $r = 1$ the result is clear. Now assume the result true for $r - 1$. Then there exists a finite orthogonal family $v_1, \ldots, v_p$ in E and scalars $\alpha_1, \ldots, \alpha_p$ such that $\Sigma_2^r \mu_j u_j = \Sigma_1^p \alpha_k v_k$. Thus $x = \mu_1 u_1 + \Sigma_1^p \alpha_k v_k$. But $v_k = v_k(1 - u_1) + v_k u_1$ and $u_1 = u_1 \Pi_1^p (1 - v_k) + \Sigma_1^p u_1 v_k$. Of course, $v_k(1 - u_1)$, $v_k u_1$ and $u_1 \Pi_1^p (1 - v_k)$ are idempotents for all $1 \leq k \leq p$, those different from zero belong to E, and x may be expressed as a linear combination of them.

THEOREM (*Van der Put*). *A non-archimedean Banach algebra X is a C-algebra if and only if the linear span of E is dense in X.*

Proof. First suppose that the linear span of E is dense in X. Then X is a Gelfand algebra and $\mathfrak{M}$ is a compact Hausdorff space in the Gelfand topology. If $x \in X$, applying the lemma, we choose (x_n) in X such that $x_n \to x$ and $r_\sigma(x_n) = \|x_n\|$. The continuity of the Gelfand transformation then implies $\hat{x}_n \to \hat{x}$ in $C(\mathfrak{M})$, and so $r_\sigma(x) = \lim r_\sigma(x_n) = \lim\|x_n\| = \|x\|$. Thus X is isometrically isomorphic to $C(\mathfrak{M})$ under the Gelfand transformation. The converse is trivial.

(See Van der Put [**6**, Prop. (5.4), p. 417] or Van Rooij [**7**, Th. (6.12), p. 215], and see also [**2**] and [**5**].)

REFERENCES

1. T. Gommers, *On maximal ideals of Banach algebras over a non-archimedean valued field*, Report 7621, Mathematisch Instituut, Katholieke Universiteit, Nijmegen, (1976).
2. G. J. Murphy, *Commutative non-archimedean C^*-algebras*, Pacific J. Math., **78** No. 2, (1978), 433–446.
3. L. Narici, E. Beckenstein, and G. Bachman, *Functional Analysis and Valuation Theory*, Marcel Dekker, New York (1971).
4. N. Shilkret, *Non-Archimedean Gelfand Theory*, Pacific J. Math., **32** (1970), 541–550.
5. ______, *Non-Archimedean Banach algebras*, Duke Math. J., **37** (1970), 315–322.
6. M. Van der Put, *Algèbres de fonctions continues p-adiques*, Indag. Math., **30** (1968), 401–420.
7. A. Van Rooij, *Non-Archimedean Functional Analysis*, Marcel Dekker, Inc., New York, 1978.

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UNIVERSIDAD DE VALLADOLID
DEPARTAMENTO DE ALGEBRA Y FUNDAMENTOS
VALLADOLID, SPAIN