# ENDOMORPHISMS OF
# RANK ONE MIXED MODULES
# OVER DISCRETE VALUATION RINGS

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Mixed modules over a discrete valuation ring with torsion-free rank one and simply presented torsion submodules are considered. It is proved that every isomorphism of endomorphism algebras of such modules is induced by an isomorphism of the modules.

We begin by recalling some relevant results. A simple antecedent of the Noether-Skolem theorem states that if $V$ is a vector space over a field $F$, then every automorphism of the endomorphism algebra of $V$ is inner. R. Baer proved a similar result for bounded $p$-primary groups which was subsequently generalized by Kaplansky to torsion modules over discrete valuation rings. Fix a discrete valuation ring $R$ (and choose a prime element $p$). If $M$ is an $R$-module, we denote the endomorphism algebra of $M$ by $E(M)$. In this paper, we shall prove the following

THEOREM. *Let $M$ be an $R$-module of torsion-free rank one with simply presented torsion submodule. If $N$ is an $R$-module of torsion-free rank one, then every algebra isomorphism $\Phi\colon E(M) \to E(N)$ is induced by an isomorphism $\theta\colon M \to N$ such that $\Phi(\alpha) = \theta\alpha\theta^{-1}$ for every $\alpha \in E(M)$.*

COROLLARY. *Every algebra automorphism of $E(M)$ is inner.*

We remark that the isomorphism $\theta$ is not obtained by applying the familiar invariants for rank one modules with simply presented torsion, but is constructed directly. We have restricted our attention to modules over discrete valuation rings in view of an example in [5] which suggests that the theorem does not generalize to modules over principal ideal domains which have infinitely many primes.

To fix notation, let $K$ denote the quotient field of $R$. Moreover, we shall always let $M$ and $N$ denote $R$-modules of torsion-free rank one, with respective torsion submodules $T(M)$ and $T(N)$. If $\Phi\colon E(M) \to E(N)$ is an $R$-algebra isomorphism, then there exists an isomorphism

(A) $$\phi\colon T(M) \to T(N)$$

such that $\Phi(\alpha)\,|_{T(N)} = \phi\alpha\phi^{-1}$ for every $\alpha \in E(M)$. This is a consequence of Kaplansky's theorem [**2**, Theorem 28] and the following observation. If $\varepsilon \in E(M)$ is an idempotent such that $\varepsilon(M) \cong K/R$, then $\Phi(\varepsilon)(N)$ cannot be torsion-free since $\text{Hom}((1 - \varepsilon)(M), \varepsilon(M)) \neq 0$; hence $\Phi(\varepsilon)(N) \cong K/R$. Throughout the paper, we shall usually denote the endomorphism $\Phi(\alpha)$ by $\alpha^*$.

The proof of the theorem will be in two cases, depending on whether $M$ is split or nonsplit. First we dispose of the split case.

LEMMA 1. *Let* $\Phi \colon E(M) \to E(N)$ *be an R-algebra isomorphism, and assume that M is a split module. Then there exists an isomorphism* $\theta$: $M \to N$ *such that* $\Phi(\alpha) = \theta\alpha\theta^{-1}$ *for every* $\alpha \in E(M)$.

*Proof.* We may write $M = F \oplus T$, where $T = T(M)$ and $F$ is isomorphic to either $R$ or $K$. Let $\varepsilon \in E(M)$ be an idempotent such that $\varepsilon(M) = F$ and $\varepsilon(T) = 0$. Then $\varepsilon^*$ yields a decomposition $N = F' \oplus T'$, where $\varepsilon^*(N) = F'$ and $\varepsilon^*(T') = 0$. Since $E(F') \cong E(F) \cong R$ or $K$, we may conclude from the observation following (A) that $F' \cong F$ and that $T' = T(N)$.

First suppose that $F \cong R$. Then we may choose generators $f \in F$ and $f' \in F'$, which in turn determine isomorphisms $E(M)\varepsilon \cong M$ and $E(N)\varepsilon^* \cong N$ by evaluation. Since $\Phi$ takes $E(M)\varepsilon$ to $E(N)\varepsilon^*$, by composition we obtain an isomorphism $\theta \colon M \to N$. Let $\alpha \in E(M)$. Any element of $M$ may be represented as $\beta(f)$ for some $\beta \in E(M)\varepsilon$. Then $\theta\alpha(\beta(f)) = (\alpha\beta)^*(f') = \alpha^*\theta(\beta(f))$. Thus $\theta\alpha = \alpha^*\theta$, and we are done.

Now suppose that $F \cong K$. Let $D$ and $D'$ denote the maximal divisible submodules of $T$ and $T'$ respectively. If $D \neq 0$, choose a nonzero map $\eta \in E(M)$ with $\eta(F) \subseteq T$, $\eta(T) = 0$, and choose a generator $f$ of the kernel of $\eta$ on $F$. If $D = 0$, take $\eta = 0$ and choose $f$ to be any nonzero element of $F$. We have a homomorphism $E(M)\varepsilon \to F \oplus D$ given by evaluation at $f$. This homomorphism is surjective, and since $\eta(M)$ is a summand of $M$, the kernel can be expressed as $E(M)\eta$. But $F' \cong K$, $\eta^*(F') \subseteq T'$ and $\eta^*(T') = 0$. Thus, in a similar fashion, we may choose a generator $f'$ of the kernel of $\eta^*$ on $F'$ (or any nonzero element of $F'$ in case $\eta^* = 0$). By evaluation at $f'$, we obtain a surjective homomorphism $E(N)\varepsilon^* \to F' \oplus D'$ with kernel $E(N)\eta^*$. Since $\Phi$ takes $E(M)\eta$ to $E(N)\eta^*$, we obtain, as above, an isomorphism $\theta_0 \colon F \oplus D \to F' \oplus D'$ such that $\theta_0\alpha(x) = \alpha^*\theta_0(x)$ for every $\alpha \in E(M)$ and $x \in F \oplus D$. By (A), we may choose an isomorphism $\phi \colon T \to T'$ such that $\phi\alpha(x) = \alpha^*\phi(x)$ for every

$\alpha \in E(M)$ and $x \in T$. Since $\alpha|_D$ ranges over $E(D)$, we see that $\phi^{-1}\theta_0|_D$ lies in the center of $E(D)$. Thus it is given by the action of a unit in the completion of $R$. Alter $\phi$ by this unit factor. Then $\phi|_D = \theta_0|_D$ and $\phi\alpha = \alpha^*\phi$ still holds. Consequently, $\theta = \theta_0 \cup \phi$ is an isomorphism of $M$ to $N$ such that $\theta\alpha = \alpha^*\theta$ for every $\alpha \in E(M)$.                    $\square$

The nonsplit case will be more substantial. Let $x$ be a torsion-free element of $M$. We call $x$ *principal* if $\langle x \rangle$ is the kernel of a family of maps in $E(M)$ (i.e. the common kernel of the individual maps in the family). We denote the $p$-length of a torsion module $T$ by $l(T)$, except that we put $l(T) = \infty$ if $T$ is not reduced.

LEMMA 2. *Assume $T(M)$ is simply presented and that $x \in M$ is a torsion-free element. Then the following are equivalent*:
  (1) *$x$ is principal*;
  (2) *$\langle x \rangle$ is the kernel of the family of endomorphisms vanishing on $x$*;
  (3) *$l(M/\langle x \rangle) = l(T(M))$, where $l$ denotes $p$-length*;
  (4) *for every $v_1, v_2 \in M$ with $v_1 \notin \langle v_2, x \rangle$, there exists $\alpha \in E(M)$ with $\alpha(v_1) \neq 0$, $\alpha(\langle v_2, x \rangle) = 0$.*

*Proof.* It is evident that (1) implies (2). Endomorphisms vanishing on $x$ map into $T(M)$, thus $l(T(M)) < l(M/\langle x \rangle)$ contradicts (2). But clearly, $l(T(M)) \leq l(M/\langle x \rangle)$, therefore (2) implies (3). Now assume (3), and suppose that $v_1$ and $v_2$ are as given in (4). Let $\overline{M} = M/\langle x \rangle$. Then there exists a map $\overline{\gamma}: \langle \overline{v}_1, \overline{v}_2 \rangle \to T(M)$ such that $\overline{\gamma}(\overline{v}_1) \neq 0$, $\overline{\gamma}(\overline{v}_2) = 0$, and $\overline{\gamma}$ does not decrease $p$-heights taken in $\overline{M}$. By applying [6, Theorem 1] and [1, Corollary 81.4], suitably extended to $R$-modules, we see that $\overline{M}$ is simply presented, $\langle \overline{v}_1, \overline{v}_2 \rangle$ is a nice submodule, and $\overline{\gamma}$ can be extended to a homomorphism on $\overline{M}$. This induces the homomorphism $\alpha$ that demonstrates (4). Finally, it is clear that (4) implies (1).                    $\square$

The principal elements are torsion-free elements which are accessible via the endomorphism ring. In the proof of the nonsplit case, we shall construct an isomorphism $\theta$ by sending principal elements to corresponding principal elements. Before making this correspondence precise, we need a brief lemma on the Ulm invariants of $M/\langle x \rangle$. If $Q$ is an $R$-module, $u \in Q$, and $\rho$ an ordinal, then $f_\rho(Q)$ will denote the $\rho$th Ulm invariant of $Q$, and $h(u)$ will denote the $p$-height of $u$. We say that the $p$-indicator of $u$ has a gap at $h(p^r u)$ if $h(p^{r+1}u) > h(p^r u) + 1$.

LEMMA 3. *Let $x \in M$ be torsion-free, $\overline{M} = M/\langle x \rangle$, and $T = T(M)$. Let $\rho$ be an ordinal. Then $f_\rho(\overline{M}) = f_\rho(T)$ except in the following cases:*

(1) *if the p-indicator of $x$ has a gap at $h(p^r x)$ and $h(p^r x) = \rho$, then $f_\rho(\overline{M}) + 1 = f_\rho(T)$;*

(2) *if either the p-indicator of $x$ has a gap at $h(p^r x)$ and $h(p^{r+1} x) = \rho + 1$, or else $h(x) = \rho + 1$, then $f_\rho(\overline{M}) = f_\rho(T) + 1$.*

*Proof.* Let $T_\rho = p^\rho T[p]$ and $S_\rho = p^\rho \overline{M}[p]$. We shall regard $T$ as naturally embedded in $\overline{M}$, thus $T_\rho \subseteq S_\rho$. It suffices to prove that $T_\rho = S_\rho$ except when either $\rho < h(x)$ or $h(p^{k-1}x) < \rho < h(p^k x)$ for some $k > 0$, in which case $S_\rho/T_\rho$ has dimension one over $R/\langle p \rangle$. First suppose that either $\rho < h(x)$ or $h(p^{k-1}x) < \rho < h(p^k x)$. We may choose $v \in M$ such that $pv = p^k x$ ($k \geq 0$), and $h(v) \geq \rho$. Thus, $\bar{v} \in S_\rho$. If $k = 0$, then $\bar{v} \notin T$; if $k > 0$, then $h(v - p^{k-1}x) < \rho$ implies that $\bar{v} \notin T_\rho$. Consequently, $S_\rho \neq T_\rho$. Now suppose that $w \in M$ satisfies $\bar{w} \in S_\rho$, $\bar{w} \notin T_\rho$. Since $\langle x \rangle$ is nice (see [1, Lemma 104.1]), we may assume that $h(w) = h(\bar{w}) \geq \rho$. Thus $w$ is torsion-free, and we have $pw = p^m ax$ ($a \in R$, $p \nmid a$). Therefore $\rho < h(p^m x)$, and if $m > 0$, then $h(p^{m-1}x) < \rho$ since $w - p^{m-1}ax \notin T_\rho$. Hence, either $\rho < h(x)$ or $h(p^{k-1}x) < \rho < h(p^k x)$, where $k = m$. If $v$ is taken as above, then $w - av \in T_\rho$, hence the dimension of $S_\rho/T_\rho$ is one. $\square$

We now make the correspondence of principal elements precise. Let $x \in M$ be torsion-free. Define $I(M) = \{\alpha \in E(M) \mid \alpha(M) \subseteq T(M)\}$ and $I(M, x) = \{\alpha \in I(M) \mid \alpha(x) = 0\}$.

LEMMA 4. *Let $\Phi \colon E(M) \to E(N)$ be an R-algebra isomorphism. Assume $M$ is non-split and that $T(M)$ is simply presented. If $x$ is a principal element of $M$, then there exists a principal element $y$ of $N$ such that $\Phi(I(M, x)) = I(N, y)$. Moreover, $y$ is unique up to a unit multiple.*

*Proof.* Choose an isomorphism by (A) to identify $T(M)$ with $T(N)$. Let us use $T$ to denote this common submodule. The proof will consist of two parts.

(1) First we shall show that $\Phi(I(M)) = I(N)$. Let $\alpha \in I(M)$. Our hypothesis is symmetric by Lemma 1, hence it suffices to show that $\alpha^* \in I(N)$. Suppose that $T$ is not reduced. If $\alpha^*(N) \nsubseteq T$, then there exists $\delta \in E(M)$ such that $\delta^*(T) = 0$ and $\delta^* \alpha^* \neq 0$. But $\delta(T) = 0$, hence $\delta\alpha = 0$. This contradiction implies that $\alpha^* \in I(N)$ in this case, hence, we may assume that $T$ is reduced. Let $l(T) = \sigma + n$ ($\sigma$ limit ordinal, $n < \omega$). We claim that there exists $\Delta \in I(M)$ such that the kernel of $\Delta$ is

$\langle x, p^{\sigma}T \rangle$, the cokernel of $\Delta$ is reduced, and $\Delta^* \in I(N)$. Since $T$ is simply presented, we may decompose it as $T = T_1 \oplus T_2$, where $\Sigma_{\tau \le \rho} f_{\rho}(T_i) = \Sigma_{\tau \le \rho} f_{\rho}(T)$ for every $\tau < \sigma$, $i = 1, 2$. It follows from Lemma 3 that $\Sigma_{\tau \le \rho} f_{\rho}(M/\langle x, p^{\sigma}T \rangle) = \Sigma_{\tau \le \rho} f_{\rho}(T_i)$, hence the main result in [4], generalized to modules over $R$, implies that there exist $\beta_i: M \to T_i$ such that the kernel of $\beta_i$ is $\langle x, p^{\sigma}T \rangle$ and the cokernel of $\beta_i$ is reduced ($i = 1, 2$). By interchanging indices if necessary, there exists $r \in R$ such that $\beta_1^* - r\beta_2^* \in I(N)$. Put $\Delta = \beta_1 - r\beta_2$. It is easy to see that $\Delta$ meets the requirements of the claim. We may now choose $m > 0$ such that $\langle x, p^{\sigma}T \rangle$ is contained in the kernel of $p^m\alpha$. Thus there exists $\gamma: \Delta(M) \to T$ such that $p^m\alpha = \gamma\Delta$. Since $M$ is nonsplit, we must have $\Delta(M)/\Delta(T) \cong K/R$. But $T/\Delta(M)$ is reduced, hence $\Delta(M)/\Delta(T)$ is the maximal divisible submodule of $T/\Delta(T)$. Similar statements apply to $N$ and $\Delta^*(N)$, hence we may conclude that $\Delta(M) = \Delta^*(N)$ since $\Delta(T) = \Delta^*(T)$. It follows that the composition $\gamma\Delta^*$ makes sense, and $\gamma\Delta^* \in I(N)$. But $(\gamma\Delta)^*$ and $\gamma\Delta^*$ agree on $T$, hence they are equal since $T$ is reduced and $N/T$ is divisible. Thus $p^m\alpha^* = (\gamma\Delta)^* = \gamma\Delta^* \in I(N)$; consequently $\alpha^* \in I(N)$.

(2) Decompose $M$ as $M = M_0 \oplus D$, where $D$ is the maximal divisible submodule of $T$. Then $T = T_0 \oplus D$, where $T_0 = T(M_0)$. Note that (1) implies that $\Phi$ takes $\operatorname{Hom}(M, T_0)$ to $\operatorname{Hom}(N, T_0)$. As in (1), we see that

$$\sum_{\tau \le \rho} f_{\rho}(M/\langle x \rangle) \le \sum_{\tau \le \rho} f_{\rho}(T \oplus T_0),$$

hence by [4] (and an isomorphism on $D$) there exists a homomorphism $\Delta: M \to T \oplus T_0$ with kernel $\langle x \rangle$ and reduced cokernel. We may represent $\Delta$ as the product map $\Delta = (\beta_1, \beta_2)$, where $\beta_1 \in I(M)$ and $\beta_2 \in \operatorname{Hom}(M, T_0)$. Define $\Delta': N \to T \oplus T_0$ by $\Delta' = (\beta_1^*, \beta_2^*)$. Since $\Delta$ is injective on $T$, it follows that $\Delta'$ is injective on $T$. Since $N$ is nonsplit, the kernel of $\Delta'$ is generated by a torsion-free element $y \in N$. Clearly $y$ is principal since it is the kernel of $\{\beta_1^*, \beta_2^*\}$. We must now show that $\Phi(I(M, x)) = I(N, y)$. If this is so, then $y$ will be unique up to a unit multiple since if $z$ is another element with the same properties, then $z$ principal and $I(N, y) = I(N, z)$ imply that $\langle y \rangle = \langle z \rangle$.

By arguments similar to those in (1) with $\Delta'$ replacing $\Delta^*$, we see that $\Delta(M) = \Delta'(N)$. Now let $\alpha \in I(M, x)$. It suffices to consider cases $\alpha(M) \subseteq T_0$ and $\alpha(M) \subseteq D$, since any element in $I(M, x)$ is the sum of two such maps. First suppose $\alpha(M) \subseteq T_0$. There exists a map $\gamma: \Delta(M) \to T_0$ such that $\alpha = \gamma\Delta$. Again, as in (1), we have $\alpha^* = (\gamma\Delta)^* = \gamma\Delta' \in I(N, y)$. Now suppose that $\alpha(M) \subseteq D$. Since $D$ is divisible and $\Delta(M) \subseteq M \oplus M$, there exists a map $\gamma: M \oplus M \to D$ such that $\alpha = \gamma\Delta$. Let $\gamma_1, \gamma_2 \in I(M)$ satisfy

$\gamma_1(z) = \gamma(z, 0)$ and $\gamma_2(z) = \gamma(0, z)$ $(z \in M)$. Then $\alpha = \gamma\Delta = \gamma_1\beta_1 + \gamma_2\beta_2$, hence $\alpha^* = \gamma_1^*\beta_1^* + \gamma_2^*\beta_2^* \in I(N, y)$. We have shown that $\Phi(I(M, x)) \subseteq I(N, y)$. The reverse inclusion follows from a symmetric argument using $\Delta'(N) = \Delta(M)$. $\qquad\qquad\square$

Next we show that there are enough principal elements to generate the module $M$ modulo torsion. Moreover, they can be chosen to satisfy certain conditions. For a torsion element $t$, we let $e(t)$ denote its exponent.

LEMMA 5. *Let $M$ be nonsplit with $T(M)$ simply presented. Then*

(1) *there exist principal elements $x_i$ such that $px_{i+1} \equiv x_i \mod T(M)$* $(i \geq 1)$;

(2) *$t_i$ $(i \geq 1)$ are independent, where $t_i = x_i - px_{i+1}$;*

(3) *$e(t_i) \leq e(pt_{i+1})$ $(i \geq 1)$.*

*Proof.* If the $p$-indicator of $M$ contains infinitely many gaps then by Lemmas 2 and 3 there exist principal elements $u_i \in M$ with $pu_{i+1} \equiv u_i$ mod $T(M)$ $(i \geq 1)$. If the $p$-indictor contains only finitely many gaps then by the nonsplitting of $M$ there exists a $u \in M$ such that $h(u)$ is a limit ordinal $\geq \omega$ and the height sequence of $u$ contains no gaps. Again we invoke the above lemmas to conclude that $u$ is principal. By (4) of Lemma 2 we may choose principal elements $u_i \in M$ such that $p^i u_i = u$ $(i \geq 1)$. Thus we may asume that the $u_i \in M$ satisfy (1). Let $x_1 = u_1$. For induction suppose that $x_1, x_2, \ldots, x_k$ have been chosen such that (1), (2), and (3) hold. Let $t = x_k - pu_{k+1}$. Since $T(M)$ is unbounded we may choose $s \in T(M)$ such that it is independent of $\langle t_1, \ldots, t_{k-1}, t \rangle$ and $\langle p^{e(s)} u_{k+1} \rangle = \langle u_{k+1} \rangle \cap \langle x_k \rangle \cap \langle x_{k-1} \rangle$. Define $x_{k+1} = u_{k+1} + s$. Clearly $px_{k+1} \equiv x_k \mod T(M)$. Moreover $\langle x_{k-1} \rangle \cap \langle x_k \rangle \cap \langle u_{k+1} \rangle = \langle p^{e(s)} u_{k+1} \rangle = \langle p^{e(s)} x_{k+1} \rangle$. By (2) of Lemma 2 it follows that $p^{e(s)} x_{k+1}$ is principal. Another application of (4) of Lemma 2 shows that $x_{k+1}$ is principal and (1) holds. Now $t_k = x_k - px_{k+1} = t - ps$. The independence of $\{t_1, \ldots, t_k\}$ will follow if $e(ps) \geq e(t)$. But $\langle p^{e(s)} u_{k+1} \rangle \subseteq \langle x_k \rangle \cap \langle u_{k+1} \rangle = \langle p^{e(t)+1} u_{k+1} \rangle$ and therefore $e(s) \geq e(t) + 1$. To show (3) we observe that $\langle p^{e(s)} u_{k+1} \rangle = \langle x_{k-1} \rangle \cap \langle x_k \rangle \cap \langle u_{k+1} \rangle = \langle p^{e(t_{k-1})+1} x_k \rangle \cap \langle u_{k+1} \rangle \subseteq \langle p^{e(t_{k-1})+2} u_{k+1} \rangle$ i.e. $e(s) \geq e(t_{k-1}) + 2$. Since $e(t_k) \geq e(ps)$ by the independence of $t$ and $s$ we have $e(pt_k) = e(ps) - 1 = e(s) - 2 \geq e(t_{k-1})$. $\qquad\square$

In our final lemma we show that the corresponding principal elements can be chosen to be compatible with $\phi$, the isomorphism on torsion given

by (A). The latter may require adjustment by a $p$-adic unit. For notational convenience we let $t^*$ denote $\phi(t)$ for $t \in T(M)$.

LEMMA 6. *Let* $\Phi$: $E(M) \to E(N)$ *be an* $R$*-algebra isomorphism, and assume that* $M$ *is a nonsplit module with* $T(M)$ *simply presented.*

(1) *Let* $x_1 \in M$, $y_1 \in N$ *be principal elements such that* $\Phi(I(M, x_1)) = I(N, y_1)$. *If* $x_2 \in M$ *is principal with* $t = x_1 - px_2$ *torsion, then there exists a principal element* $y_2 \in N$ *such that* $y_1 - py_2 = kt^*$, $p \nmid k$, *and* $\Phi(I(M, x_2)) = I(N, y_2)$.

(2) *Let* $x_1, x_2, x_3 \in M$ *with* $x_1$ *principal and let* $t_i = x_i - px_{i+1}$ ($i = 1, 2$) *be independent torsion elements with* $e(t_1) \le e(pt_2)$. *Suppose further that* $y_i \in N$ *are such that* $\Phi(I(M, x_i)) = I(N, y_i)$ ($i = 1, 2, 3$) *and* $y_i - py_{i+1} = k_i t_i^*$ ($i = 1, 2$). *Then* $k_1 \equiv k_2 \mod p^{e(t_1)}$.

*Proof.* (1) By Lemma 4 there exists a principal element $y_2 \in N$ such that $\Phi(I(M, x_2)) = I(N, y_2)$. Let $\langle x_1 \rangle \cap \langle x_2 \rangle = \langle p^r x_1 \rangle$. Since $x_2$ is principal we conclude that $r$ is minimal such that $I(M, x_2) \subseteq I(M, p^r x_1)$. Since $I(M, p^r x_1) = \{\alpha \in E(M) \mid p^r \alpha \in I(M, x_1)\}$ and $\Phi(I(M, x_1)) = I(N, y_1)$, it follows that $\Phi(I(M, p^r x_1)) = I(N, p^r y_1)$. Hence $r$ is minimal such that $I(N, y_2) \subseteq I(N, p^r y_1)$. This together with $y_2$ principal implies that $\langle y_1 \rangle \cap \langle y_2 \rangle = \langle p^r y_1 \rangle$. But $t = x_1 - px_2$, therefore $\langle x_1 \rangle \cap \langle x_2 \rangle = \langle p^{r+1} x_2 \rangle$. By an argument similar to the above and interchanging the roles of $x_1, x_2$ and $y_1, y_2$, we obtain $\langle y_1 \rangle \cap \langle y_2 \rangle = \langle p^{r+1} y_2 \rangle$. We may now modify $y_2$ by a unit to get $p^r y_1 = p^{r+1} y_2$, or $y_1 - py_2$ torsion. It remains to show that $y_1 - py_2 = kt^*$, $p \nmid k$. From Lemma 2 and the fact that $x_1$ is principal it follows that the torsion of the kernel of $I(M, x_1) \cap I(M, px_2)$ is $\langle t \rangle$. Thus the torsion of the kernel of $I(N, y_1) \cap I(N, py_2)$ is $\langle t^* \rangle$. Therefore $y_1 - py_2 = kt^*$, where $p \nmid k$ since $o(y_1 - py_2) = o(t)$.

(2) As above, using the fact that $x_1$ is principal we have that the torsion of the kernel of $I(M, x_1) \cap I(M, p^2 x_3)$ is $\langle t_1 + pt_2 \rangle$. Thus the torsion of the kernel of $I(N, y_1) \cap I(N, p^2 y_3)$ is $\langle t_1^* + pt_2^* \rangle$. But $y_1 - p^2 y_3 = k_1 t_1^* + pk_2 t_2^*$, therefore $k_1 t_1^* + pk_2 t_2^* = l(t_1^* + pt_2^*)$ for some $l$. This implies that $k_1 \equiv l \mod p^{e(t_1)}$, $k_2 \equiv l \mod p^{e(pt_2)}$ thus $k_1 \equiv k_2 \mod p^{e(t_1)}$.  $\square$

We are now ready for the

*Proof of the Theorem.* We use $\phi$ in (A) to identify the torsion submodules of $M$ and $N$ and denote this common submodule by $T$. By Lemma 1 we may assume $M$ to be nonsplit. Let $\varepsilon$ be an idempotent with

$\varepsilon(M) = D$, the maximal divisible submodule of torsion, and $(1 - \varepsilon)(M) = M_0$. Choose $x_i \in M$ ($i \geq 1$) satisfying (1)–(3) of Lemma 5. By Lemmas 4 and 6 we choose principal $y_i \in N$ inductively such that

    (a) $\Phi(I(M, x_i)) = I(N, y_i)$ ($i \geq 1$),

    (b) $y_i - py_{i+1} = k_i t_i^*, p \nmid k_i$ ($i \geq 1$),

    (c) $k_i \equiv k_{i+1} \bmod p^{e(t_i)}$ ($i \geq 1$).

Since $\{e(t_i)\}$ is monotonic there exists a unit $\eta$ in the $p$-adic completion of $R$ such that $\eta t_i^* = k_i t_i^*$ ($i \geq 1$). Define $R$-isomorphisms $\theta_i : T \oplus \langle x_i \rangle \to T \oplus \langle y_i \rangle$ by $\theta_i |_T = \eta\phi$ and $\theta_i(x_i) = y_i$. We claim that $\theta_{i+1} |_{T \oplus \langle x_i \rangle} = \theta_i$. It suffices to show that $\theta_{i+1}(x_i) = \theta_i(x_i)$. However

$$\theta_{i+1}(x_i) = \theta_{i+1}(px_{i+1} + t_i) = py_{i+1} + \eta t_i^*$$

$$= py_{i+1} + k_i t_i^* = y_i = \theta_i(x_i).$$

Put $\theta = \bigcup_i \theta_i$. Since $\bigcup_i (T \oplus \langle x_i \rangle) = M$ and $\bigcup_i (T \oplus \langle y_i \rangle) = N$, $\theta$ is an $R$-isomorphism from $M$ to $N$. We note for later use that for $\alpha \in E(M)$, $\Phi(\alpha)|_T = \phi\alpha\phi^{-1}|_T = \theta\alpha\theta^{-1}|_T$.

    We now show that $E(M) = R + I(M)$. This will hold if we can prove that no $\alpha \in E(M)$ induces multiplication by $1/p$ on $M/T$. If such an $\alpha$ exists, then for torsion-free $x \in M$, $p\alpha(x) = x + t$ for some $t \in T$. Thus $\alpha(p^{n+1}x) = p^n x$ for some $n$, whence $h(p^n x) = h(\alpha(p^{n+1}x)) \geq h(p^{n+1}x) > h(p^n x)$, a contradiction. In view of this we need to show $\alpha^* = \theta\alpha\theta^{-1}$ only for $\alpha \in I(M)$. In fact if we put $T(M_0) = T_0$ then it suffices to consider $\alpha$ such that either $\alpha(M_0) \subseteq T_0$ or $\alpha(M_0) \subseteq D$. Before we treat these cases we prove a claim: $\varepsilon^* = \theta\varepsilon\theta^{-1}$. Since $\theta\varepsilon\theta^{-1}$ is an idempotent yielding the decomposition $\theta(M_0) \oplus D$ it suffices to show that $\varepsilon^*$ yields this decomposition also. Clearly $\varepsilon^*|_D = 1_D$. Note that $M_0 = \langle \{x_i\}, T_0 \rangle$ and therefore $\theta(M_0) = \langle \{y_i\}, T_0 \rangle$. Since $\varepsilon^*(T_0) = \varepsilon(T_0) = 0$ and from (a) $\varepsilon^* \in I(N, y_i)$ ($i \geq 1$) we have $\varepsilon^*\theta(M_0) = 0$. Thus our claim is shown and we return to our two cases.

    First suppose that $\alpha(M_0) \subseteq T_0$. This is equivalent to $\alpha\varepsilon = \varepsilon\alpha$. Therefore $\alpha^*\varepsilon^* = \varepsilon^*\alpha^*$ and by part (1) of the proof of Lemma 4 it follows that $\alpha^*(N_0) \subseteq T_0$ where $N_0 = (1 - \varepsilon^*)(N)$. Now $\theta\alpha\theta^{-1}\varepsilon^* = \theta\alpha\varepsilon\theta^{-1} = \theta\varepsilon\alpha\theta^{-1} = \varepsilon^*\theta\alpha\theta^{-1}$ and hence $\theta\alpha\theta^{-1}(N_0) \subseteq T_0$. However this together with $\alpha^*(N_0) \subseteq T_0$, $\alpha^*|_T = \theta\alpha\theta^{-1}|_T$, and $\mathrm{Hom}(N/T, T_0) = 0$ allows us to conclude that $\alpha^* = \theta\alpha\theta^{-1}$. Now we suppose $\alpha(M_0) \subseteq D$. Let $x \in M$ be principal. Then $p^r\alpha(x) = 0$ for some $r$. By an argument in part (2) of the proof of Lemma 4, $p^r\alpha = \gamma_1\beta_1 + \gamma_2\beta_2$ where $\gamma_i(M_0) \subseteq D$ and $\beta_i(M_0) \subseteq T_0$ ($i = 1, 2$). By the previous case and the fact that $\gamma_i$ in the above equation are applied only to torsion elements, we have $p^r\alpha^* = \gamma_1^*\beta_1^* + \gamma_2^*\beta_2^* = \theta^{-1}(\gamma_1\beta_1 + \gamma_2\beta_2)\theta = p^r\theta\alpha\theta^{-1}$. Now $\varepsilon\alpha = \alpha$ implies $\varepsilon^*\alpha^* = \alpha^*$ and therefore $\alpha^*(N) \subseteq D$. Clearly $\theta\alpha\theta^{-1}(N) \subseteq D$. Since $\alpha^*|_T = \theta\alpha\theta^{-1}|_T$ and

Hom($N/T, D$) is torsion-free we have $\alpha^* = \theta\alpha\theta^{-1}$ to complete the proof. □

We close the paper with two questions. Does the theorem hold for arbitrary torsion submodules? Here, different methods may be required since our proof relies to a large degree on the result in [4] proved for simply presented torsion. A key step would be to reprove Lemma 4 in this new setting. The second question concerns higher rank. Does the theorem hold for modules of countable torsion-free rank over complete discrete valuation rings? The ring needs to be complete to eliminate pathological examples of torsion-free modules of finite rank that occur over incomplete discrete valuation rings. In addition the rank must be countable in view of an example in [3, §4].

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