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A CONSTRUCTION OF INNER MAPS PRESERVING THE HAAR MEASURE ON SPHERES

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We show, for $n \geq m$, the existence of non-trivial inner maps f: $B^n \to B^m$ with boundary values f_*: $S^n \to S^m$ such that $f_*^{-1}(A)$ has a positive Haar measure for every Borel subset A of S^m which has a positive Haar measure. Moreover, if $n = m$, the equality $\sigma(f_*^{-1}(A)) = \sigma(A)$ holds, where σ is the Haar measure of S^m.

In this paper $\mathbf{C}^n$ is an n-dimensional complex space with inner product defined by $\langle z^1, z^2 \rangle = \sum z_i^1 \bar{z}_i^2$, where $z^j = (z_1^j, z_2^j, \ldots, z_n^j)$ for $j = 1, 2$, and the norm $|z| = \langle z, z \rangle^{1/2}$. Let us introduce some notation:

$$B^n = \{z \in \mathbf{C}^n : |z| < 1\}, \qquad S^n = \partial B^n;$$

let d be the metric on S^n:

$$d(z, z^*) = (1 - \operatorname{Re}\langle z, z^* \rangle)^{1/2} = \frac{1}{\sqrt{2}} |z - z^*| \quad \text{for } z, z^* \in S^n,$$

and finally

$$B(z, r) = \{z^* \in S^n : d(z, z^*) < r\} \quad \text{for } z \in S^n \text{ and } r > 0.$$

For every complex function h: $X \to \mathbf{C}$ we define $Z(h) = h^{-1}(0)$. A holomorphic map f: $B^n \to B^m$ is called inner if

$$f_*(z) = \lim_{r \to 1} f(rz) \in S^m \quad \text{for almost every } z \in S^n$$

with respect to the unique, rotation-invariant Borel measure σ_n on S^n such that $\sigma_n(S^n) = 1$. If a continuous function g: $\bar{B}^n \to \mathbf{C}^m$, defined on the closure of B^n, is holomorphic on B^n, we write $g \in A_m(B^n)$ or $g \in A(B^n)$ when $m = 1$. The theorem stated below is a generalization of the result of Aleksandrov [**1**]. Corollary 1 answers the problem given by Rudin [**3**]. Corollary 4 is a result of Aleksandrov obtained independently by the author.

THEOREM. *Let $n \geq m$ and let $g = (g_1, \ldots, g_m) \in A_m(B^n)$, $h \in A(B^n)$ be maps such that $|g(z)| + |h(z)| \leq 1$ and $h(z) \neq 0$ for some $z \in B^n$. Then there exists an inner map $f = (f_1, f_2, \ldots, f_m)$: $B^n \to B^m$ such that $f(z) = g(z)$ for every $z \in Z(h)$ and $f_i(z) = g_i(z)$ for every $z \in B^n$ and $i = 1, 2, \ldots, m - 1$.*

COROLLARY 1. *For every $n \geq m$ there exist inner maps f: $B^n \to B^m$ such that for every Borel subset $A \subset S^m$ the inequality $\sigma_n(f_*^{-1}(A)) > 0$ holds provided $\sigma_m(A) > 0$. Moreover, if $m = n$, the equality $\sigma_n(f_*^{-1}(A)) = \sigma_n(A)$ holds and f is not an automorphism of B^n.*

COROLLARY 2. *For every $n \geq 1$ there exist inner maps f: $B^n \to B^m$, not automorphisms of B^n, such that*

$$\int_{S^n} (h \circ f_*)\, d\sigma_n = \int_{S^n} h\, d\sigma_n$$

for every continuous function h on S^n.

Corollary 2 is an immediate consequence of Corollary 1. Let us assume that $n \geq m$ and $n \geq 2$. To deduce the assertion of Corollary 1 from the Theorem let us take a holomorphic function $k \in A(B^1)$ and the map $g \in A_m(B^n)$, $g(z) = p(z) + \frac{1}{4}z_n^2 r(z_n)$, where $p(z) = (z_1, z_2, \ldots, z_{m-1}, 0)$, $r(z) = (0, \ldots, 0, k(z_n))$ for $z \in B^n$. Define $h(z) = \frac{1}{4}z_1 z_n^2$. Then

$$|g(z)| + |h(z)| \leq |p(z)| + \frac{1}{4}|z_n^2| + \frac{1}{4}|z_n^2| \leq \sqrt{1 - z_n^2} + \frac{1}{2}|z_n^2| \leq 1.$$

By virtue of the Theorem there exists an inner map $f = (f_1, f_2, \ldots, f_m)$: $B^n \to B^m$ such that

$$f_j(z_1, z_2, \ldots, z_n) = z_j \quad \text{for } j = 1, 2, \ldots, m-1, \tag{1}$$

$$f_m(0, 0, \ldots, 0, z_n) = \frac{1}{4}z_n^2 r(z_n), \tag{2}$$

$$f(z_1, z_2, \ldots, z_{n-1}, 0) = (z_1, z_2, \ldots, z_{m-1}, 0). \tag{3}$$

For any $z \in B^{m-1}$ and $l \geq m$ let

$$B_z^l = \left\{ z^* \in B^l : z_j^* = z_j \text{ for } j = 1, 2, \ldots, m-1 \right\},$$

$$S_z^l = \left\{ z^* \in S^l : z_j^* = z_j \text{ for } j = 1, 2, \ldots, m-1 \right\},$$

let σ_z^l be the rotation-invariant measure on the sphere S_z^l such that $\sigma_z^l(S_z^l) = 1$ and let f_z, f_z^* be the restrictions of f, f_* to the sets B_z^n and S_z^n respectively. From (1) it follows that f_z: $B_z^n \to B_z^m$ and (2) says that $f_z(w_1) = w_2$, where w_1, w_2 are the centers of the balls B_z^n, B_z^m respectively. Since B_z^m is a one-dimensional complex ball, the equality $\sigma_z^n((f_z^*)^{-1}(C)) = \sigma_z^m(C)$ holds for every Borel subset C of S_z^m and every z for which f_z is an inner map (see **[4]** p. 405). The function f_z is inner for almost every $z \in B^{m-1}$ (with respect to the usual Lebesgue measure λ on B^{m-1}) because the map f is inner. Let us notice that there are positive functions

$s_1, s_2\colon B^{m-1} \to R_+$ such that for all Borel subsets $C^1 \subset S^n$, $C^2 \subset S^m$ we have

$$\sigma_n(C^1) = \int_{B^{m-1}} s_1(z) \cdot \sigma_z^1(C_z^1)\, d\lambda(z),$$

$$\sigma_m(C_2) = \int_{B^{m-1}} s_2(z) \cdot \sigma_z^m(C_z^2)\, d\lambda(z),$$

where $C_z^1 = C^1 \cap S^n$, $C_z^2 = C^2 \cap S^m$. Substituting $C_1 = (f^*)^{-1}(C_2)$ and using the equality $\sigma_z^n(C_z^1) = \sigma_z^m(C_z^2)$ (which holds for almost every z), it is easy to see that both of the above integrals are positive or equal to 0. If $n = m$ then $s_1 = s_2$ and the equality holds. This ends the proof of Corollary 1.

The following proof of the assertion of the Theorem is based on Löw's construction of inner functions [3]. Let g and h be maps satisfying the assumptions of the Theorem. Then $\sigma_n(F) = 0$, where $F = Z(h) \cap S^n$. (This fact can be proved by induction. For $n = 1$ it is well-known theorem.) For $\delta > 0$ let

$$F_\delta = \{z \in S^n : d(z, F) < \delta\} \quad \text{and} \quad |||s|||_\delta = \sup_{z \in F_\delta} |s(z)|,$$

where $s\colon S^n \to \mathbf{C}^m$ is a continuous map. Observe that there exist constants A_1, A_2 such that for every $0 < r < \sqrt{2}$,

$$A_1 r^{2n-1} \le A(r) \le A_2 r^{2n-1}, \tag{4}$$

where $A(r) = \sigma_n(B(z, r))$ for any $z \in S^n$.

Let $S \subset S^n$ be any closed subset of S^n, $\sigma_n(S) > 0$. Assume that for some number $r > 0$,

$$\sigma_n(S_r) \le 2\sigma_n(S), \tag{5}$$

where $S_r = \{z \in S^n : d(z, S) < r\}$. Let $\{B(z^j, r)\}_{j=1}^{N(r)}$ be a maximal family of disjoint balls with centers $z^j \in S$. Since $S_r \supset \bigcup_{j=1}^{N(r)} B(z^j, r)$ and $S \subset \bigcup_{j=1}^{N(r)} B(z^j, 2r)$, applying inequalities (4) and (5), we get

$$\begin{aligned} 2\sigma_n(S) \ge \sigma_n(S_r) &\ge \sigma_n\left(\bigcup_{j=1}^{N(r)} B(z^j, r)\right) = \sum_{j=1}^{N(r)} \sigma_n(B(z^j, r)) \\ &= N(r) \cdot A(r) \ge A_1 r^{2n-1} \cdot N(r) \end{aligned}$$

and

$$\sigma_n(S) \le \sigma_n\left(\bigcup_{j=1}^{N(r)} B(z^j, 2r)\right) = \sum_{j=1}^{N(r)} A(2r) = N(r)\cdot A(2r)$$
$$\le N(r)\cdot A_2 \cdot (2r)^{2n-1} = N(r)\cdot A_2 \cdot 2^{2n-1}\cdot r^{2n-1}.$$

So we have proved the existence of positive constants C_1 and C_2 ($C_1 = 1/2^{2n-1}$, $C_2 = 2/A_1$) such that

$$\text{(6)}\qquad \frac{C_1}{r^{2n-1}}\cdot\sigma_n(S) \le N(r) \le \frac{C_2}{r^{2n-1}}\cdot\sigma_n(S).$$

Let us assume now that $r > 0$, $z \in B^n$, k is a natural number and M_k is the maximal number of disjoint balls of radius r and with centers in $B(z,(k+1)r)$. Because these balls are included in $B(z,(k+2)r)$, an argument similar to the above gives the estimate

$$\text{(7)}\qquad M_k \le C_3 k^{2n-1}$$

for some constant C_3. Let φ: $(0,1) \to R$ be the continuous, positive function defined by

$$\varphi(a) = \frac{1}{4\pi}\cdot C_1\cdot A_1\cdot \arccos(a)\cdot\left[\log\frac{1}{a}\right]^{(2n-1)/2}.$$

LEMMA 1. *Let* $0 < 2\varepsilon < a < b$, $0 < \delta < 2C_3\cdot a$, $\varepsilon < C_3 e^{-2n}$, $R < 1$. *Let P be a closed subset of F_δ and let v be a continuous map v: $S^n \to \mathbf{C}^m$ such that $|v(z)| > a$ for $z \in P$. There exists a closed subset K of F_δ and a holomorphic map u: $\mathbf{C}^n \to \mathbf{C}^m$ such that*:

(a) $\quad |||v + h\cdot u|||_{\delta/2} \le \max(1, |||f|||_{\delta/2}) + 3\varepsilon$;

(b) $\quad \|u\|_R = \sup_{|z|\le R} |u(z)| \le \varepsilon$;

(c) $\quad |v(z) + h(z)\cdot u(z)| > a - 3\varepsilon$ *for* $z \in K \cup P$;

(d) $\quad K \subset F_\delta$, $\ K \cap P = 0$ *and*
$$\sigma_n(K) \ge \varphi(a)\cdot[\log(4C_3/\delta\varepsilon)]^{-(2n-1)/2}\cdot\sigma_n(F_\delta - P);$$

(e) $\quad |g(z)| < \varepsilon$ *for* $z \in B^n - F_{\delta/2}$;

(f) $\quad u_j \equiv 0$ *for* $j = 1, 2, \ldots, m-1$, *where* $u = (u_1, u_2, \ldots, u_m)$.

Proof. If $\sigma_n(P) = \sigma_n(F_\delta)$ then the map $u = (0, 0, \ldots, 0)$ and the set $K = \varnothing$ satisfy conditions (a)–(e). Let us assume that $\sigma_n(P) < \sigma_n(F_\delta)$.

There exists a positive number γ such that $\gamma < \delta/2$ and

$$(8) \qquad \sigma_n(S) \geq \tfrac{1}{2} \cdot \sigma_n(F_\delta - P),$$

where $S = S^n - [(S^n - F_\delta) \cup P]_\gamma$.

Since v, h are uniformly continuous maps and S is a closed subset, there exists a positive number γ^* such that

$$(9) \quad |g(z) - g(z')| < \varepsilon\delta, \quad |v(z) - v(z')| < \varepsilon, \quad \sigma_n(S_r) \leq 2 \cdot \sigma_n(S)$$
$$\text{for } z, z' \in S^n, d(z, z') < \gamma^* \text{ and } r < \gamma^*.$$

Let r, m be positive numbers such that $r \leq \frac{1}{2}\min(\gamma, \gamma^*)$, m is an integer and $mr^2 = \log(2C_3/\delta\varepsilon)$. Moreover we assume m is large so that

$$(10) \qquad C_2 \cdot m^{(2n-1)/2} \cdot e^{-m(1-R)} < \varepsilon.$$

Choose a maximal family $\{B(z^j, r)\}_{j=1}^{N(r)}$ of pairwise disjoint balls with centers $z^j \in S^n$. Because of (9), condition (5) is satisfied, so inequalities (6) also hold. For $k = 1, 2, \ldots, [\sqrt{2}/r]$ and $z \in S^n$ let

$$V_k(z) = \{z^j : kr \leq d(z, z^j) < (k+1)r\}$$

and let $N_k(z)$ be the number of elements of the set V_k. Since $V_k(z) \subset B(z, (k+1)r)$, from the definition of M_k, we have $N_k(z) \leq M_k$ and (7) gives us

$$(11) \qquad N_k(z) \leq C_3 k^{2n-1}.$$

Let $g(z) = \sum_{j=1}^{N(r)} \beta_j e^{-m(1-(\langle z, z^j\rangle))}$, where $\beta_j = (0, 0, \ldots, 0, \alpha_j) \in \mathbf{C}^m$ is defined by $\beta_j = (0, 0, \ldots, 0, 0)$ if $|f(z^j)| \geq b$. If $|f(z^j)| < b$, then let β_j be of the previous form, such that

$$|f(z^j) + h(z) \cdot \beta_j| = b \quad \text{and} \quad |f(z^j) + \alpha \cdot h(z) \cdot \beta_j| \leq b$$

for every $\alpha \in \mathbf{C}$, $|\alpha| = 1$. Let us notice that for every j, $|\beta_j| \leq 1/|h(z^j)| \leq 1/\delta$ and that

$$\begin{aligned} g(z) &= \vec{k} \cdot \sum_{j=1}^{N(r)} |\beta_j| \cdot e^{-md^2(z, z^j)} \cdot e^{iQ_{m,j}(z)} \\ &= \vec{k} \cdot \sum_{k=0}^{[\sqrt{2}/r]} \sum_{z^j \in V_k(z)} |\beta_j| e^{-md^2(z, z^j)} e^{iQ_{m,j}(z)} \end{aligned}$$

for some real functions $Q_{m,j}$ and $\vec{k} = (0, 0, \ldots, 0, 1) \in \mathbf{C}^m$.

If $V_0(z) = \varnothing$ or $z \in B(z^j, r)$ with $\beta_j = 0$ then, because of (11) and the inequality $mr^2 > 2n$, we have

$$(12)\quad |g(z)| \leq \sum_{k=1}^{[\sqrt{2}/r]} \sum_{z^j \in V_k(z)} \frac{1}{\delta} e^{-md^2(z,z^j)} \sum_{k=1}^{[\sqrt{2}/r]} \frac{1}{\delta} |V_k(z)| e^{-mk^2r^2}$$

$$\leq \sum_{k=1}^{\infty} \frac{C_3}{\delta} k^{2n-1} e^{-k^2mr^2} \leq \frac{C_3}{\delta} \sum_{k=1}^{\infty} e^{-kmr^2} \leq 2\frac{C_3}{\delta} e^{-mr^2} = \varepsilon.$$

This proves part (e) of Lemma 1.If $z \in B(z^j, r)$ with $\beta_j \neq 0$ then

$$(13)\quad |v(z) + h(z) \cdot u(z)|$$

$$\leq \left| v(z^j) + h(z^j) \cdot \beta_j \cdot e^{-md^2(z,z^j)} \cdot e^{iQ_{m,j}(z)} \right|$$

$$+ \left| [h(z) - h(z^j)] \cdot \beta_j \cdot e^{-md^2(z,z^j)} \cdot e^{iQ_{m,j}(z)} \right| + |v(z) - v(z^j)|$$

$$+ \left| h(z) \cdot \sum_{z^j \notin V_0(z)} \beta_j \cdot e^{-md^2(z,z^j)} \cdot e^{iQ_{m,m}(z)} \right|$$

$$= \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}.$$

Because of (9)

$$\mathrm{III} \leq \varepsilon \quad \text{and} \quad \mathrm{II} \leq |h(z) - h(z^j)| \cdot |\beta_j| < \delta \cdot \varepsilon \cdot \frac{1}{\delta} = \varepsilon.$$

By the same argument as in (12) we can prove that IV $\leq \varepsilon$. Moreover, we have I $\leq |v(z^j)| + |h(z^j) \cdot \beta_j| = b$. This altogether gives us

$$(14)\qquad |v(z) + h(z) \cdot u(z)| \leq b + 3\varepsilon.$$

Inequalities (12) and (14) prove part (a) of Lemma 1. Now we shall determine a certain subset V of $W = \bigcup_{j=1}^{N(r)} B(z^j, r)$. To do this let us fix j, $1 \leq j \leq N(r)$, and let us take $\alpha = |v(z_j)|$, $s(z) = e^{-md^2(z,z^j)}$, $Q(z) = \arg(e^{-m(1-(\langle z, z^j \rangle))}) = m \cdot \operatorname{Im}\langle z, z^j \rangle$.

Let us assume at first that $\alpha < 1$. We define

$$V_j = \{ z \in B(z^j, r) : s(z) \geq a \text{ and } \cos Q(z) \geq a \}.$$

Using the same notation as in (13) we can write

$$(15)\qquad |v(z) + h(z) \cdot u(z)| \geq \mathrm{I} - \mathrm{II} - \mathrm{III} - \mathrm{IV}.$$

As before, II $\leq \varepsilon$, III $\leq \varepsilon$ and IV $\leq \varepsilon$. Assuming $z \in V_j$, we have

$$\text{(16)} \qquad \begin{aligned} \mathrm{I} &= \left| v(z^j) + h(z^j) \cdot \beta_j \cdot e^{-m \cdot d^2(z, z^j)} \cdot e^{iQ(z)} \right| \\ &\geq \left| \alpha + (1 - \alpha) \cdot s(z) \cdot e^{iQ(z)} \right| \\ &= \sqrt{\alpha^2 + 2\alpha(1 - \alpha) \cdot s(z) \cdot \cos Q(z) + (1 - \alpha)^2} \geq a \end{aligned}$$

because of our assumption about $s(z)$ and $\cos Q(z)$, the definition of β_j and simple geometry.

Combining (15) and (16) we get

$$\text{(17)} \qquad |v(z) + h(z) \cdot u(z)| > a - 3\varepsilon \quad \text{for } z \in V_j.$$

Let $\rho > 0$ be defined by $m\rho^2 = \log(1/a)$. Then $\rho \leq r$ because $mr^2 = 2C_3/\delta\varepsilon$ and $2C_3/\delta \geq 1/a$. So $B(z^j, \rho) \subset B(z^j, r)$, and if $z \in B(z^j, \rho)$ then $s(z) \geq a$. The set $\{z \in B(z^j, \rho) : \cos Q \geq a\}$ consists of certain strips in the ball $B(z^j, \rho)$. An easy geometric argument shows that these strips have a total area at least

$$\frac{1}{2\pi} \cdot \arccos a \cdot \sigma_n\big(B(z^j, \rho)\big) = \frac{1}{2\pi} \cdot \arccos a \cdot A(\rho).$$

Moreover $V_j \subset B(z^j, r) \subset F_\delta$. Using inequality (4) and the fact that the above strips are included in V_j, we get

$$\text{(18)} \qquad \sigma_n(V_j) \geq \frac{1}{2\pi} \cdot \arccos a \cdot A(\rho) \geq \frac{1}{2\pi} \cdot A_1 \cdot \arccos a \cdot \rho^{2n-1}.$$

If $\alpha \geq 1$, we define $V_j = B(z^j, \rho)$. Because $\beta_j = 0$, it follows from (12) that

$$\text{(19)} \qquad \begin{aligned} |v(z) + h(z) \cdot u(z)| &\geq |v(z^j)| - |v(z) - v(z^j)| - |h(z) \cdot u(z)| \\ &\geq a - \varepsilon - |u(z)| \geq a - 2\varepsilon \end{aligned}$$

for $z \in V_j$.

Finally, we define $K = \bigcup_{j=1}^{N(r)} \overline{V_j}$. We observe that inequality (17) holds for $z \in K$. If $z \in P$, then $V_0(z) = \varnothing$ and inequality (12) gives us

$$|v(z) + h(z) \cdot u(z)| \geq |v(z)| - |u(z)| \geq a - \varepsilon.$$

This altogether proves part (c) of Lemma 1. It is easy to check that $K \cap P = \varnothing$. Inequalities (18), (6), (9) and the definitions of ρ and mr^2 yield

$$\begin{aligned}
\sigma_n(K) &\geq \sigma_n\left(\bigcup_{j=1}^{N(r)} V_j\right) = \sum_{j=1}^{N(r)} \sigma_n(V_j) \\
&\geq N(r) \cdot \frac{1}{2\pi} \cdot A_1 \cdot \arccos a \cdot \rho^{2n-1} \\
&\geq \frac{C_1}{r^{2n-1}} \cdot \sigma_n(S) \cdot \frac{1}{2\pi} \cdot A_1 \cdot \arccos a \cdot \rho^{2n-1} \\
&\geq \frac{1}{4\pi} \cdot C_1 \cdot A_1 \cdot \arccos a \cdot (mr^2)^{-(2n-1)/2} \cdot (m\rho^2)^{2n-1} \cdot \sigma_n(F_\delta - P) \\
&= \varphi(a) \cdot \log(4C_3/(\delta\varepsilon))^{-(2n-1)/2} \cdot \sigma_n(F_\delta - P).
\end{aligned}$$

This proves part (d) of Lemma 1. Finally, if $|z| \leq R$ then $\operatorname{Re}(1 - \langle z, z^j \rangle) \leq 1 - R$ for $j = 1, 2, \ldots, N(r)$. Because of the inequalities $mr^2 \geq 1$, (10) and (6), we have

$$\begin{aligned}
|u(z)| \leq N(r) \cdot e^{-m(1-R)} &\leq C_2 \cdot \frac{1}{r^{2n-1}} \cdot e^{-m(1-R)} \\
&= C_2 \cdot m^{(2n-1)/2} \cdot e^{-m(1-R)} \cdot (mr^2)^{-(2n-1)/2} \\
&\leq C_2 \cdot m^{(2n-1)/2} \cdot e^{-m(1-R)} \leq \varepsilon.
\end{aligned}$$

This proves part (d) of Lemma 1 and ends the proof.

LEMMA 2. *Let v be a continuous map $v\colon S^n \to \mathbf{C}^m$ such that $|||v|||_\delta < b < 1$ for some $\delta < C_3$. Let $\frac{1}{4} > \varepsilon > 0$, $R < 1$. Then there exists a holomorphic map $u\colon \mathbf{C}^n \to \mathbf{C}^m$ and a closed set $K \subset F_\delta$ such that:*

(a)′ $|||v + h \cdot u|||_\delta < b + \varepsilon$;

(b)′ $\|u\|_R \leq \varepsilon$;

(c)′ $|v(z) + h(z) \cdot u(z)| > b - \varepsilon$;

(d)′ $\sigma_n(K) \geq \sigma_n(F_\delta) - \varepsilon$;

(e)′ $|u(z)| \leq \varepsilon$ *for* $z \in S^n - F_\delta$;

(f)′ $u_j \equiv 0$ *for* $j = 1, 2, \ldots, m - 1$, *where* $u = (u_1, u_2, \ldots, u_m)$.

Proof. Let $a = b - \frac{1}{2}\varepsilon$ and choose a sequence $\{\varepsilon_j\}$ satisfying the assumptions of Lemma 1 and such that $6\sum_{j=1}^{\infty} \varepsilon_j < \varepsilon$. We can assume $\varepsilon_j = A \cdot \exp\{-(\tau \cdot j)^{2/(2n-1)}\}$, $A = 2C_3/\delta$ and τ is some large number.

Apply Lemma 1 to the data a, ε_1, R, v, $P = \varnothing$ to produce a holomorphic map u_1: $\mathbf{C}^n \to \mathbf{C}^m$ and a closed set $K_1 \subset F_\delta$ such that:

(a)$_1$ $\quad |||v + h \cdot u_1|||_\delta \le b + 3\varepsilon_1$;

(b)$_1$ $\quad \|v_1\|_R \le \varepsilon_1$;

(c)$_1$ $\quad |v(z) + h(z) \cdot u_1(z)| \ge a - 3\varepsilon_1 \quad$ for $z \in K_1$;

(d)$_1$ $\quad \alpha_1 = \sigma_n(K_1) \ge \varphi(a) \cdot [\log(A/\varepsilon_1)]^{-(2n-1)/2} \cdot \sigma_n(F_\delta)$;

(e)$_1$ $\quad |u_1(z)| \le \varepsilon_1 \quad$ for $z \in S^n - F_\delta$;

(f)$_1$ $\quad u_j^1 \equiv 0 \quad$ for $j = 1, 2, \dots, m-1$, where $u_1 = (u_1^1, u_2^1, \dots, u_m^1)$.

Suppose that holomorphic maps $u_1, u_2, \dots, u_{p-1}$ (u_j: $\mathbf{C}^n \to \mathbf{C}^m$ for $j = 1, 2, \dots, p-1$) have been chosen together with closed sets $K_1, K_2, \dots, K_{p-1}$ such that if $W_i = \bigcup_{j=1}^i K_j$ then $K_{i+1} \cap W_i = \varnothing$ and $\sigma_n(K_i) = \alpha_i$, $K_i \subset F_\delta$. A map u_p: $\mathbf{C}^n \to \mathbf{C}^m$ and a closed set K_p is then obtained by applying Lemma 1 to the data $a - 3\sum_{i=1}^{p-1}\varepsilon_i$, ε_p, R, $v + h(z) \cdot (u_1 + u_2 + \cdots + u_{p-1})$, W_{p-1}. This produces a sequence $\{v_k\}$ of holomorphic maps (v_k: $\mathbf{C}^n \to \mathbf{C}^m$ for $k = 1, 2, \dots$) and a sequence $\{K_k\}$ of disjoint closed sets such that $K_k \subset F_\delta$, $\sigma_n(K_k) = \alpha_k$ and:

(a)$_p$
$$|||v + h \cdot \sum_{k=1}^p u_k|||_\delta \le b + 3 \cdot \sum_{k=1}^p \varepsilon_k < b + \varepsilon;$$

(b)$_p$
$$\left\| \sum_{k=1}^p u_k \right\|_R \le \sum_{k=1}^p \|u_k\|_R \le \sum_{k=1}^p \varepsilon_k < \varepsilon;$$

(c)$_p$
$$\left| v(z) + h(z) \cdot \sum_{k=1}^p u_k(z) \right| \ge a - 3 \cdot \sum_{k=1}^p \varepsilon_k$$
$$\ge a - \tfrac{1}{2}\varepsilon = b - \varepsilon \quad \text{for } z \in W_p;$$

(d)$_p$
$$\begin{aligned} \alpha_p &= \sigma_n(K_p) \\ &\ge \varphi\left(a - 3 \cdot \sum_{k=1}^{p-1} \varepsilon_i\right) \cdot \left[\log \frac{A}{\varepsilon_p}\right]^{-(2n-1)/2} \cdot \left(\sigma_n(F_\delta) - \sum_{k=1}^{p-1} \alpha_k\right) \\ &\ge \varphi(a) \cdot \left[\log \frac{A}{\varepsilon_p}\right]^{-(2n-1)/2} \cdot \left(\sigma_n(F_\delta) - \sum_{k=1}^{p-1} \alpha_k\right); \end{aligned}$$

(e)$_p$
$$\left| \sum_{k=1}^p u_k(z) \right| \le \sum_{k=1}^p |u_k(z)| \le \sum_{k=1}^p \varepsilon_k < \varepsilon \quad \text{for } z \in S^n - F_\delta;$$

(f)$_p$
$$u_j^k \equiv 0 \quad \text{for } k = 1, 2, \dots, p \text{ and } j = 1, 2, \dots, m-1,$$
$$\text{where } u_k = (u_1^k, u_2^k, \dots, u_m^k).$$

If $\sum_{k=1}^{\infty} \alpha_k < \sigma_n(F_\delta)$, (d) shows that there is a constant C_4 such that for every positive integer k,

$$\alpha_p \geq C_4 \cdot \left[\log \frac{A}{\varepsilon_p}\right]^{-(2n-1)/2} = \left[C_4 \cdot (\tau p)^{2/(2n-1)}\right]^{-(2n-1)/2} = \frac{C_4}{\tau p}.$$

This is impossible, because then $\sum_{p=1}^{\infty} \alpha_p = \infty$ and α_p are the measures of the disjoint sets. Hence, we may assume that $\sum_{k=1}^{\infty} \alpha_k = \sigma_n(F_\delta)$. It follows that for p sufficiently large and $P = W_p$ we have $\sigma_n(P) = \sum_{k=1}^{p} \alpha_k > 1 - \varepsilon$, which is part (d)′ of Lemma 2. Letting $h = \sum_{k=1}^{p} u_k$, parts (a)′, (b)′, (c)′, (e)′, (f)′ are just $(\mathrm{a})_p$, $(\mathrm{b})_p$, $(\mathrm{c})_p$, $(\mathrm{e})_p$, $(\mathrm{f})_p$. So we have proved the assertion of Lemma 2.

Assume now that g and h satisfy the assumptions of the Theorem. Then $|||g|||_\delta \leq 1 - \delta$. To prove the Theorem, take a sequence $\delta_1, \delta_2, \ldots$ of positive numbers such that $\delta_1 < C_3$ and $\delta_{i+1} < \delta_i/2$ and let $a_1 = b_1 = 1 - \frac{1}{2}\delta_1$, $\varepsilon_1 = \min(\frac{1}{16}, \frac{1}{4}\delta_1)$, $R_1 = \frac{1}{2}$. Apply Lemma 2 to the data $g_1 = g$, b_1, δ_1, R_1 to get a map u_1 and a set $K_1 \subset F_{\delta_1}$ such that, for $p = 1$ and $g_1 = g$:

$(\mathrm{i})_p$ $\quad |||g_p + h \cdot u_p|||_{\delta_p} < b_p + \varepsilon_p < 1;$

$(\mathrm{ii})_p$ $\quad \|u_p\|_{R_p} \leq \varepsilon_p;$

$(\mathrm{iii})_p$ $\quad |g_p(z) + h(z) \cdot u_p(z)| > b_p - \varepsilon_p \quad$ for $z \in K_p$;

$(\mathrm{iv})_p$ $\quad \sigma_n(K_p) \geq \sigma_n(F_{\delta_p}) - \varepsilon_p;$

$(\mathrm{v})_p$ $\quad 1 - |g_p(z) + h(z) \cdot u_p(z)|$

$$\geq \left(1 - \sum_{i=1}^{p} \varepsilon_i\right)|h(z)| \quad \text{for } z \in S^n - F_{\delta_p};$$

$(\mathrm{vi})_p$ $\quad u_j^p \equiv 0 \quad$ for $j = 1, 2, \ldots, m-1$ where $u_p = (u_1^p, u_2^p, \ldots, u_m^p)$.

Inequality (v) follows from (e)′ of Lemma 2, because for $z \in S^n - F_{\delta_1}$, we have $|u_1(z)| < \varepsilon_1$, so

$$1 - |v(z) + h(z) \cdot u_1(z)| \geq 1 - |v(z)| - |u_1(z) \cdot h(z)|$$
$$\geq |h(z)| - \varepsilon_1 \cdot |h(z)| = (1 - \varepsilon_1) \cdot |h(z)|.$$

Since $g_1 + h \cdot u_1$ is a continuous map on $\overline{B}^n$, there exists an R_2 such that $\frac{1}{2} + \frac{1}{2}R_1 < R_2 < 1$ and, for $p = 1$,

$(\mathrm{vii})_p$ $\quad |g_p(R_{p+1} \cdot z) + h(R_{p+1} \cdot z) \cdot u_p(R_{p+1} \cdot z)| > b_p - 2\varepsilon_p$

for $z \in K_p$.

Suppose we have inductively found holomorphic maps $u_1, u_2, \ldots, u_p$, closed sets $K_1, K_2, \ldots, K_p$, real numbers $R_1, R_2, \ldots, R_{p+1}$, $b_1, b_2, \ldots, b_p$, $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_p$ such that $\frac{1}{2} + \frac{1}{2}R_i < R_{i+1}$, $\varepsilon_i > 0$ for $i = 1, 2, \ldots, p$ and $\sum_{i=1}^{p} \varepsilon_i < 1/8$. Let us assume $g_{j+1} = g + h \cdot \sum_{i=1}^{j} u_i$ and conditions (i)$_j$–(vii)$_j$ are satisfied for $j = 1, 2, \ldots, p$. We also assume that $1 - 1/j \leq b_j < b_j + \varepsilon_j < 1$. If $z \in (F_{\delta_{p+1}} - F_{\delta_p})$ then according to (v)$_p$, we have

$$1 - |g_{p+1}(z)| \geq \left(1 - \sum_{i=1}^{p} \varepsilon_i\right) \cdot |h(z)| \geq \frac{1}{2} \cdot \delta_{p+1}$$

since $|h(z)| \geq \delta_{p+1}$. This, together with (i)$_p$, shows that $|||g_{p+1}|||_{\delta_{p+1}} < 1$. Take any $b_{p+1} > 1 - 1/(p+1)$ and ε_{p+1} satisfying the inequalities $1 > b_{p+1} + \varepsilon_{p+1} > b_{p+1} > |||g_{p+1}|||_{\delta_{p+1}}$ and $\sum_{i=1}^{p+1} \varepsilon_i < 1/8$. Since the map g_{p+1} is continuous on $\overline{B}^n$, we can find a number R_{p+2} such that $\frac{1}{2} + \frac{1}{2}R_{p+1} < R_{p+2} < 1$ and such that condition (vii)$_{p+1}$ is satisfied. Now we can apply Lemma 2 to the data $g_{p+1}, b_{p+1}, \varepsilon_{p+1}, R_{p+1}$. We get some map u_{p+1} and a set K_{p+1}. It follows from Lemma 2 that conditions (i)$_{p+1}$–(iv)$_{p+1}$ and (vi)$_{p+1}$ are satisfied. For $z \in S^n - F_{\delta_{p+1}}$, by the virtue of (e)′ and (v)$_p$, we have

$$\begin{aligned}
1 - &|g_{p+1}(z) + h(z) \cdot u_{p+1}(z)| \\
&\geq 1 - |g_p(z) + h(z) \cdot u_p(z)| - |h(z) \cdot u_{p+1}(z)| \\
&\geq \left(1 - \sum_{i=1}^{p} \varepsilon_i\right) \cdot |h(z)| - |h(z)| \cdot \varepsilon_{p+1} \\
&= \left(1 - \sum_{i=1}^{p+1} \varepsilon_i\right) \cdot |h(z)|.
\end{aligned}$$

So we have also proved that condition (v)$_{p+1}$ is satisfied. Conditions (ii)$_p$ ($p = 1, 2, 3 \ldots$) and the definition of g_p say that the sequence $\{g_p\}$ is convergent uniformly on every ball $R_p \cdot B^n$, and since $\lim_{p \to 1} R_p = 1$, this sequence is pointwise convergent to some holomorphic map f on the ball B^n. From conditions (i)$_p$ and (v)$_p$ it follows that each map g_p is bounded by 1 on B^n. So, also $\|f\|_\infty \leq 1$. For $\delta > 0$ let $L_p = F_\delta \cap \bigcap_{j>p} K_j$. Then, for q large enough, $F_\delta \subset F_{\delta_p}$ for $p > q$. We have

$$\begin{aligned}
\sigma_n(F_\delta) - \sigma_n(L_q) &= \sigma_n\left(\bigcup_{j>q} (F_\delta - (F_\delta \cap K_q))\right) \\
&\leq \sum_{j>q} \sigma_n(F_\delta - (F_\delta \cap K_j)) \leq \sum_{j>q} \sigma_n(F_{\delta_j} - K_j) < \sum_{j>q} \varepsilon_j.
\end{aligned}$$

Hence $\lim_{q\to\infty}\sigma_n(L_q) = \sigma_n(F_\delta)$. It is obvious from $(\text{iii})_p$ and the equality $\lim_{p\to\infty} b_p = 1$ that $\lim_{R\to 1} f(Rz) = 1$ for $z \in L_q$, provided this limit exists. Since δ was arbitrary, this proves that the map f is inner, since $\sigma_n(\bigcap_p(S^n - F_{\delta_p})) = 0$. Now it is easy to check that f satisfies the Theorem.

COROLLARY 3. *Let $m < n$ and let $g \in A_m(B^m)$, $\|g\|_\infty \leq 1$. There exists an inner map f: $B^n \to B^m$ such that*

$$f(z_1, z_2,\ldots,z_m, 0, 0,\ldots,0) = g(z_1, z_2,\ldots,z_m).$$

Proof. Let Φ: $B^m \to B^m$ be an automorphism of B^m such that $\Phi(g(0,\ldots,0)) = (0,\ldots,0)$. Take $\tilde{g}$: $B^m \to B^m$, $\tilde{g}(z) = \Phi(g(z_1, z_2,\ldots,z_m))$, $h(z) = \frac{1}{2}\cdot z_n^2$. By virtue of Schwartz's lemma,

$$\tilde{g}(z) \leq \left(|z_1|^2 + |z_2|^2 + \cdots + |z_m|^2\right)^{1/2}.$$

So we have

$$|\tilde{g}(z)| + |h(z)| \leq \left(1 - |z_n|^2\right)^{1/2} + \tfrac{1}{2}\cdot|z_n|^2 \leq 1.$$

We can apply the Theorem for g and h to get an inner map $\tilde{f}$. The inner map $f = \Phi^{-1}(\tilde{f})$ will satisfy Corollary 3.

COROLLARY 4. *There exists an inner function f: $B^n \to D$ such that*

$$\frac{\partial f}{\partial z_1}(0, 0,\ldots,0) = 1.$$

Proof. Take $m = 1$ in Corollary 3 and a function g: $B^1 \to D$, $g(z) = z$.

REMARK. The assumption $g \in A_m(B^m)$ in Corollary 3 is not necessary: we can take any holomorphic map g: $B^m \to B^m$. Then the map $\tilde{g}$, defined as before, can be prolonged to a continuous map on $\overline{B}^n - A$, where $A \subset S^n$ and $\sigma_n(A) = 0$. One can check that the Theorem is still valid for such maps.

References

[1] A. B. Aleksandrov, *The existence of inner functions on the unit ball*, Mat. Sb. 1982, v. 118 (160) N2(6), 147–163 (in Russian).
[2] E. Löw, *A construction of inner functions on the unit ball of* $\mathbf{C}^p$, Invent. Math., **67** (1982), 223–229.
[3] W. Rudin, *Inner functions in the unit ball*, preprint.
[4] ______, *Function Theory in the Unit Ball of* $\mathbf{C}^n$, Springer, New York, 1980.

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