PACIFIC JOURNAL OF MATHEMATICS
Vol. 108, No. 1, 1983

A NOTE ON M_1-SPACES

KUO-SHIH KAO

A mapping f: $X \to Y$ is called quasi-open if the interior of $f(U)$ is non-void for any non-void open subsets U of X. The main result in this paper is that the image of an M_1-space under a quasi-open, countably bi-quotient closed mapping is an M_1-space; it follows that the locally finite regular closed sum of M_1-spaces is an M_1-space.

In 1961, J. Ceder [**4**] defined the M_i-spaces ($i = 1, 2, 3$). From the definitions, it is clear that $M_1 \to M_2 \to M_3$. Recently, G. Gruenhage [**6**] and H. Junnila [**8**] independently proved that the stratifiable (M_3-) spaces coincide with the M_2-spaces. Whether stratifiable spaces are M_1-spaces still remains open. Moreover, it is still unknown if the closed image of an M_1-space is an M_1-space. It is known that irreducible perfect mappings preserve M_1-spaces (Borges-Lutzer [**2**]). The main result in this paper is that the quasi-open (Definition 1), countably bi-quotient closed mappings preserve M_1-spaces (Theorem 1), which improves the above result as well as the result of R. F. Gittings [**5**], and from the main result it follows that the locally finite regular closed sum of M_1-spaces is an M_1-space which partially answers the problem posed by Ceder [**4**]. On the other hand, we generalize the theorem of Gruenhage [**7**], which proves that σ-discrete stratifiable spaces are M_1.

In this paper, regular, normal spaces are assumed to be T_1, and all mappings are continuous and surjective. Let $\mathcal{U}$ be a collection of subsets of X, the union $\cup\{U: U \in \mathcal{U}\}$ is denoted by $\mathcal{U}^*$.

A collection $\mathcal{U}$ of subsets of X is closure preserving if for any $\mathcal{U}' \subset \mathcal{U}$, $\overline{\mathcal{U}'^*} = \cup\{\overline{U}: U \in \mathcal{U}'\}$. $\mathcal{U}$ is hereditarily closure preserving if for any choice of a subset $S(U) \subset U$, $U \in \mathcal{U}$, the resulting collection $\{S(U): U \in \mathcal{U}\}$ is closure preserving.

A space X is an M_1-space if X is regular and has a σ-closure preserving base.

DEFINITION 1. A mapping f: $X \to Y$ is called quasi-open if the interior of $f(U)$ (denoted by Int $f(U)$) is non-void for any non-void open subsets U of X.

Clearly, open mappings are quasi-open and quasi-open mappings are preserved by composition and cartesian products.

DEFINITION 2. A mapping f: $X \to Y$ is called pseudo-open if for any $y \in Y$ and any open subset $U \supset f^{-1}(y)$, $y \in \operatorname{Int} f(U)$.

It is well known that every closed mapping is pseudo-open.

DEFINITION 3. A mapping f: $X \to Y$ is called irreducible if f maps no proper closed subspace of X onto Y.

LEMMA 1. *Irreducible pseudo-open mappings are quasi-open.*

Proof. Let f: $X \to Y$ be an irreducible pseudo-open mapping. Let U be any non-void open subset of X. Since f is irreducible, $U \supset f^{-1}(y)$ for some $y \in Y$, otherwise $f(X - U) = Y$ would be contrary to the irreducibility of the mapping f. Since f is pseudo-open, $y \in \operatorname{Int} f(U)$. This shows that f is a quasi-open mapping.

LEMMA 2. *Let f: $X \to Y$ be a quasi-open closed mapping. Let $\mathfrak{B}$ be a closure preserving collection of open subsets of X. Then $\mathcal{C} = \{\operatorname{Int} f(U)$: $U \in \mathfrak{B}\}$ is a closure preserving collection of open subsets of Y.*

Proof. Let $\mathfrak{B}' \subset \mathfrak{B}$ and let $y \in \overline{\cup \{\operatorname{Int} f(U) \colon U \in \mathfrak{B}'\}}$. Since $f(U) \supset \operatorname{Int} f(U)$, we have

$$f(\overline{\mathfrak{B}'^*}) \supset f(\mathfrak{B}'^*) \supset \cup \{\operatorname{Int} f(U) \colon U \in \mathfrak{B}'\}.$$

Since f is a closed mapping, $f(\overline{\mathfrak{B}'^*})$ is a closed set; therefore

$$f(\overline{\mathfrak{B}'^*}) \supset \overline{\cup \{\operatorname{Int} f(U) \colon U \in \mathfrak{B}'\}}.$$

It follows $f^{-1}(y) \cap \overline{\mathfrak{B}'^*} \neq \varnothing$. Because $\mathfrak{B}'$ is closure preserving, there exists $U' \in \mathfrak{B}'$ such that $f^{-1}(y) \cap \overline{U'} \neq \varnothing$. Let V be any open neighborhood of y. Then $f^{-1}(V) \cap U' \neq \varnothing$. Since f is quasi-open, the interior of the image of the non-void open set $f^{-1}(V) \cap U'$ is non-void. According to

$$\operatorname{Int} f(f^{-1}(V) \cap U') \subset \operatorname{Int}[V \cap f(U')] = V \cap \operatorname{Int} f(U'),$$

$V \cap \operatorname{Int} f(U')$ is non-void. It shows that any open neighborhood V of y intersects $\operatorname{Int} f(U')$. Therefore $y \in \overline{\operatorname{Int} f(U')}$. Thus we have proved that $\mathcal{C}$ is a closure preserving collection of open subsets of Y.

DEFINITION 4. A mapping f: $X \to Y$ is called bi-quotient if, whenever $y \in Y$ and $\mathfrak{U}$ is a collection of open subsets of X such that $\mathfrak{U}^* \supset f^{-1}(y)$, there exists finite subcollection $\mathfrak{U}' \subset \mathfrak{U}$ such that $y \in \operatorname{Int} f(\mathfrak{U}'^*)$. If $\mathfrak{U}$ is any countable collection of open subsets then the mapping f is called countably bi-quotient.

It is well known that

$$\begin{array}{ccccccc} \text{open} & \rightarrow & \text{bi-quotient} & \rightarrow & \text{countably bi-quotient} & \rightarrow & \text{pseudo-open} \\ & & \uparrow & & \uparrow & & \uparrow \\ & & \text{perfect} & \rightarrow & \text{quasi-perfect} & \rightarrow & \text{closed} \end{array}$$

and all the implications cannot be reversed.

THEOREM 1. *The image of an M_1-space under a quasi-open, countably bi-quotient closed mapping is an M_1-space.*

Proof. Let f be a quasi-open, countably bi-quotient closed mapping from an M_1-space X onto a topological space Y. Let $\mathfrak{B} = \bigcup_{i=1}^{\infty} \mathfrak{B}_i$ be a σ-closure preserving base for X. Note that if $\mathfrak{U}$ is a closure preserving collection of sets and $\tilde{\mathfrak{U}}$ is the collection of all unions of all subcollections of $\mathfrak{U}$ then $\tilde{\mathfrak{U}}$ is also closure preserving. Therefore we may assume that the union of any subcollection of $\mathfrak{B}_i$ is a member of $\mathfrak{B}_i$. Moreover, without loss of generality, we also assume $\mathfrak{B}_i \subset \mathfrak{B}_{i+1}$ $(i = 1, 2, \ldots)$. Put $\mathcal{C} = \{\operatorname{Int} f(B)\colon B \in \mathfrak{B}\}$. According to Lemma 2, $\mathcal{C}$ is a σ-closure preserving collection of open subsets of Y.

For each $y \in Y$, let V be an open neighborhood of y. Since $\mathfrak{B}$ is a base for X, there exists $\mathfrak{B}' \subset \mathfrak{B}$ such that $f^{-1}(y) \subset \mathfrak{B}'^{*} \subset f^{-1}(V)$. Put $\mathfrak{B}'_i = \mathfrak{B}' \cap \mathfrak{B}_i$, then $\mathfrak{B}' = \bigcup_{i=1}^{\infty} \mathfrak{B}'_i$, $f^{-1}(y) \subset \bigcup_{i=1}^{\infty} \mathfrak{B}'^{*}_i \subset f^{-1}(V)$. According to $\mathfrak{B}_i \subset \mathfrak{B}_{i+1}$ $(i = 1, 2, \ldots)$, the sequence $\{\mathfrak{B}'^{*}_i\}$ is increasing. Since f is a countably bi-quotient mapping, there exists a natural number n such that $y \in \operatorname{Int} f(\mathfrak{B}'^{*}_n) \subset V$. By hypothesis, there exists $B \in \mathfrak{B}_n \subset \mathfrak{B}$ such that $B = \mathfrak{B}'^{*}_n$. Therefore $\operatorname{Int} f(B) \in \mathcal{C}$ and $y \in \operatorname{Int} f(B) \subset V$. So $\mathcal{C}$ is a base for Y, which is σ-closure preserving. Clearly, Y is regular (closed mappings preserve T_1 and normality). Therefore Y is an M_1-space.

According to Lemma 1 and the fact that perfect mappings are countably bi-quotient closed mappings, we obtain the following result.

COROLLARY 1 (*Borges-Lutzer* [2]). *The image of an M_1-space under an irreducible perfect mapping is an M_1-space.*

There exists an open (hence quasi-open, countably bi-quotient), closed mapping which is neither irreducible nor perfect (let X be a countably compact but non-compact space, Y be a space satisfying first axiom of countability and f be the projection of the product space $X \times Y$ onto Y). Therefore Theorem 1 improves Borges-Lutzer's theorem.

COROLLARY 2. *The image of an M_1-space under an open, closed mapping is an M_1-space.*

A mapping f: $X \to Y$ is called k-to-one, if for each $y \in Y$, $f^{-1}(y)$ consists of exactly k points in X.

COROLLARY 3. (*R. F. Gittings* [**5**]). *The image of an M_1-space under a k-to-one, open mapping is an M_1-space.*

Proof. Let f be a k-to-one, open mapping from an M_1-space X onto a space Y. According to Lemmas 1 and 2 of Arhangelskii [**1**], f is closed, and hence by Corollary 2, Y is an M_1-space.

D. Burke, R. Engelking and D. Lutzer [**3**] proved that a regular space X is metrizable if and only if X has a σ-hereditarily closure preserving base. Using the above theorem we may easily obtain E. Michael's elegant theorem which effectively improved the famous theorem of Morita-Hanai-Stone (see [**10**]).

COROLLARY 4 (*E. Michael* [**9**]). *The image of a metrizable space under a countably bi-quotient closed mapping is a metrizable space.*

Proof. By the same argument in the proof of Theorem 1 we need only prove that if f: $X \to Y$ is a closed mapping, $\mathfrak{B}$ is a hereditarily closure preserving collection of open subsets of X, then $\mathcal{C} = \{\text{Int } f(U): U \in \mathfrak{B}\}$ is a hereditarily closure preserving collection of open subsets of Y.

Whenever $S(U) \subset \text{Int } f(U)$ is chosen for each $U \in \mathfrak{B}$, let $R(U) = U \cap f^{-1}(S(U))$. Then $R(U) \subset U$ and $f(R(U)) = S(U)$. Since the collection $\{R(U): U \in \mathfrak{B}\}$ is closure preserving and f is a continuous closed mapping, the collection $\{S(U): U \in \mathfrak{B}\}$ is also closure preserving. Therefore $\mathcal{C} = \{\text{Int } f(U): U \in \mathfrak{B}\}$ is a hereditarily closure preserving collection of open subsets of Y.

THEOREM 2. *Let X be a paracompact σ-space. Let f: $X \to Y$ be a quasi-open, closed mapping. If $f^{-1}(F)$ has a σ-closure preserving neighborhood base for each closed subset F of Y, then Y is an M_1-space.*

Proof. Since f is closed, the space Y is a paracompact σ-space. Let F be an arbitrary closed subset of Y, let $\mathfrak{B}$ be a σ-closure preserving neighborhood base of $f^{-1}(F)$. By the Lemma 2, $\mathcal{C} = \{\text{Int } f(U): U \in \mathfrak{B}\}$ is a σ-closure preserving collection of open subsets of Y. For any open subset $V \supset F, f^{-1}(F) \subset f^{-1}(V)$, there exists $U \in \mathfrak{B}$ such that $f^{-1}(F) \subset U \subset f^{-1}(V)$. Since f is closed, there exists an open subset U' such that $f^{-1}(y) \subset U' \subset U$ and $f(U')$ is an open subset of Y. Hence $f(U') \subset \operatorname{Int} f(U) \subset f(U) \subset V$, and $F \subset \operatorname{Int} f(U) \subset V$. Therefore $\mathcal{C}$ is a σ-closure preserving neighborhood base of the closed subset F.

Thus we have proved that every closed subset F of the paracompact σ-space Y has a σ-closure preserving neighborhood base. According to Borges-Lutzer's result (Remark 2.7 of [**2**]), Y is an M_1-space.

COROLLARY. *Let X be an M_1-space with every closed subset having a σ-closure preserving neighborhood base. Let $f: X \to Y$ be a quasi-open closed mapping. Then Y is an M_1-space.*

This corollary improves a result of Borges-Lutzer (Remark 3.5 of [**2**]).

Ceder [**4**] proved the locally finite closed sum theorem for M_2 and M_3 spaces (Theorem 2.8 of [**4**]), and asked if this theorem remained valid for M_1-spaces. In the following, we give two locally finite sum theorems for M_1-spaces. Theorem 3 improves Ceder's theorem for locally M_1-spaces (Theorem 2.6 of [**4**]). Theorem 4 gives a partial answer to Ceder's question.

THEOREM 3. *Let X be a normal space. Let $\mathfrak{U} = \{U_\alpha\}_{\alpha \in A}$ be a locally finite open covering of X. If each U_α $(\alpha \in A)$ be an M_1-space then X is an M_1-space.*

Proof. Let $\mathfrak{B}^\alpha = \bigcup_{i=1}^\infty \mathfrak{B}_i^\alpha$ be a σ-closure preserving base for open subspace U_α $(\alpha \in A)$. By the regularity of X, we may assume $\bar{B} \subset U_\alpha$ for each $B \in \mathfrak{B}^\alpha$.

By the normality of X, there exists an open covering $\{V_\alpha\}_{\alpha \in A}$ of X such that $\overline{V_\alpha} \subset U_\alpha$ $(\alpha \in A)$. Since the open subspace of an M_1-space is an M_1-space, V_α $(\alpha \in A)$ is an M_1-space, and we may choose

$$\mathcal{C}^\alpha = \bigcup_{i=1}^\infty \mathcal{C}_i^\alpha$$

as the base for subspace V_α, where

$$\mathcal{C}_i^\alpha = \{B : B \in \mathfrak{B}_i^\alpha, \bar{B} \subset V_\alpha\}$$

is closure preserving in subspace V_α. We are going to prove $\mathcal{C}_i^\alpha$ is also closure preserving in space X.

Let $\mathcal{C}' \subset \mathcal{C}_i^\alpha$, we need to prove

$$(1) \qquad \bigcup \{\bar{B} : B \in \mathcal{C}'\} = \overline{\bigcup \{B : B \in \mathcal{C}'\}}.$$

Since U_α is an M_1-space, $V_\alpha \subset U_\alpha$, $\mathcal{C}' \subset \mathcal{C}_i^\alpha \subset \mathcal{B}_i^\alpha$, and $\mathcal{B}_i^\alpha$ is closure preserving in subspace U_α, therefore

$$\bigcup \{\bar{B}: B \in \mathcal{C}'\} = \overline{\bigcup \{B: B \in \mathcal{C}'\}} \cap U_\alpha.$$

According to $\bigcup\{B: B \in \mathcal{C}'\} \subset V_\alpha$, $\overline{\bigcup \{B: B \in \mathcal{C}'\}} \subset \overline{V_\alpha} \subset U_\alpha$. It follows

$$\overline{\bigcup \{B: B \in \mathcal{C}'\}} \cap U_\alpha = \overline{\bigcup \{B: B \in \mathcal{C}'\}}.$$

Hence (1) is proved. Since $\{V_\alpha\}_{\alpha \in A}$ is locally finite, we can easily prove $\mathcal{C}_i = \bigcup_{\alpha \in A} \mathcal{C}_i^\alpha$ is a closure preserving collection of space X. Moreover, it is easy to verify $\mathcal{C} = \bigcup_{i=1}^\infty \mathcal{C}_i$ is a base for X. Therefore X is an M_1-space.

COROLLARY (*Ceder* [4]). *Let X be a paracompact and locally M_1-space. Then X is an M_1-space.*

THEOREM 4. *Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be a locally finite open covering of space X. If each $\overline{U_\alpha}$ $(\alpha \in A)$ be an M_1-space, then X is an M_1-space.*

Proof. For each $\alpha \in A$, let X_α be a copy of $\overline{U_\alpha}$ and f_α be the homeomorphism from X_α onto $\overline{U_\alpha}$. Let

$$X^* = \sum_{\alpha \in A} X_\alpha$$

be the (disjoint) topological sum of X_α's. Evidently X^* is an M_1-space. Let f: $X^* \to X$ be the mapping defined as follows: for each $x \in X^*$, $f(x) = f_\alpha(x)$, if $x \in X_\alpha$. By the local finiteness of $\{\overline{U_\alpha}\}_{\alpha \in A}$, it can be easily verified that f is a finite to one, closed continuous mapping. Moreover, f is quasi-open, it is proved as follows. Because of the definition of topological sum, we need only prove that the interior of the image of non-void subset E $(E \subset X_\alpha)$ which is relatively open in subspace X_α is non-void. Since f_α is the homeomorphism from X_α onto $\overline{U_\alpha}$, $f_\alpha(E)$ is relatively open in $\overline{U_\alpha}$. There exists an open subset G such that $f_\alpha(E) = G \cap \overline{U_\alpha}$. Let $x \in f_\alpha(E) \subset G$. There exists an open neighborhood $V(x)$ of x such that $V(x) \subset G$. On the other hand, $x \in f_\alpha(E) \subset \overline{U_\alpha}$, $V(x) \cap U_\alpha \neq \varnothing$. Since $V(x) \cap U_\alpha \subset f_\alpha(E)$ and $V(x) \cap U_\alpha$ is a non-void open set, therefore Int $f_\alpha(E) \neq \varnothing$.

Thus f is a quasi-open, finite to one, closed continuous mapping from X^* onto X. According to Theorem 1, X is an M_1-space.

Subset F of space X is called regular closed, if $F = \overline{\text{Int } F}$. Evidently, F is regular closed if and only if F is the closure of an open subset. By means of this concept, above Theorem 4 may be stated as follows:

"Let $\{F_\alpha\}_{\alpha \in A}$ be a locally finite regular closed covering of space X. If each F_α ($\alpha \in A$) is an M_1-space, then X is an M_1-space."

Whether every stratifiable space is an M_1-space, the partial result in this direction is due to G. Gruenhage [7].

THEOREM (*Gruenhage*). *Every stratifiable space which has a countable covering consisting of closed discrete subsets of X, is an M_1-space.*

Gruenhage's theorem may be stated in a more general form as follows.

THEOREM 5. *Every stratifiable space, which has a σ-hereditarily closure preserving covering consisting of closed discrete subsets of X, is an M_1-space.*

The proof of Theorem 5 follows from the following lemmas.

LEMMA 3. *Let F be a closed discrete subset of X. Then $\{\{x\}: x \in F\}$ is a discrete collection of subsets of X. If the space X is T_1, the converse is also true.*

LEMMA 4. *If X is T_1 space, the subset of a closed discrete subset of X is a closed discrete subset.*

LEMMA 5. *Let $\mathfrak{F}$ be a discrete collection of closed discrete subsets of X. Then $\mathfrak{F}^*$ is a closed discrete subset of X.*

The proofs of above lemmas are simple and direct.

LEMMA 6. *Let X be a T_1 space which has a σ-hereditarily closure preserving covering $\mathfrak{F}$ consisting of closed discrete subsets of X. Then X has a countable covering consisting of closed discrete subsets of X.*

Proof. Let $\mathfrak{F} = \bigcup_{n=1}^{\infty} \mathfrak{F}_n$, each $\mathfrak{F}_n$ ($n = 1, 2, \ldots$) being a hereditarily closure preserving collection consisting of closed discrete subsets of X. Let $\mathfrak{F}_n = \{F_{n,\alpha_n}\}_{\alpha_n \in A_n}$, each F_{n,α_n} is closed discrete subset. For each n, put

$$H_n = \mathfrak{F}_n^* - \bigcup_{i=1}^{n-1} \mathfrak{F}_i^*, \qquad H_{n,\alpha_n} = H_n \cap F_{n,\alpha_n} \quad (\alpha_n \in A_n).$$

By well ordering the index set A_n, put

$$F'_{n,\alpha_n} = H_{n,\alpha_n} - \bigcup_{\beta_n < \alpha_n} H_{n,\beta_n}.$$

Clearly $F'_{n,\alpha_n} \subset F_{n,\alpha_n}$. According to Lemma 4, F'_{n,α_n} is a closed discrete subset. $\mathfrak{F}'_n = \{F'_{n,\alpha_n}\}_{\alpha_n \in A_n}$ being closure preserving and pairwise disjoint is a discrete collection of closed discrete subsets. Hence, by the Lemma 5, $\mathfrak{F}'^*_n$ is a closed discrete subset of X. Furthermore

$$\bigcup_{n=1}^{\infty} \mathfrak{F}'^*_n = \bigcup_{n=1}^{\infty} \left(\bigcup_{\alpha_n \in A_n} F'_{n,\alpha_n} \right) = \bigcup_{n=1}^{\infty} H_n = \bigcup_{n=1}^{\infty} \mathfrak{F}^*_n = X.$$

Therefore $\{\mathfrak{F}'^*_n\}$ is a countable covering of X.

Proof of the Theorem 5. The proof follows from Lemma 6 and Gruenhage's theorem.

References

[1] A. Arhangelskii, *Test for the existence of a bicompact element in a continuous decomposition. Theorem on the invariance of weight in open closed finitely multiple mappings*, Dokl. SSSR, **166** (1966), 1263–1266.

[2] C. R. Borges and D. J. Lutzer, *Characterizations and Mappings of M_i-spaces*, Lecture Notes in Math. No. 375 (Springer-Verlag, Berlin, 1974), 34–40.

[3] D. K. Burke, R. Engelking and D. J. Lutzer, *Hereditarily closure-preserving collections and metrization*, Proc. Amer. Math. Soc., **51** (1975), 483–488.

[4] J. Ceder, *Some generalizations of metric spaces*, Pacific J. Math., **11** (1961), 105–125.

[5] R. F. Gittings, *Open Mapping Theory*, in Set-Theoretic Topology (Academic Press, 1977), 141–191.

[6] G. Gruenhage, *Stratifiable spaces are M_2*, Topology Proc., **1** (1976), 221–226.

[7] ——, *Stratifiable σ-discrete spaces are M_1*, Proc. Amer. Math. Soc., **72** (1978), 189–190.

[8] H. Junnila, *Neighbornets*, Pacific J. Math., **76** (1978), 83–108.

[9] E. Michael, *A quintuple quotient quest*, General Topology and Appl., **2** (1972), 91–138.

[10] F. Siwiec, *On the Theorem of Morita and Hanai, and Stone*, Lecture Notes in Math. No. 378 (Springer-Verlag, Berlin, 1974), 449–454.

Received December 22, 1981 and in revised form May 10, 1982.

KAINGSU TEACHERS' COLLEGE
SUCHOW, KAINGSU
CHINA