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RAMIFICATION AND UNINTEGRATED VALUE DISTRIBUTION

J. R. QUINE

For a holomorphic map f from the complex plane into the Riemann sphere, the ramification term $n_1(f, r)$ is studied. A geometric version of ramification is defined in terms of the intersection points of $f(z) \times f(z + h)$ with the diagonal Δ for a suitable vector field h. Estimates of a counting function for this intersection number are given in terms of the mean covering number.

1. Introduction. Let S be the Riemann sphere normalized with radius $1/2\sqrt{\pi}$ and area 1. Suppose that f: $\mathbf{C} \to S$ is a non-constant holomorphic mapping (meromorphic function). Let $B(r)$ denote the ball $|z| \leq r$ in the complex plane. Let $n_1(r)$ and $N_1(r)$ be the unintegrated and integrated counting functions for ramification as in the value distribution theories of Ahlfors and Nevanlinna ([**1**], [**6**]). Let $L(r) = L(f, r)$ denote the length of $f(\partial B(r))$ and $A(r) = A(f, r)$ the area of $f(B(r))$ counting multiplicity (also called the mean covering number).

If f is rational, the total ramification is $2A - 2$ where A is the area or degree. In general, as a consequence of Nevanlinna's second main theorem, we know that there is a set E of finite logarithmic measure such that

$$N_1(r) \leq 2T(r) + o(T(r)) \tag{1}$$

as $r \to \infty$ in $\tilde{E}$, where T is the Nevanlinna characteristic. (A derivation of this estimate directly from the Gauss-Bonnet theorem is given in Griffiths [**3**].) In the unintegrated theory of Ahlfors [**1**], the term $n_1(r)$ disappears from the second main theorem (Nevanlinna [**6**], p. 350). Although the ramification at the points $a_1, \ldots, a_q$ is still counted, an inequality analogous to (1) cannot be proven. Terms of the form $o(A(r))$ in Ahlfors theory are given in the form $cL(r)$ where c is a constant. In the class of functions dealt with in this theory, ramification can be added topologically to any given $f|_{B(r)}$ while $L(r)$ changes very little. One can imagine adding "loops" of arbitrarily small length to $f(\partial B(r))$. This does suggest, however, that ramification "near" $\partial B(r))$ should not be counted. In the theory of Rickman and the treatment of the Ahlfors theory by Pesonen [**7**], none of the ramification term is included. (This seems to be an advantage in dealing with higher dimensions.)

The purpose of this paper is to investigate a modified ramification term in the unintegrated theory and obtain some estimates of the type (1). To do so we abandon the classical notion of ramification and go to a more geometric version. Consider a map of the form $(f(z), f(z+h))$ from $\mathbf{C}$ into $S \times S$ where $h = h(z)$ is some suitably chosen vector (difference) field on $\mathbf{C}$. We will consider h only of the form c or cz, where c is a complex constant. Let $f_h(z) = f(z+h)$. A point z_0 where $(f \times f_h)(z_0)$ is in the diagonal Δ of $S \times S$ is a geometric version of a ramification point. As $h \to 0$, in fact, these points approach the ramification points of f. The advantage of the geometric version is that we can think of the chordal distance from f to f_h, $[f, f_h]$, as being a measure of "proximity" to Δ. To use the chordal distance, choose h as above and choose $\alpha > 0$. Consider the subset of $\mathbf{C}$ determined by the inequality $[f(z), f(z+h)] < \alpha$. This will be a region of $\mathbf{C}$ bounded by the piecewise smooth curve $[f(z), f(z+h)] = \alpha$. Let $P(r, h, \alpha)$ be the union of the components of this set which intersect $\partial B(r)$. These are analogous to the peninsulas in the Ahlfors theory of the counting function for regions in the plane. Let $n_1(r, h, \alpha)$ count the number of intersections of $f \times f_h$ with Δ for z in $B(r) \cap \tilde{P}(r, h, \alpha)$. This counts the intersection "far" from $\partial B(r)$. Our main estimate is:

THEOREM 1. *Suppose $f \times f_h$ does not intersect Δ on $\partial B(r)$, then*

$$n_1(r, h, \alpha) \leq A(f_h, r) + A(f, r) + \frac{1}{\pi\alpha}(L(f_h, r) + L(f, r)).$$

As an easy corollary, we get

COROLLARY 1. *Let $h(z) = (e^{i\beta} - 1)z$ for β real such that the hypothesis of Theorem 1 is satisfied, then for fixed $\alpha > 0$ there is a set E of finite logarithmic measure such that*

$$n_1(r, h, \alpha) \leq 2A(r) + o(A(r))$$

as $r \to \infty$ in $\tilde{E}$.

The proof is based on an estimate of the form $|\sigma| \leq 2\,ds'$ where σ is a 1-form on $S \times S - \Delta$ such that $d\sigma$ represents the Poincaré dual of Δ, and ds' is the naturally defined metric on $S \times S$ (Lemma 1).

2. Definitions. Let w be the usual coordinate system for the finite part of S, with $1/w$ used as a local coordinate near ∞. The metric on S is given by

$$ds = \frac{|dw|}{\sqrt{\pi}\,(1 + |w|^2)} \tag{2}$$

and the associated area form is

$$\omega = \frac{i}{2\omega} \frac{dw \wedge d\bar{w}}{(1 + |w|^2)^2}.$$

We consider $S \times S$ as a complex 2 manifold with the product, (w_1, w_2), of the usual coordinate system on S, plus $(1/w_1, w_2)$, $(w_1, 1/w_2)$, $(1/w_1, 1/w_2)$ as coordinate patches covering $S \times S$. All computations will be done in the first one. Let $d = \bar{\partial} + \partial$ and $d^{\perp} = i(\bar{\partial} - \partial)$ be the usual differential operators. Recall that $d^{\perp}$ commutes with a holomorphic map. The main rule for computing with these is that $d^{\perp} \operatorname{re} \phi = d \operatorname{im} \phi$ for ϕ analytic. The chordal distance is defined on S by

$$[w_1, w_2] = \frac{1}{\sqrt{\pi}} \frac{|w_1 - w_2|}{\sqrt{1 + |w_1|^2}\sqrt{1 + |w_2|^2}}, \tag{4}$$

which can be thought of as a function from $S \times S$ into the reals.

We consider the pullback (pseudo) metric $f^*(ds)$ on $\mathbf{C}$, which we will call ds for simplicity. This metric gives a coordinate-free way of expressing the ramification. If f has ramification number k at z_0, then $ds/|dz| = |z - z_0|^k \phi(z)$ where $\phi(z_0) \neq 0$, and ϕ is smooth at z_0. Hence

$$\lim_{\varepsilon \to 0} \frac{1}{2\pi} \int_{|z - z_0| = \varepsilon} d^{\perp} \log \frac{ds}{|dz|} = k. \tag{5}$$

(See Cowen and Griffiths [2].) If $f \times f_h(z_0) \in \Delta$ is an isolated intersection point, define

$$\lim_{\varepsilon \to 0} \frac{1}{2\pi} \int_{|z - z_0| = \varepsilon} d^{\perp} \log[f, f_h] = \text{intersection number of } f \times f_h \text{ with } \Delta \text{ at } z_0. \tag{6}$$

This is in accord with the usual definition from intersection theory (Guillamin and Pollak [5]). If $f(z_0)$ is finite, then this is also the order of the zero of $w_1(z) - w_2(z)$ at z_0.

Let $n_1(r, h)$ denote the total number of isolated intersection points, counting multiplicity, of $f \times f_h$ with Δ in $B(r)$. Clearly $\lim_{h \to 0} [f, f_h]/|h| = ds/|dz|$, hence by (5) and (6) $\lim_{h \to 0} n_1(r, h) = n_1(r) -$ number of zeros of h in $B(r)$. This last quantity is 0 or 1 by the way h was chosen. In this sense, the ramification points of f are limit points of the intersection points of $f \times f_h$ with Δ.

Since $f \times f_h$ is holomorphic, the integrand in (6) is

$$(f \times f_h)^* d^{\perp} \log[w_1, w_2],$$

or the pullback of the 1-form $d^{\perp}\log[w_1, w_2]$ defined on $S \times S - \Delta$. We have

$$(7)\qquad \begin{aligned} d^{\perp}\log[w_1,w_2] &= d^{\perp}\log|w_1 - w_2| \\ &\quad - \frac{1}{2}d^{\perp}\left(\log\left(1+|w_1|^2\right) + \log\left(1+|w_2|^2\right)\right) \\ &= d^{\perp}\log|w_1 - w_2| - \frac{|w_1|^2}{1+|w_1|^2}d^{\perp}\log|w_1| \\ &\quad - \frac{|w_2|^2}{1+|w_2|^2}d^{\perp}\log|w_2|. \end{aligned}$$

Now taking the differential of (7) and using the fact that

$$dd^{\perp}\log|w_1 - w_2| = 0,$$

get

$$(8)\qquad \begin{aligned} dd^{\perp}\log[w_1,w_2] &= -\frac{d|w_1|^2 \wedge d\arg w_1}{\left(1+|w_1|^2\right)^2} - \frac{d|w_2|^2 \wedge d\arg w_2}{\left(1+|w_1|^2\right)^2} \\ &= -i\frac{dw_1 \wedge d\bar{w}_1}{\left(1+|w_1|^2\right)^2} - i\frac{dw_2 \wedge d\bar{w}_2}{\left(1+|w_2|^2\right)^2} \\ &= -2\pi(\omega_1 + \omega_2) \end{aligned}$$

on $S \times S - \Delta$, where ω_1 is the pullback of ω by projection on the first coordinate and similarly for ω_2.

We remark that as $h \to 0$, (8) becomes $dd^{\perp}\log ds = -2\omega$ on S. This expresses the fact that the Gaussian curvature of S is 2. Equations (7) and (8) together show $\omega_1 + \omega_2$ is Poincaré dual to Δ in $S \times S$, or that $dd^{\perp}\log[w_1, w_2]$ as a distribution equal to $\Delta - \omega_1 - \omega_2$ (see Griffiths and Harris [**4**] for the relevant cohomology theory).

3. A preliminary estimate. The key to the proof is an estimate of $|d^{\perp}\log[w_1, w_2]|$ on $S \times S$ in terms of the metric

$$(9)\qquad (ds')^2 = \frac{|dw_1|^2}{\pi\left(1+|w_1|^2\right)^2} + \frac{|dw_2|^2}{\pi\left(1+|w_2|^2\right)}.$$

The basic idea is exemplified by the differential $d^{\perp}\log|z| = \operatorname{im}(dz/z)$ in the plane minus the origin. Clearly no global estimate of the form $|d^{\perp}\log|z|| \le C|dz|$ is possible, but since $|d^{\perp}\log|z|| \le |dz|/|z|$ we have

$|d^{\perp} \log|z|| \leq |dz|/r_0$ in $|z| \geq r_0$. The following lemma enables us to estimate $d^{\perp} \log[w_1, w_2]$ away from Δ:

LEMMA 1. *On* $S \times S - \Delta$,

$$[w_1, w_2]\,|d^{\perp} \log[w_1, w_2]| \leq 2\, ds'. \tag{10}$$

Proof. By (7),

$$d^{\perp} \log[w_1, w_2] = \mathrm{im}\left(\frac{dw_2 - dw_2}{w_2 - w_1} - \frac{\overline{w}_1\, dw_1}{1 + |w_1|^2} - \frac{\overline{w}_2\, dw_2}{1 + |w_2|^2}\right) \tag{11}$$

$$= \mathrm{im}\left(\frac{(1 + |w_2|^2)(1 + \overline{w}_1 w_2)(-dw_1) + (1 + |w_1|^2)(1 + \overline{w}_2 w_1)\, dw_2}{(w_2 - w_1)(1 + |w_1|^2)(1 + |w_2|^2)}\right)$$

$$= \mathrm{im}(\sigma_1 - \sigma_2)$$

where

$$\sigma_1 = \frac{(1 + \overline{w}_1 w_2)}{w_1 - w_2} \frac{dw_1}{(1 + |w_1|^2)}$$

and σ_2 is defined similarly with w_1 and w_2 switched.

Now we have

$$\frac{[w_1, w_2]\,|\sigma_1|}{ds'} \leq \frac{|w_1 - w_2|}{\sqrt{1 + |w_1|^2}\sqrt{1 + |w_2|^2}} |\sigma_1| \frac{1 + |w_1|^2}{|dw_1|} \tag{12}$$

$$= \frac{|1 + \overline{w}_1 w_2|}{\sqrt{1 + |w_1|^2}\sqrt{1 + |w_2|^2}} \leq 1$$

where the last inequality follows from the Cauchy-Schwarz inequality. Similarly, we have

$$\frac{[w_1, w_2]\,|\sigma_2|}{ds'} \leq 1. \tag{13}$$

Now by (11), (12), and (13) we get

$$[w_1, w_2]\,|d^{\perp} \log[w_1, w_2]| \leq [w_1, w_2](|\sigma_1| + |\sigma_2|) \leq 2\, ds'.$$

This completes the proof of the lemma.

4. Proof of Theorem 1. We now proceed with the proof of Theorem 1. Let $D(r, h, \alpha) = B(r) \cap \tilde{P}(r, h, \alpha)$. We have $\partial D = \partial' D + \partial'' D$ where $\partial' D = \partial B \cap \tilde{P}$ and $\partial'' D = B \cap \partial \tilde{P}$. On $\partial'' D$, $[f, f_h] = \alpha$, and the region $[f, f_h] < \alpha$ lies to the right. Thus the directional derivative in the direction of vectors pointing to the right is non-positive. Hence $d^{\perp}\log[f, f_h] \le 0$ along $\partial D''$ and

$$(14) \qquad \int_{\partial'' D} d^{\perp}\log[f, f_h] \le 0.$$

By (6), (7), (8) and Stokes' theorem,

$$\begin{aligned}
(15) \quad n_1(r, h, \alpha) &= \int_D (f \times f_h)^* \omega_1 + \int_D (f \times f_h)^* \omega_2 \\
&\quad + \frac{1}{2\pi}\int_{\partial D} (f \times f_h)^*\big(d^{\perp}\log[w_1, w_2]\big) \\
&= \int_D f^*\omega + \int_D f_h^*\omega + \frac{1}{2\pi}\int_{\partial D} d^{\perp}\log[f, f_h] \\
&\le A(f, r) + A(f_h, r) + \frac{1}{2\pi}\int_{\partial D} d^{\perp}\log[f, f_h].
\end{aligned}$$

Using Lemma (1), (14) and $[f, f_h] \ge \alpha$ on $\partial' D$, get

$$\begin{aligned}
(16) \quad \int_{\partial D} d^{\perp}\log[f, f_h] &= \int_{\partial' D} d^{\perp}\log[f, f_h] + \int_{\partial'' D} d^{\perp}\log[f, f_h] \\
&\le \int_{\partial' D} d^{\perp}\log[f, f_h] \le \frac{2}{\alpha}\int_{\partial' D} (f \times f_h)^* ds' \\
&= \frac{2}{\alpha}\int_{\partial' D} \frac{1}{\sqrt{\pi}}\left(\frac{|df|^2}{(1+|f|^2)^2} + \frac{|df_h|^2}{(1+|f_h|^2)^2}\right)^{1/2} \\
&\le \frac{2}{\alpha}\int_{\partial' D} \frac{1}{\sqrt{\pi}}\frac{|df|}{1+|f|^2} + \frac{2}{\alpha}\int_{\partial' D} \frac{1}{\sqrt{\pi}}\frac{|df_h|}{1+|f_h|^2} \\
&\le \frac{2}{\alpha}\int_{\partial B} \frac{1}{\sqrt{\pi}}\frac{|df|}{1+|f|^2} + \frac{2}{\alpha}\int_{\partial B} \frac{1}{\sqrt{\pi}}\frac{|df_h|}{1+|f_h|^2} \\
&= \frac{2}{\alpha}\big(L(f, r) + L(f_h, r)\big).
\end{aligned}$$

Now (15) and (16) combined give Theorem 1.

To prove the Corollary, note that $f_h(z) = f(ze^{i\beta})$ so that in this case $A(f_h, r) = A(f, r)$ and $L(f_h, r) = L(f, r)$. The estimate on $L(r)$ is obtained in the usual manner (Nevanlinna **[6]**, p. 350).

5. Conclusion. The above gives, at least in principle, a way to derive bounds on a term $n_1(r, h, \alpha)$ related to ramification and dependent on two parameters h and α. In Corollary 1 since the right-hand-side of the inequality is independent of h, we can choose $h = h_r$ such that $[f, f_h]/|h| \to ds/|dz|$ in $B(r)$ as $r \to \infty$. If $\alpha = \alpha_r$ and $\alpha_r/|h_r| \to 0$ as $r \to \infty$ then $n_1(r, h_r, \alpha_r) \to n_1(r)$ as $r \to \infty$, however α_r must remain bounded below to get the uniform estimate on the remainder term.

The purpose of the paper was to establish two facts: first that by looking at maps from $\mathbf{C} \times \mathbf{C}$ to $S \times S$, the corresponding counting function n_1 can be treated in a way analogous to the counting function for domains in the Ahlfors theory; secondly, that it is possible to obtain bounds of the form cL on the remainder term in the unintegrated theory by proving an inequality of the form $|d^{\perp} \log[w_1, w_2]| \leq ds'$ on $S \times S$. The hope is that such an approach will establish a basis for proving the Ahlfors defect relation in a way that can be extended to higher dimensions and for which a treatment of the n_1 term is possible.

References

[1] L. V. Ahlfors, *Zur Theorie der Uberlagerungsflächen*, Acta. Math., **65** (1935).

[2] M. Cowen and P. Griffiths, *Holomorphic curves and metrics of negative curvature*, J. D'Anal. Math., **29** (1976), 93–153.

[3] P. Griffiths, *Entire Holomorphic Mappings in One and Several Complex Variables*, Princeton, 1976.

[4] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Wiley, 1978.

[5] V. Guillemin and A. Pollack, *Differential Topology*, Prentice-Hall, 1974.

[6] R. Nevanlinna, *Analytic Functions*, Springer-Verlag, 1970.

[7] M. Pesonen, *A path family approach to Ahlfor's value distribution theory*, Ann. Acad. Sci. Fenn. Ser. AI Math. Dissertationes 39, 1982.

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THE FLORIDA STATE UNIVERSITY
TALLAHASSEE, FL 32306

AND

THE INSTITUTE FOR ADVANCED STUDY
PRINCETON, NJ 08540