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RANDOM PERMUTATIONS AND BROWNIAN MOTION

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Consider the cycles of the random permutation of length n. Let $X_n(t)$ be the number of cycles with length not exceeding n^t, $t \in [0,1]$. The random process $Y_n(t) = (X_n(t) - t\ln n)/\ln^{1/2} n$ is shown to converge weakly to the standard Brownian motion $W(t)$, $t \in [0,1]$. It follows that, as a process, the empirical distribution function of "loglengths" of the cycles weakly converges to the Brownian Bridge process. As another application, an alternative proof is given for the Erdös-Turán Theorem: it states that the group-order of random permutation is asymptotically $e^{\mathscr{U}}$, where $\mathscr{U}$ is Gaussian with mean $\ln^2 n/2$ and variance $\ln^3 n/3$.

1. Introduction. Results. Consider S_n, the symmetric group of permutations of a set $\{1,\ldots,n\}$ endowed with the uniform distribution, $P(\sigma) = 1/n!$ for each $\sigma \in S_n$. Since a pioneering work by Goncharov [**10**], [**11**], a considerable attention has been paid to the asymptotic study of the order sequence of cycles lengths for the random permutation (r.p.), and of components sizes for the random mapping (Kolchin, et al. [**13**], [**14**], Shepp and Lloyd [**20**], Balakrishnan, et al. [**1**], Stephanov [**21**], Vershik and Shmidt [**22**]). Let $X_{ns} = X_{ns}(\sigma)$ designate the random number of cycles of length s in the r.p. σ. It is known [**11**] that X_n, the total number of cycles, it asymptotically normal with mean and variance $\ln n$. A similar result holds true for the total number of cycles whose lengths are divisible by a given number, [**4**], [**20**]. In this paper, we study the asymptotical behavior of the *joint* distribution of $X_{n1},\ldots,X_{nn}$.

For each $t \in [0,1]$, consider

$$(1.1) \qquad X_n(t) = \sum_{1\le s\le n^t} X_{ns}, \qquad Y_n(t) = \big(X_n(t) - t\ln n\big)/\ln^{1/2} n;$$

so, $X_n(t)$ is the total number of cycles of the r.p. with lengths not exceeding n^t. Clearly, each sample function of $Y_n(\cdot)$ belongs to $D[0,1]$ the space of functions on $[0,1]$ which are right-continuous at each $t \in [0,1)$ and have left limits at each $t \in (0,1]$. Introduce $W(t)$, $t \in [0,1]$, the standard Brownian motion defined on a complete probability space with continuous sample paths. Let $\mathscr{H}$ be a class of functionals on $D[0,1]$ continuous in the sup-norm metric.

THEOREM. *$Y_n(\cdot)$ converges to $W(\cdot)$ in terms of finite dimensional distributions. Moreover, for each $H \in \mathscr{H}$, the random variable $H(Y_n(\cdot))$ converges weakly to $H(W(\cdot))$; in short, $Y_n \Rightarrow W$.*

Notes. Since $(X_n - \ln n)/\ln^{1/2} n = Y_n(1)$, the Goncharov result is a direct corollary of the theorem.

(2) To each cycle of the r.p., let us assign its "loglength" which is the logarithm of the cycle length with base n. Clearly, all the loglengths are in $[0, 1]$. Introduce the empirical distribution function (e.d.f.) $F_n(t)$, $t \in [0, 1]$, of the loglengths, that is, $F_n(t) = X_n(t)/X_n$. The theorem yields, after simple manipulations, that, as a process, $\ln^{1/2} n(F_n(t) - t)$, $t \in [0, 1]$, converges weakly ($\Rightarrow$) to $W(t) - tW(1)$, $t \in [0, 1]$. Thus, the asymptotical behavior of the loglengths is very nearly the same as of that for a sequence of $[\ln n]$ *independent* random variables each *uniformly* distributed on $[0, 1]$, **[9]**.

(3) Consider Z_n and P_n respectively the order and the product of the cycle lengths of the r.p. Erdös and Turán **[5]** proved that $\ln P_n$ and $\ln Z_n$ are relatively close in probability, as $n \to \infty$. Later **[6]**, they established, via very complicated argument, asymptotic normality of $\ln P_n$, whence of $\ln Z_n$. Best **[4]** found a simpler proof of closeness of $\ln P_n$ and $\ln Z_n$, but his proof that $\ln P_n$ is nearly normal remains rather technical. We are aware of, but have not seen, two other published proofs (Kolchin **[15]**, Pavlov **[18]**) of the Erdös-Turán theorem.

Let us show how this theorem follows from our result.

First, we prove that, for each $\alpha > 2$,

$$P\left(\Delta_n \geq \ln n(\ln\ln n)^{\alpha}\right) \to 0, \qquad n \to \infty, \tag{1.2}$$

where $\Delta_n = \ln P_n - \ln Z_n$. (Our proof resembles the Best argument, but is much simpler.) Introduce

$$D_{nk} = \sum_{s=1}^{n} \theta_s(k) X_{ns}, \qquad \theta_s(k) = \begin{cases} 1 & \text{if } k|s, \\ 0 & \text{otherwise.} \end{cases}$$

Since $E(X_{ns}) = 1/s$, $E(X_{ns}(X_{ns'} - \delta_{ss'})) = 1/ss'$, $s + s' \leq n$, a simple computation leads to

$$\begin{gathered} E(D_{nk}) = \sum_{s=1}^{n} \theta_s(k)/s = O(\ln n/k), \\ E(D_{nk}(D_{nk} - 1)) \leq \left(\sum_{s=1}^{n} \theta_s(k)/s\right)^2 = O(\ln^2 n/k^2), \end{gathered} \tag{1.3}$$

the estimates being uniform in $k \leq n$.

Denote $D^*_{nk} = \min(1, D_{nk})$. Since the multiplicity of a prime factor p in P_n (resp. Z_n) is $\sum_{s\geq 1} D_{np^s}$ (resp. $\sum_{s\geq 1} D^*_{np^s}$), we have

$$\ln P_n = \sum_p \sum_{s\geq 1} D_{np^s} \ln p, \qquad \ln Z_n = \sum_p \sum_{s\geq 1} D^*_{np^s} \ln p,$$

so that

$$\Delta_n \leq \sum_{k\geq 1} \left(D_{nk} - D^*_{nk}\right) \ln k = \sum_{k\geq 1} \Delta_{nk} \ln k.$$

As $\Delta_{nk} \leq D_{nk}$, $\Delta_{nk} \leq D_{nk}(D_{nk} - 1)/2$, we obtain (see (1.3)),

$$E(\Delta_n) \leq c\left(\ln n \sum_{k=1}^{[\ln n]} \ln k/k + \ln^2 n \sum_{k>[\ln n]} \ln k/k^2\right) = O\left(\ln n (\ln\ln n)^2\right).$$

Since $\Delta_n \geq 0$, the last estimate implies (1.2).

Second, we prove that $\ln P_n$ is asymptotically normal with mean $2^{-1}\ln^2 n$ and variance $3^{-1}\ln^3 n$. (Then, in view of (1.2), $\ln Z_n$ has the same limiting distribution.) Introducing $t_{ns} = \ln s/\ln n$, $1 \leq s \leq n$, and summing up by parts, we have (see (1.1))

$$\begin{aligned}\ln P_n &= \sum_{1\leq s\leq n} X_{ns} \ln s = \ln^2 n\left[1 - \sum_{1\leq s\leq n-1} t_{ns}\left(t_{n,s+1} - t_{ns}\right)\right]\\ &\quad + \ln^{3/2} n\left[Y_n(1) - \sum_{1\leq s\leq n-1} Y_n(t_{ns})\left(t_{n,s+1} - t_{ns}\right)\right]\\ &= \ln^2 n\left[2^{-1} + O(\ln^{-2} n)\right] + \ln^{3/2} n\left[Y_n(1) - \int_0^1 Y_n(t)\,dt\right].\end{aligned}$$

So, by the theorem,

$$\left(\ln P_n - 2^{-1}\ln^2 n\right)/\ln^{3/2} n \Rightarrow \int_0^1 \left(W(1) - W(t)\right) dt \overset{\mathcal{D}}{=} \int_0^1 W(t)\,dt.$$

It remains to observe that the last integral is normal with zero mean and variance 3^{-1}.

(4) For $\sigma \in S_n$, let $i \leq i_1 < \cdots < i_\nu \leq n$, $\nu = \nu(\sigma)$, be the locations of all the (upper forward) record values in σ. Consider the inter-record times $\Delta_j = i_{j+1} - i_j$, $1 \leq j \leq \nu$, $\Delta_{\nu+1} = n + 1 - i_\nu$, and let $R_{ns} = R_{ns}(\sigma)$ stand for the number of Δ's equal to s, $1 \leq s \leq n$. Since there exists a one-to-one mapping T of S_n onto itself such that

$$\{X_{ns}(\sigma)\}_{s=1}^n = \{R_{ns}(T(\sigma))\}_{s=1}^n,$$

([**12**], [**16**]), the sequences $\{R_{ns}\}_{s=1}^n$ and $\{X_{ns}\}_{s=1}^n$ are equidistributed. Thus, with no other proof needed, we could have formulated the analogues of the theorem, and the statement in (2), in terms of the inter-record times. The correspondinig results appear to be new, though the (inter)record times have been studied by many authors, [2], [8]. (For example, Neuts [17] proved asymptotic normality of the nth interrecord time in the (infinite) r.p. associated with a sequence of independent random variables with a common continuous distribution function.)

2. Proof of the theorem. The joint distribution of X_{ns}, $1 \leq s \leq n$, is given by Cauchy's formula [3]:

$$P(X_{ns} = \alpha_s, 1 \leq s \leq n) = \begin{cases} \prod_{s=1}^{n} \left((1/s)^{\alpha_s}/\alpha_s!\right), & \text{if } \sum_{s=1}^{n} s\alpha_s = n, \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

Introduce a bounded sequence $z = \{z_s\}_{s=1}^{\infty}$ and the sequence of generating functions (g.f.) $f_n(z) = E(\prod_{1 \leq s \leq n} z_s^{X_{ns}})$, $n \geq 1$, $f_0(z) \equiv 1$. It follows from (2.1) that, for $|t| < 1$,

$$\sum_{n \geq 0} t^n f_n(z) = \exp\left[\sum_{s \geq 1} z_s t^s/s\right], \tag{2.2}$$

[19], (cf. [20]). Fix the positive integers r, $l_1, \ldots, l_r$, and introduce the r-dimensional g.f. $g_n(y) = E(\prod_{\nu=1}^{r} y_\nu^{X_{nl_\nu}})$, $(X_{n_l} = 0$, for $l > n)$. Choosing in (2.2) $z_s = y_\nu$, if $s = l_\nu$ $(1 \leq \nu \leq r)$, and $z_s = 1$ otherwise, we obtain that $(g_0(y) \equiv 1)$

$$\begin{aligned} \sum_{n \leq 0} t^n g_n(y) &= \exp\left[\sum_{\nu=1}^{r} y_\nu t^{l_\nu}/l_\nu + \sum_{s \neq l_1, \ldots, l_\nu} t^s/s\right] \\ &= \exp\left[\sum_{\nu=1}^{r} (y_\nu - 1) t^{l_\nu}/l_\nu\right]/(1-t). \end{aligned} \tag{2.3}$$

Hence, by Cauchy's integral formula,

$$g_n(y) = (2\pi i)^{-1} \int_C \exp\left[\sum_{\nu=1}^{r} (y_\nu - 1) z^{l_\nu}/l_\nu\right]/\left((1-z)z^{n+1}\right) dz, \tag{2.4}$$

where C is any circle with radius less than one surrounding the origin in the complex plane. It is important that (2.4) holds for each set of positive integers n, r, $l_1, \ldots, l_r$.

Introduce a process

$$Y_n^*(t) = \left(\sum_{1 \leq s \leq [n^t]} (X_{ns} - 1/s)\right)/\ln^{1/2} n, \quad t \in [0,1]. \tag{2.5}$$

Since $\sum_{1 \leq s \leq \nu} 1/s - \ln \nu = 0(1)$, $\nu \to \infty$, it suffices to prove that $Y_n^* \Rightarrow W$. (Centering of X_{ns} by $1/s$ is natural since $E(X_{ns}) = 1/s$, $1 \leq s \leq n$.)

LEMMA 1. *For each fixed* k *and* $0 = t_0 < \cdots < t_k = 1$, *the random vector* $\{Y_n^*(t_j)\}_{j=1}^k$ *converges to* $\{W(t_j)\}_{j=1}^k$ *in distribution.*

Proof. For $1 \le j \le k$, let $n_j = [n^{t_j}]$ so that $n_0 = 1$, $n_k = n$. Denote

$$\mathscr{X}_{nj} = \sum_{s=n_{j-1}+1}^{n_j} X_{ns}, \qquad \bar{\mathscr{X}}_{nj} = \sum_{s=n_{j-1}+1}^{n_j} 1/s.$$

We have to show that $\{(\mathscr{X}_{nj} - \bar{\mathscr{X}}_{nj})/\ln^{1/2} n\}_{j=1}^k$ converges weakly to the Gaussian vector with k independent components having parameters $(0, t_j - t_{j-1})$, $1 \le j \le k$.

Introduce $x_j = \exp(u_j/\ln^{1/2} n)$, $u_j > 0$ and is fixed, $1 \le j \le k$. Setting $r = n$, $l_s = s$ for each s, and $y_\nu = x_j$ for $n_{j-1} + 1 \le \nu \le n_j$ in (2.4), and choosing the radius of C equal to $\rho = 1 - n^{-1}$, we have

$$\begin{aligned} (2.6) \qquad h_n(x) &= E\left(\prod_{j=1}^n x_j^{\mathscr{X}_{nj}}\right) \\ &= (2\pi\rho^n)^{-1} \exp\left[\sum_{j=1}^k (x_j - 1) \sum_{s=n_{j-1}+1}^{n_j} \rho^s/s\right] \cdot I, \end{aligned}$$

where

$$(2.7) \quad I = \int_{[-\pi,\pi)} e^{-in\phi} b_n(\phi)\, d\phi, \qquad b_n(\phi) = (1 - \rho e^{i\phi})^{-1} \exp[a_n(\phi)],$$

$$(2.8) \qquad a_n(\phi) = \sum_{j=1}^k (x_j - 1) \sum_{s=n_{j-1}+1}^{n_j} (\rho^s/s)(e^{is\phi} - 1).$$

To estimate I, we proceed as follows. Break $[-\pi, \pi)$ into $[-\phi_0, \phi_0]^c$, $[-\phi_0, \phi_0]$, $\phi_0 = n^{-3/4}$; let the corresponding integrals be I_1, I_2. First, we estimate I_1. In I_2, we replace $b_n(\phi)$ by $\tilde{b}_n(\phi)$, which is close to $b_n(\phi)$ for $\phi \in [-\phi_0, \phi_0]$, and nicely manageable if $\phi \in (-\infty, \infty)$. The resulting integral $\tilde{I}_2$ is a difference of two integrals J_1 and J_2, over respectively $(-\infty, \infty)$ and $(-\infty, \infty) - [-\phi_0, \phi_0]$. We estimate J_2. J_1, whose contribution in the value of I is dominant, is asymptotically *evaluated* by means of the inversion formula for an L_1-integrable characteristic function.

The proof follows.

(1) Show that

$$(2.9) \qquad I_1 = O(n^{-1/4}).$$

Integrating once by parts, we have

$$\left|\int_{[\phi_0,\pi]} e^{-in\phi} b_n(\phi)\, d\phi\right| \le n^{-1}\left[|b_n(\pi)| + |b_n(\phi_0)| + \int_{[\phi_0,\pi]} |b_n'(\phi)|\, d\phi\right].$$

Here, (see (2.7), (2.8)),

$$|b_n(\pi)| = O(\exp(\operatorname{Re} a_n(\pi))) = O(1),$$

since

$$\operatorname{Re}(a_n(\phi)) \le \sum_{j=1}^{k} (x_j - 1) \sum_{s=n_{j-1}+1}^{n_j} (\rho^s/s)(\cos s\phi - 1) \le 0,$$

$(x_j \ge 1, j = 1, \dots, k)$. Also,

$$(2.10) \quad |b_n(\phi_0)| \le |1 - \rho e^{i\phi_0}|^{-1} = \left[(1-\rho)^2 + 2\rho(1 - \cos\phi_0)\right]^{-1/2}$$

$$= O\left[(1 - \cos\phi_0)^{-1/2}\right] = O\left(\phi_0^{-1}\right).$$

Further, since

$$b_n'(\phi) = \rho i(1 - \rho e^{i\phi})^{-2} \exp[a_n(\phi)] + i(1 - \rho e^{i\phi})^{-1}$$

$$\times \exp[a_n(\phi)] \sum_{j=1}^{k} (x_j - 1) \sum_{s=n_{j-1}+1}^{n_j} (\rho e^{i\phi})^s,$$

estimate

$$|b_n'(\phi)| \le |1 - \rho e^{i\phi}|^{-2} + |1 - \rho e^{i\phi}|^{-2} \sum_{j=1}^{k} (x_j - 1)\left|1 - (\rho e^{i\phi})^{n_j - n_{j-1}}\right|$$

$$= O\left(|1 - \rho e^{i\phi}|^{-2}\right) = O\left[(1 - \cos\phi)^{-1}\right] = O(\phi^{-2}),$$

for $\phi \in [\phi_0, \pi]$. Therefore

$$\int_{[\phi_0, \pi]} |b_n'(\phi)| d\phi = O\left(\phi_0^{-1}\right),$$

and, together with (2.10), it yields

$$\left| \int_{[\phi_0, \pi]} e^{-in\phi} b_n(\phi)\, d\phi \right| = O\left((n\phi_0)^{-1}\right) = O(n^{-1/4}).$$

The case of $[-\pi, -\phi_0]$ is similar.

(2) To estimate I_2, compare it with

$$(2.11) \qquad \tilde{I}_2 = \int_{[-\phi_0, \phi_0]} e^{-in\phi}\, \tilde{b}_n(\phi)\, d\phi,$$

$$\tilde{b}_n(\phi) = [1 - \rho(1 + i\phi)]^{-1}(1 - i\phi)^{-1} \exp[a_n(\phi)].$$

Since $\rho = 1 - n^{-1}$ and $|e^{i\phi} - (1 + i\phi)| \le 2^{-1}\phi^2$,

$$\left|(1 - \rho e^{i\phi})^{-1} - [1 - \rho(1 + i\phi)]^{-1}(1 - i\phi)^{-1}\right|$$

$$\le 2^{-1}n^2\phi^2 + |1 - \rho(1 + i\phi)|^{-1}\left|1 - (1 - i\phi)^{-1}\right|$$

$$\le 2^{-1}n^2\phi^2 + n|\phi|.$$

Subsequently ($\phi_0 = n^{-3/4}$),

$$(2.12) \qquad \begin{aligned} |I_2 - \tilde{I}_2| &\le \int_{[-\phi_0, \phi_0]} |b_n(\phi) - \tilde{b}_n(\phi)| d\phi \\ &\le 2^{-1}n^2 \int_{[-\phi_0, \phi_0]} \phi^2 \, d\phi + n \int_{[-\phi_0, \phi_0]} |\phi| d\phi \\ &= O\left(n^2\phi_0^3 + n\phi_0^2\right) = O\left(n^{-1/4}\right). \end{aligned}$$

Thus, it suffices to estimate $\tilde{I}_2$. Notice first that $\tilde{b}_n(\cdot) \in L_1(-\infty, \infty)$; (one reason why the factor $(1 - i\phi)^{-1}$ is included in $\tilde{b}_n(\phi)$ is to have this happen). If so,

$$(2.13) \quad \begin{aligned} \tilde{I}_2 &= \int_{(-\infty, \infty)} e^{-in\phi}\tilde{b}_n(\phi) \, d\phi - \int_{[-\phi_0, \phi_0]^c} e^{-in\phi}\tilde{b}_n(\phi) \, d\phi \\ &= n^{-1}\int_{(-\infty, \infty)} e^{-iu}\tilde{b}_n(u/n) \, du - n^{-1}\int_{|u| \ge n\phi_0} e^{-iu}\tilde{b}_n(u/n) \, du \\ &= J_1 - J_2. \end{aligned}$$

(2a) Evaluate J_1. By (2.8), (2.11), we have

$$(2.14) \quad \begin{aligned} n^{-1}\tilde{b}_n(u/n) &= (1 - i\alpha_i u)^{-1}(1 - i\alpha_2 u)^{-1}\exp[a_n(u/n)] \\ &= (1 - i\alpha_1 u)^{-1}(1 - i\alpha_2 u)^{-1} \prod_{j=1}^{k} \prod_{s=n_{j-1}+1}^{n_j} \exp\left[\Lambda_{js}(e^{isu/n} - 1)\right] \end{aligned}$$

where

$$(2.15) \qquad \alpha_1 = \rho = 1 - n^{-1}, \qquad \alpha_2 = n^{-1},$$

$$(2.16) \qquad \Lambda_{js} = \left(x_j - 1\right)\rho^s/s.$$

Notice that

$$(1 - iu)^{-1} = E[\exp(iuV)], \qquad \exp\left[\Lambda(e^{iu} - 1)\right] = E[\exp(iu\mathscr{P}(\Lambda))],$$

where $V \ge 0$ is exponentially distributed with parameter 1, and $\mathscr{P}(\Lambda)$ is Poisson distributed with parameter Λ. Hence, a crucial observation:

$$(2.17) \qquad n^{-1}\tilde{b}_n(u/n) = E[\exp(iuM_n)],$$

$$M_n = \alpha_1 V_1 + \alpha_2 V_2 + \sum_{j=1}^{k} \sum_{s=n_{j-1}+1}^{n_j} (s/n)\mathscr{P}(\Lambda_{js}), \tag{2.18}$$

where V_1, V_2, $\{\mathscr{P}(\Lambda_{js})\}_{j,s}$ are all independent. Since $n^{-1}\tilde{b}_n(u/n) \in L_1(-\infty, \infty)$, M_n has a (bounded) continuous density f_{M_n} (of course, it is seen directly from (2.18)). Moreover, by the inversion formula [7], for each x

$$f_{M_n}(x) = (2\pi)^{-1}\int_{(-\infty,\infty)} e^{-iux}E\left[\exp(iuM_n)\right] du,$$

so

$$J_1 = n^{-1}\int_{(-\infty,\infty)} e^{-iu}\tilde{b}_n(u/n)\, du = 2\pi f_{M_n}(1).(!)$$

The density of $\alpha_1 V_1$ is $\alpha_1 \exp(-\alpha_1 x)$, $x \geq 0$; denote F_n the distribution function of

$$\tilde{M}_n = \alpha_2 V_2 + \sum_{j=1}^{k} \sum_{s=n_{j-1}+1}^{n_j} (s/n)\mathscr{P}(\Lambda_{js}).$$

Then

$$f_{M_n}(1) = \int_{(-\infty,1]} \alpha_1 \exp(-\alpha_1(1-x))\, dF_n(x). \tag{2.19}$$

Now (see (2.18)),

$$\begin{aligned} E(\tilde{M}_n) &= \alpha_2 + \sum_{j=1}^{k} \sum_{s=n_{j-1}+1}^{n_j} (s/n)\Lambda_{js} \\ &\leq n^{-1} + \left[\max_{1\leq j\leq k}(x_j - 1)\right] n^{-1}\sum_{s\geq 1}\rho^s \\ &= O\left(n^{-1} + \max_{1\leq j\leq k}(x_j - 1)\right) = O(\ln^{-1/2} n). \end{aligned}$$

Therefore, for each $\varepsilon > 0$,

$$\lim_{n\to\infty}\left[F_n(\varepsilon) - F_n(-\varepsilon)\right] = 1. \tag{2.20}$$

Since $\alpha_1 \to 1$ as $n \to \infty$, we get from (2.19), (2.20) that

$$\lim_{n\to\infty} f_{M_n}(1) = \lim_{n\to\infty} \alpha_1 \exp(-\alpha_1) = e^{-1}.$$

Thus, $J_1 \to 2\pi e^{-1}$. More precisely, since $\alpha_1 = 1 - n^{-1}$ and $E(\tilde{M}_n) = O(\ln^{-1/2} n)$,

$$J_1 = 2\pi e^{-1} + O(\ln^{-\delta} n), \quad \forall\delta \in (0, 1/2). \tag{2.21}$$

(2b). Estimate J_2. Integrating by parts, we have ($B_n(u) = n^{-1}\tilde{b}_n(u/n)$)

$$\left|\int_{u\geq n\phi_0} e^{-iu}B_n(u)\,du\right| \leq |B_n(n\phi_0)| + \int_{u\geq n\phi_0} |B_n'(u)|du.$$

Here (see (2.8), (2.14)),

$$(2.22)\quad |B_n(n\phi_0)| = O\left(|1 - i\alpha_1 n\phi_0|^{-1}\right) = O\left((n\phi_0)^{-1}\right) = O(n^{-1/4}),$$

$$|B_n'(u)| = O\left(u^{-2} + n^{-1}u^{-1}|1 - i\alpha_2 u|^{-1} + n^{-1}|a_n'(u/n)|nu^{-2}\right)$$
$$= O\left(u^{-2} + |a_n'(u/n)|u^{-2}\right),$$

and

$$(2.23)\qquad |a_n'(u/n)| = \left|\sum_{j=1}^{k}(x_j - 1)\sum_{s=n_{j-1}+1}^{n_j}(\rho e^{iu/n})^s\right|$$
$$\leq 2k\max_{1\leq j\leq k}(x_j - 1)|1 - \rho e^{iu/n}|^{-1},$$

$$(2.24)\qquad |1 - \rho e^{iu/n}|^{-1} \leq \left[n^{-2} + 1 - \cos(u/n)\right]^{-1/2}$$
$$= \begin{cases} O(nu^{-1}), & \text{if } u/n \leq \pi, \\ O(n), & \text{always.}\end{cases}$$

Putting together (2.22)–(2.24), we obtain

$$\int_{u\geq n\phi_0}|B_n'(u)|du = O\left(\int_{u>n\phi_0}u^{-2}du\right)$$
$$+O\left(\max_{1\leq j\leq k}(x_j - 1)\left(n\int_{[n\phi_0,\,n\pi]}u^{-3}\,du\right.\right.$$
$$\left.\left.+n\int_{u\geq\pi n}u^{-2}\,du\right)\right)$$
$$= O\left((n\phi_0)^{-1} + \max_{1\leq j\leq k}(x_j - 1)\right) = O(\ln^{-1/2} n).$$

Therefore (the case $u \leq -n\phi_0$ is similar),

$$(2.25)\qquad J_2 = O(\ln^{-1/2} n).$$

(3) Combining (2.9), (2.12), (2.13), (2.21) and (2.25), we can conclude:

$$I = J_1 + [I_1 + (I_2 - \tilde{I}_2) - J_2] = 2\pi e^{-1} + O(\ln^{-\delta} n), \qquad \delta \in (0, 1/2).$$

Hence, by (2.6) and $\rho^n = e^{-1}(1 + O(n^{-1}))$,

$$h_n(x) = E\left(\prod_{j=1}^{k} x_j^{\mathscr{X}_{nj}}\right)$$

$$= \exp\left[\sum_{j=1}^{k}(x_j - 1)\sum_{s=n_{j-1}+1}^{n_j}\rho^s/s\right](1 + O(\ln^{-\delta} n)).$$

What remains is to evaluate the first factor on the right. Since

$$x_j - 1 = \exp\left(u_j/\ln^{1/2} n\right) - 1 = u_j/\ln^{1/2} n + u_j^2/2\ln n + O(\ln^{-3/2} n),$$

$$\left|\sum_{s=n_{j-1}+1}^{n_j}(\rho^s/s - 1/s)\right| \le 1, \tag{2.26}$$

$$\left|\sum_{s=n_{j-1}+1}^{n_j} 1/s - \left(\ln n_j - \ln n_{j-1}\right)\right| \le 1, \tag{2.27}$$

$$\left|\left(\ln n_j - \ln n_{j-1}\right) - \left(t_j - t_{j-1}\right)\ln n\right| \le 1, \tag{2.28}$$

we have

$$E\left\{\exp\left[\left(\sum_{j=1}^{k} u_j\mathscr{X}_{nj}\right)/\ln^{1/2} n\right]\right\}$$

$$= \exp\left[\left(\sum_{j=1}^{k} u_j \sum_{s=n_{j-1}+1}^{n_j} 1/s\right)/\ln^{1/2} n\right]$$

$$\times\exp\left[2^{-1}\sum_{j=1}^{k} u_j^2\left(t_j - t_{j-1}\right)\right](1 + O(\ln^{-\delta} n)).$$

So, by definition of $\bar{\mathscr{X}}_{nj}$,

$$\lim_{n\to\infty} E\left\{\exp\left[\left(\sum_{j=1}^{k} u_j\left(\mathscr{X}_{nj} - \bar{\mathscr{X}}_{nj}\right)/\ln^{1/2} n\right)\right]\right\}$$

$$= \prod_{j=1}^{k}\exp\left[2^{-1}u_j^2\left(t_j - t_{j-1}\right)\right].$$

It follows from this relation that $\{(\mathscr{X}_{nj} - \bar{\mathscr{X}}_{nj})/\ln^{1/2} n\}_{j=1}^{k}$ converges in distribution to $\{W(t_j) - W(t_{j-1})\}_{j=1}^{k}$. Lemma 1 is proven.

To complete the proof of the Theorem, it suffices to show **[9]** that the processes $Y_n^*(\cdot)$ are equicontinuous, or more precisely, that for each $\varepsilon > 0$,

$$\lim_{c\downarrow 0}\ \limsup_{n\to\infty} P\left\{\sup_{|t''-t'|\le c}\left|Y_n^*(t'') - Y_n^*(t')\right| \ge \varepsilon\right\} = 0.$$

A method we shall use to prove it is inspired by a proof of equicontinuity of the e.d.f. processes $\xi_n(\cdot)$ on $[0,1]$ for a sequence of n independent random variables uniformly distributed on $[0,1]$ (see Introduction), which is given in [**9**].

By definition of $Y_n^*(\cdot)$ (see (2.5), (2.6)), $Y_n^*(t) + (\sum_{s=1}^{[n^t]} 1/s)/\ln^{1/2} n$ is a nondecreasing function of t. Hence, for $0 \le t_1 \le t_2 \le t_3 \le t_4 \le 1$,

$$(2.29) \qquad \begin{aligned} -\Delta_n(t_1, t_4) &\le Y_n^*(t_3) - Y_n^*(t_2) \\ &\le Y_n^*(t_4) - Y_n^*(t_1) + \Delta_n(t_1, t_4), \end{aligned}$$

$$\Delta_n(t_1, t_4) = \left(\sum_{s=[n^{t_1}]+1}^{[n^{t_4}]} 1/s \right) / \ln^{1/2} n,$$

where (see (2.27), (2.28)),

$$(2.30) \qquad \Delta_n(t_1, t_4) \le (t_4 - t_1)\ln^{1/2} n + 2\ln^{-1/2} n.$$

The proof quoted above is based only on (2.29) (with $\xi_n(\cdot)$ instead of $Y_n^*(\cdot)$, of course), where

$$(2.31) \qquad \Delta_n(t_1, t_4) \le (t_4 - t_1) n^{1/2},$$

and an inequality

$$(2.32) \qquad E\left[(\xi_n(t+h) - \xi_n(t))^4\right] \le ch^2, \quad \text{for } h \ge n^{-1}.$$

No changes would have been necessary, had the inequality (2.31) contained on its right-hand side an extra term $o(1)$, which is present in (2.30). Thus, in our case it would be sufficient, (compare (2.30) with (2.31), (2.32)), to prove an inequality analogous to (2.32) with restriction on h of the form: $h \ge \ln^{-1} n$. Fortunately, it is exactly the case here.

LEMMA 2. *There is*

$$(2.33) \qquad E\left[(Y_n^*(t+h) - Y_n^*(t))^4\right] \le 174h^2, \quad \text{if } h \ge \ln^{-1} n.$$

Proof. Fix $1 \le \nu_1 \le \nu_2$. Introduce $\sum_{s=\nu_1}^{\nu_2} X_{ns}$, the total number of cycles with lengths from ν_1 to ν_2. Denote it just C_n, for simplicity of subsequent expressions. We shall prove

$$(2.34) \qquad E\left[(C_n - E(C_n))^4\right] \le 15E^2(C_n) + 13E(C_n),$$

where

$$(2.35) \qquad E(C_n) = \sum_{\{s \le n:\, \nu_1 \le s \le \nu_2\}} 1/s.$$

But let us show first how (2.34), (2.35) lead to (2.33). We have:

$$(2.36)\qquad Y_n^*(t+h) - Y_n^*(t) = \left(\sum_{s=[n^t]+1}^{[n^{t+h}]} (X_{ns} - 1/s) \right) \Big/ \ln^{1/2} n$$

$$= (C_n - E(C_n)) / \ln^{1/2} n,$$

with $\nu_1 = [n^t] + 1$, $\nu_2 = [n^{t+h}]$. Then, (see (2.27), (2.28)),

$$E(C_n) = \sum_{s=[n^t]+1}^{[n^{t+h}]} 1/s \le h \ln n + 2 = \ln n (h + 2 \ln^{-1} n) \le 3h \ln n,$$

if $h \ge \ln^{-1} n$. Since (2.34), (2.36), we conclude that

$$E\left[\left(Y_n^*(t+h) - Y_n^*(t) \right)^4 \right] \le (135 h^2 \ln^2 n + 39 h \ln n) / \ln^2 n$$

$$= 135 h^2 + 39 h \ln^{-1} n \le 174 h^2.$$

In order to prove (2.34), notice first that by (2.3),

$$\sum_{n \ge 0} t^n E(y^{C_n}) = \exp\left[\sum_{s=\nu_1}^{\nu_2} (y-1) t^s / s \right] \Big/ (1-t).$$

Taking the jth order derivative of both sides of this relation at $y = 1$, we obtain

$$(2.37)\qquad \sum_{n \ge 0} t^n m_n^{(j)} = \left(\sum_{s=\nu_1}^{\nu_2} t^s / s \right)^j \Big/ (1-t),$$

where

$$m_n^{(j)} = E\left[C_n (C_n - 1) \cdots (C_n - j + 1) \right]$$

is the jth order factorial moment of C_n. Equating coefficients by the same powers of t on both sides of (2.37) yields

$$(2.38)\qquad m_n^{(j)} = \sum_{s_1 + \cdots + s_j \le n} \prod_{\mu=1}^{j} 1/s_\mu;$$

(here and everywhere below, the restrictions $\nu_1 \le s_\mu \le \nu_2$ $(1 \le \mu \le j)$, are silently assumed; the same goes for $s_\mu \le n$ $(1 \le \mu \le j)$, though in this case these restrictions are redundant). In case $j = 1$, (2.38) gives (2.35). A direct corollary of (2.38) is

$$(2.39)\qquad m_n^{(j)} \le \left(m_n^{(1)} \right)^j = E^j(C_n),$$

or, more generally,

$$m_n^{(j_2)} \le m_n^{(j_1)}\left(m_n^{(1)}\right)^{j_2-j_1}, \qquad j_2 \ge j_1 \ge 1. \tag{2.40}$$

Now, a simple argument shows that

$$E\left[\left(C_n - E(C_n)\right)^4\right] = E_1 + E_2 + E_3 + E_4,$$

$$E_1 = m_n^{(1)}, \qquad E_2 = 7m_n^{(2)} - 4\left(m_n^{(1)}\right)^2, \tag{2.41}$$

$$E_3 = 6m_n^{(3)} - 12m_n^{(2)}m_n^{(1)} + 6\left(m_n^{(1)}\right)^3, \tag{2.42}$$

$$E_4 = m_n^{(4)} - 4m_n^{(3)}m_n^{(1)} + 6m_n^{(2)}\left(m_n^{(1)}\right)^2 - 3\left(m_n^{(1)}\right)^4. \tag{2.43}$$

Estimate E_2, E_3, E_4. By (2.39), (2.41),

$$E_2 \le 3\left(m_n^{(1)}\right)^2. \tag{2.44}$$

Then, by (2.40), (2.42),

$$\begin{aligned} E_3 &= 6\left(m_n^{(3)} - m_n^{(2)}m_n^{(1)}\right) + 6m_n^{(1)}\left[\left(m_n^{(1)}\right)^2 - m_n^{(2)}\right] \\ &\le 6m_n^{(1)}\left[\left(m_n^{(1)}\right)^2 - m_n^{(2)}\right]; \end{aligned}$$

here (see (2.38)),

$$\begin{aligned} \left(m_n^{(1)}\right)^2 - m_n^{(2)} &= \sum \frac{1}{s_1 s_2} - \sum_{s_1+s_2\le n} \frac{1}{s_1 s_2} \\ &= \sum_{s_1+s_2>n} \frac{1}{s_1 s_2} = \sum_{s_1+s_2>n} (s_1+s_2)^{-1}\left(\frac{1}{s_1} + \frac{1}{s_2}\right) \\ &\le \left(\frac{2}{n}\right) \sum_{s_1+s_2\ge n} \frac{1}{s_1} \le \left(\frac{2}{n}\right)\sum_s \left(\frac{1}{s}\right)s \le 2, \end{aligned}$$

so

$$E_3 \le 12m_n^{(1)}. \tag{2.45}$$

Consider finally E_4. Write (see (2.40), (2.43))

$$\begin{aligned} E_4 &= \left(m_n^{(4)} - m_n^{(3)}m_n^{(1)}\right) + 3\left(m_n^{(1)}\right)^2\left[m_n^{(2)} - \left(m_n^{(1)}\right)^2\right] \\ &\quad + 3m_n^{(1)}\left(m_n^{(2)}m_n^{(1)} - m_n^{(3)}\right) \\ &\le 3m_n^{(1)}\left(m_n^{(2)}m_n^{(1)} - m_n^{(3)}\right) = 3m_n^{(1)}\sum. \end{aligned} \tag{2.46}$$

Here (see (2.38))

$$(2.47)\qquad \sum = \sum_{s_1+s_2\le n} \frac{1}{s_1 s_2 s_3} - \sum_{s_1+s_2+s_3\le n} \frac{1}{s_1 s_2 s_3}$$

$$= \sum_{\substack{s_1+s_2\le n\\ s_1+s_2+s_3>n}} (s_1+s_2+s_3)^{-1}\left[\frac{1}{s_1 s_2} + \frac{1}{s_1 s_3} + \frac{1}{s_2 s_3}\right]$$

$$\le n^{-1} \sum_{\substack{s_1+s_2\le n\\ s_1+s_2+s_3>n}} \frac{1}{s_1 s_2} + 2n^{-1} \sum_{\substack{s_1+s_2\le n\\ s_1+s_2+s_3>n}} \frac{1}{s_1 s_3}$$

$$= \sum{}' + \sum{}''.$$

Since in Σ', for each (s_1, s_2), s_3 can assume at most $s_1 + s_2$ values,

$$(2.48)\qquad \sum{}' \le n^{-1} \sum_{s_1+s_2\le n} \frac{s_1+s_2}{s_1 s_2} = 2n^{-1} \sum_{s_1+s_2\le n} \frac{1}{s_1}$$

$$\le 2n^{-1} \sum_s \frac{n}{s} = 2m_n^{(1)};$$

similarly,

$$(2.49)\qquad \sum{}'' \le 2n^{-1} \sum \frac{s_3}{s_1 s_3} = 2n^{-1} \sum_s \frac{n}{s} = 2m_n^{(1)}.$$

Hence, (see (2.46)–(2.49)),

$$(2.50)\qquad E_4 \le 12\left(m_n^{(1)}\right)^2.$$

Collecting together (2.44), (2.45), and (2.50), we arrive at (2.34) (remember, $E(C_n) = m_n^{(1)}$).

The theorem is proven.

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