PACIFIC JOURNAL OF MATHEMATICS
Vol 109, No. 2, 1983

AN INTERPOLATION THEOREM FOR H_E^∞

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We prove a synthesis of Carleson's interpolation theorem, the Rudin-Carleson theorem and an interpolation theorem of S. A. Vinogradov.

Let D be the open unit disc in $\mathbf{C}$ and let T be its boundary. By $A(D)$ we mean the set of functions continuous on $\bar{D}$ analytic on D. H^∞ is the set of bounded analytic functions on D, and if E is a subset of T, H_E^∞ is the set of functions continuous on $D \cup E$ bounded and analytic on D.

The Rudin-Carleson theorem states that if K is a closed subset of T of measure zero, then $A(D) \mid K = C(K)$. This was proved independently by W. Rudin and L. Carleson [**8**], [**3**].

A sequence $\{z_n\} \subset D$ is said to be uniformly separated if

$$\inf_n \prod_{m \neq n} \left| \frac{z_n - z_m}{1 - \bar{z}_n z_m} \right| = \delta > 0.$$

Carleson's interpolation theorem states that $H^\infty \mid \{z_n\} = l^\infty$ if and only if $\{z_n\}$ is uniformly separated. This was first proved in [**2**]. Other proofs can be found in [**5**] and [**10**].

Let $F \subset \mathbf{N} \cup \{0\}$. A function $f(z) = \sum a_n z^n \in H^1$ is said to be an F function if $a_n = 0$ for $n \notin F$. For a definition and properties of the H^p spaces see [**4**]. F is said to be of type $\Lambda(s)$ if for every $r < s$ there is a constant K depending on F, r and s only such that $\|f\|_s \leq K \|f\|_r$ for every F function. If $F = \{n_k\}$ satisfies $n_{k+1}/n_k > \lambda > 1$, then F is of type $\Lambda(s)$ for every $s \in \langle 0, \infty \rangle$. Other sets of type $\Lambda(s)$ exist. See [**7**]. Let $\{n_k\}$ be of type $\Lambda(2)$ and let R be the operator from $A(D) \to l^2$ defined by $R(\sum a_n z^n) = \{a_{n_k}\}$. S. A. Vinogradov proved that R is onto. In fact he proved much more. See [**11**].

These results do not live their own lives separate from each other. In [**6**] E. A. Heard and J. H. Wells proved that if E is an open subset of T and S is a relatively closed subset of $D \cup E$ such that $S \cap E$ has measure zero and $S \cap D$ is uniformly separated, then $H_E^\infty \mid S = C_b(S)$, the space of all bounded continuous functions on S. Vinogradov proved in [**11**] that if K is a closed subset of T of measure zero, $g \in C(K)$ and $\{b_k\} \in l^2$, then there is an $f \in A(D)$ such that $f|K = g$ and $R(f) = \{b_k\}$. We intend to prove:

THEOREM. *Let E be an open subset of T and assume that S is a relatively closed subset of $D \cup E$ such that $S \cap E$ has measure zero, $S \cap D$ is uniformly separated and $0 \notin S$. Assume $F = \{n_k\}$ is an increasing sequence of integers of type $\Lambda(2)$ such that $\lim_{k\to\infty}(n_{k+1} - n_k) = \infty$. If $\beta(S) \in C_b(S)$ and $\{b_k\} \in l^2$, there is a function $f(z) = \sum a_n z^n \in H_E^\infty$ such that $f|S = \beta$ and $a_{n_k} = b_k$ for all k.*

REMARK. $0 \notin S$ represents no loss of generality since we may have $0 \in \{n_k\}$.

Before proving the theorem, we are going to develop some background material. Let $S \cap D = \{z_n\}$ and let

$$\inf_n \prod_{m \neq n} \left| \frac{z_m - z_n}{1 - \bar{z}_m z_n} \right| = \delta > 0.$$

Then there exists a real number M with the following property: Given $\{w_n\} \in \text{ball } l^\infty$, we can find a real number α and a Blaschke product $B(z)$ such that $Me^{i\alpha}B(z_n) = w_n$ for all n. The zeros $\{\xi_n\}$ of $B(z)$ can be chosen to satisfy $\psi(z_n, \xi_n) < \delta$ where $\psi(a, b) = |(a - b)/(1 - \bar{a}b)|$ is the pseudo-hyperbolic metric on D. This shows that $B(z)$ has analytic continuation across $T \backslash \{z_n\}$. The result is due to J. Earl [5]. We want to prove that the mass of the Taylor coefficients of $B(z)$ regarded as an element of H^2 is concentrated on the first coefficients.

LEMMA 1. *Let $B(z) = \sum a_n z^n$ be as above. If $\varepsilon > 0$ then there is an integer $N = N(\varepsilon)$ independent of $\{\xi_n\}$ such that $\sum_{n=N}^\infty |a_n|^2 < \varepsilon$.*

Proof. ε is now fixed. Let

$$B_K(z) = \prod_{n=K}^\infty \frac{|\xi_n|}{\xi_n} \cdot \frac{\xi_n - z}{1 - \bar{\xi}_n z}.$$

Since $\psi(\xi_n, z_n) < \delta$, a calculation shows that

$$1 - |\xi_n| \leq (2/(1 - \delta))(1 - |z_n|).$$

Hence $\lim_{K\to\infty} \sum_{n=K}^\infty (1 - |\xi_n|) = 0$ uniformly in $\{\xi_n\}$. This shows that $B_K(0) \underset{K\to\infty}{\to} 1$. Since $\|B_K\|_2 = 1$, $B_K(z) = \sum_{n=0}^\infty a_{n,K} z^n$ satisfies $\sum_{n=N_K}^\infty |a_{n,K}|^2 < \varepsilon/2$ for $N_K = 1$ if K is chosen large.

$$B_{K-1}(z) = B_K(z) \cdot \frac{|\xi_{K-1}|}{\xi_{K-1}} \cdot \frac{\xi_{K-1} - z}{1 - \bar{\xi}_{K-1} z}.$$

We have

$$B_K(z) = \sum_{n=0}^{N_K} a_{n,K} z^n + \sum_{n=N_K+1}^{\infty} a_{n,K} z^n = p(z) + \varepsilon_p(z)$$

where $\|\varepsilon_p\|_2^2 < \varepsilon/2$ and $\|p\|_2 \leq 1$.

$$\frac{|\xi_{K-1}|}{\xi_{K-1}} \cdot \frac{\xi_{K-1} - z}{1 - \bar{\xi}_{K-1} z} = \sum_{n=0}^{\infty} b_n(\xi_{K-1}) z^n.$$

Since $\psi(z_{K-1}, \xi_{K-1}) < \delta$ this converges uniformly on D independent of ξ_{K-1}. Choose R such that

$$\sum_{n=0}^{R} b_n(\xi_{K-1}) z^n + \sum_{n=R+1}^{\infty} b_n(\xi_{K-1}) z^n = q(z) + \varepsilon_q(z)$$

satisfies $\|\varepsilon_q\|_\infty < \eta$, $\|q\|_\infty < 1 + \eta$ where η is to be chosen below. We have

$$B_{K-1} = (p + \varepsilon_p)(q + \varepsilon_q) = pq + \varepsilon_p q + p\varepsilon_q + \varepsilon_p \varepsilon_q.$$

pq is a polynomial of degree $N_K + R$. It is not the $(N_K + R)$-partial sum of the Taylor series of B_{K-1}, but deleting coefficients decreases the $\|\ \|_2$ norm. For $B_{K-1}(z) = \sum C_n z^n$ we therefore have

$$\begin{aligned}\left(\sum_{n=R+N_K+1}^{\infty} |C_n|^2\right)^{1/2} &= \|B_{K-1}(z) - \sum_{n=0}^{R+N_K} C_n z^n\|_2 \\ &\leq \|\varepsilon_p \cdot q\|_2 + \|p\varepsilon_q\|_2 + \|\varepsilon_p \varepsilon_q\|_2 \\ &\leq \|\varepsilon_p\|_2 \cdot \|q\|_\infty + \|p\|_2 \cdot \|\varepsilon_q\|_\infty + \|\varepsilon_p\|_2 \cdot \|\varepsilon_q\|_\infty \\ &\leq \sqrt{\varepsilon/2}\,(1 + \eta) + \eta + \sqrt{\varepsilon/2} \cdot \eta < \sqrt{3\varepsilon/4}\end{aligned}$$

if η is chosen small. Continuing in the same way, the lemma is proved in a finite number of steps.

We are now going to take a look at Vinogradov's theorem. If $F = \{n_k\}$ is of type $\Lambda(2)$, the mapping R: $A(D) \to l^2$: $\sum a_n z^n \to \{a_{n_k}\}$ is onto. The open mapping theorem gives that R(ball $A(D)$) $\supseteq c$ ball l^2 for some $c > 0$. To obtain an estimate for c we need a result of Smirnov. Let $f(\xi)$ be integrable over the unit circle and let

$$h(z) = \frac{1}{2\pi} \int_T \frac{f(\xi)}{\xi - z} d\xi.$$

Then $h \in H^{1/2}$ and $\|h\|_{1/2} \leq K_1 \|f\|_1$. For a proof see p. 35 of **[4]** or **[11]**. Since F is of type $\Lambda(2)$, we have $\|f\|_2 \leq K_2 \|f\|_{1/2}$ for every F function in

H^2. Vinogradov proves his theorem by showing that the adjoint mapping $R^*\colon (l^2)^* = l^2 \to A(D)^*$ satisfies

$$\|R^*(x)\| \geq (1/2\pi K_1 K_2)\|x\|.$$

This is proved more generally on the first seven pages of [**11**]. Using a result of Banach, Lemma 4.13 of [**9**], we get $R(\text{ball } A(D)) \supseteq (1/2\pi K_1 K_2)\text{ball } l^2$ if by ball we mean open ball. Our balls are open from now on.

If $F = \{n_k\}_{k=1}^{\infty}$, consider the set $F' = \{n_k - n_K\}_{k=K+1}^{\infty}$. F' is also of type $\Lambda(2)$, and it is not difficult to see that the associated constant $K_2' \leq K_2$. If R' is the operator from $A(D)$ to l^2 associated with F' we see that $R'(\text{ball } A(D)) \supseteq (1/2\pi K_1 K_2)\text{ball } l^2$.

The proof of the theorem will also make use of

LEMMA 2. *Let $T\colon X \to Y$ be a continuous linear mapping between Banach spaces. Assume there are constants $\varepsilon < 1$ and M such that for all $y \in$ ball B there is $x \in X$ such that $\|x\| < M$ and $\|Tx - y\| < \varepsilon$. Then T is onto.*

For a proof see [**1**]. We now prove the theorem. Assume first that $S \cap E = \varnothing$. Choose an integer K such that $f_K(z) = B_K(z)/B_K(0) = 1 + \varepsilon(z)$ satisfies $\|f_K\|_\infty < 2$ and $\|\varepsilon\|_2 < 1/4\pi K_1 K_2$. Let $B_K \cdot H_E^\infty$ be the subspace of H_E^∞ consisting of the functions that vanish at z_n for $n \geq K$. Given $\{b_k\} \in$ ball l^2, choose $g(z) = \sum a_n z^n \in A(D)$ such that $a_{n_k} = b_k$ for all k and $\|g(z)\|_\infty \leq 2\pi K_1 K_2$. Let

$$g_K(z) = g(z) f_K(z) = \sum c_n z^n \in B_K H_E^\infty,$$
$$\|g_K(z)\|_\infty \leq 4\pi K_1 K_2$$

and

$$\|\{b_k - c_{n_K}\}\|_2 \leq \|\varepsilon(z) g(z)\|_2 \leq \|\varepsilon(z)\|_2 \cdot \|g(z)\|_\infty < 1/2.$$

Lemma 2 now proves that $R(B_K H_E^\infty) = l^2$. Let $\{w_n\}_{n=K}^{\infty} \in l^\infty$ and $\{b_k\} \in l^2$ be given. Choose $h(z) = \sum d_n z^n \in H_E^\infty$ such that $h(z_n) = w_n$ for $n \geq K$ and choose $j(z) = \sum l_n z^n \in B_K H_E^\infty$ such that $l_{n_k} = b_k - d_{n_k}$ for all k. The function $r(z) = h(z) + j(z) = \sum t_n z^n$ satisfies $r(z_n) = w_n$ for $n \geq K$ and $t_{n_k} = b_k$ for all k. This proves the theorem for $\{z_n\}_{n=1}^{\infty}$ replaced by $\{z_n\}_{n=K}^{\infty}$. The proof will be complete if we can prove that K can be replaced by $K - 1$. To obtain this it is enough to find a function $f(z) = \sum a_n z^n \in B_K \cdot H_E^\infty$ such that $a_{n_k} = 0$ for all k and $f(z_{K-1}) = 1$.

Such a function is likely to exist because it is easy to prove that there are many functions in $B_K H_E^\infty$ with F coefficients zero. All these functions could, however, vanish at z_{K-1} (a black hole). In that case, then for every $f(z) = \Sigma a_n z^n \in B_K \cdot H_E^\infty$, $f(z_{K-1})$ would be a function of $\{a_{n_k}\}$ alone.

Let $f(z) = \Sigma a_n z^n \in B_K H_E^\infty$. Look at $f(z_{K-1}) \leftarrow f(z) \overset{R}{\rightarrow} \{a_{n_k}\} \in l^2$. $\{a_{n_k}\} \rightarrow f(z_{K-1})$ is now seen to be a well-defined linear functional on l^2 since R is onto. This functional is continuous since every $x \in$ ball l^2 comes from a function of norm $< C$ as an application of the open mapping theorem shows. Therefore there exists a unique $\{\lambda_k\} \in l^2$ such that

$$(*) \qquad f(z_{K-1}) = \sum_k a_{n_k}\lambda_k \quad \text{for every } f(z) = \sum a_n z^n \in B_K \cdot H_E^\infty .$$

Infinitely many $\lambda_k \neq 0$. If this were not so, let λ_M be the largest. If $f(z) \in B_K H_E^\infty$. Then $z^{n_{M+1}} f(z)$ would vanish at z_{K-1}. This is clearly impossible. Since $\{\lambda_k\}$ is unique, the relation $(*)$ is impossible if we delete some n_N from F for which $\lambda_N \neq 0$. If we do so, K can be replaced by $K - 1$. We may choose n_N arbitrary large. Doing so we have pushed the problem from $\{z_n\}$ to F. We now prove that n_N can be replaced.

Let $\{z_n^*\} = \{z_n\}_{n=K-1}^\infty \cup \{0\}$. Every sequence $\{w_n\} \in$ ball l^∞ can be interpolated at $\{z_n^*\}$ by a function of the form $Me^{i\alpha}B(z) = \Sigma l_n z^n$ as pointed out above. Choose an integer 0 independent of $\{w_n\}$ such that

$$(**) \qquad \left(\sum_{n=Q}^\infty |l_n|^2 \right)^{1/2} < \frac{1}{10\pi K_1 K_2} .$$

This is possible by Lemma 1.

Choose n_N such that $\lambda_N \neq 0$ and $n_{N+1} - n_N > Q$. Let $F' = \{n_k - n_N\}_{k=N+1}^\infty$ and let

$$\mathfrak{B} = \left\{ f(z) = \sum a_n z^n \in H_E^\infty : a_n = 0 \text{ for } n \in F' \right\}.$$

We want to prove that $\mathfrak{B} \mid \{z_n^*\} = l^\infty$. Let $\{w_n\} \in$ ball l^∞ be given. Choose α and $B(z)$ as above such that $Me^{i\alpha}B(z_n^*) = w_n$ for all n. Choose $h(z) = \Sigma b_n z^n \in A(D)$ such that $b_n = l_n$ for $n \in F'$ and such that $\|h(z)\| \leq \frac{1}{5}$. This is possible by $(**)$ and the remark following Vinogradov's theorem. $f(z) = Me^{i\alpha}B(z) - h(z)$ has the following properties: $f \in \mathfrak{B}$, $\|f\| \leq M + \frac{1}{5}$, $|f(z_n^*) - w_n| < \frac{1}{5}$. Lemma 2 now proves that $\mathfrak{B} \mid \{z_n^*\} = l^\infty$. n_N can now be replaced: Let $\{w_n\} \in l^\infty$, $\{b_k\} \in l^2$. Take $f(z) = \Sigma a_n z^n \in H_E^\infty$ such that $f(z_n) = w_n$ for $n \geq K - 1$ and $a_{n_k} = b_k$ for $n_k \in F \setminus \{n_N\}$.

Choose $g(z) \in \mathfrak{B}$ such that $g(0) = 1$, $g(z^*) = 0$ for $z_n^* \neq 0$. Let $r(z) = z^{n_N} g(z) = \sum t_n z^n$. We have: $r(z_n) = 0$ for $n \geq K - 1$, $t_n = 0$ for $n \in F \setminus \{n_N\}$, $t_{n_N} = 1$. Our interpolation problem is now solved by the function $f(z) + \lambda r(z)$ for a proper choice of λ.

The proof is now complete except we assumed $S \cap E = \varnothing$. Using the Heard and Wells result, we may assume $\beta | S = 0$. Let $E' = E \setminus S$, $\mathcal{C} = \{f \in H_E^\infty \colon f | S = 0\}$, $\mathcal{C}' = \{f \in H_{E'}^\infty \colon f | S = 0\}$. The proof will be complete if we can prove $R(\mathcal{C}) = l^2$. By what we have just proved and the open mapping theorem, $R(k \cdot \text{ball } \mathcal{C}') \supseteq \text{ball } l^2$ for some constant k. Now choose $g \in H_E^\infty$ such that $g = 0$ on $S \cap T$, $\|g\| \leq 1$ and $g(z) = 1 + \varepsilon(z)$ satisfies $\|\varepsilon(z)\|_2 < 1/2k$. This is possible by Lemma 4 of **[6]**. Let $\{b_k\} \in \text{ball } l^2$. Take $f(z) = \sum a_n z^n \in \mathcal{C}'$ such that $\|f\| \leq k$ and $a_{n_k} = b_k$ for all k. $h(z) = f(z) g(z) = \sum c_n z^n$ satisfies: $h \in \mathcal{C}$, $\|h\| \leq k$,

$$\|\{c_{n_k} - b_k\}_k\|_2 \leq \|\varepsilon(z)\|_2 \cdot \|f(z)\|_\infty < 1/2.$$

Lemma 2 now proves $R(\mathcal{C}) = l^2$ and the proof is complete.

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Received February 25, 1981.

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