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ON THE SATO-SEGAL-WILSON SOLUTIONS OF THE K-dV EQUATION

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We discuss the class of solutions of the K-dV equation found by Sato, Segal, and Wilson. We relate this class of solutions to properties of the Weyl *m*-functions, and of the Floquet exponent for the random Schrödinger equation.

1. Introduction. In a series of recent papers, Date, Jimbo, Kashiwara, and Miwa [**5, 6, 7, 8, 9**] have developed ideas of M. and Y. Sato [**23, 24**] for finding solutions of the Kadomtsev-Petviashvili (K-P) hierarchy. The solutions of the K-P hierarchy discussed in these papers are expressed in terms of the so-called τ-function, which can be viewed as a generalization of the Riemann Θ-function.

Even more recently, Segal and Wilson [**25**] have given a careful formulation of the work of the Kyoto group. A consequence of their analysis is the following. Recall that one equation of the K-P hierarchy is the Korteweg-de Vries (K-dV) equation:

$$\frac{\partial u}{\partial t} = 6u\frac{\partial u}{\partial x} - \frac{\partial^3 u}{\partial x^3}, \qquad u(0,x) = u_0(x), \tag{1}$$

viewed as an evolution equation with initial data $u_0(x)$. Segal and Wilson produce a class $\mathscr{C}^{(2)}$ of initial conditions (or "potentials") $u_0(x)$ for which (1) admits a solution $u(t,x)$ which is meromorphic in t and x. The class $\mathscr{C}^{(2)}$ contains the solitons (see, e.g., [**1**]) and the algebro-geometric potentials [**11, 18, 21**]. We will call the elements of $\mathscr{C}^{(2)}$ Sato-Segal-Wilson potentials.

The purpose of the present note is to describe in some detail a subclass LP (for "limit-point"; see below) of the class $\mathscr{C}^{(2)}$. Namely, consider the Schrödinger equation

$$L\phi = \left(\frac{-d^2}{dx^2} + u_0(x)\right)\phi = \lambda\phi \tag{2}$$

with potential $u_0(x)$. Define $\mathrm{LP} \subset \mathscr{C}^{(2)}$ to be the set of Sato-Segal-Wilson potentials which are real and finite for all real x, and for which L is in the limit-point case $x = \pm\infty$ ([**26**]; [**3**, Ch. 9]). Let $m_+(\lambda)$ be the corresponding Weyl m-functions; they are defined and holomorphic for $\operatorname{Im}\lambda \neq 0$. Define

$$\mathscr{M}(z) = \begin{cases} m_+(z^2), & \operatorname{Im} z > 0, \operatorname{Re} z \neq 0, \\ m_-(z^2), & \operatorname{Im} z < 0, \operatorname{Re} z \neq 0. \end{cases}$$

We show that, if u_0 is in LP, then there exists $r > 0$ such that $\mathscr{M}$ extends to a holomorphic function on $|z| > r$ with a simple pole at $z = \infty$. Conversely, if $u_0(x)$ is a locally-integrable, real function of $x \in \mathbf{R}$ such that $L = -d^2/dx^2 + u_0(x)$ is in the limit-point case at $x = \pm\infty$, and if $m_\pm(\lambda)$ form branches of a function $\mathscr{M}(z)$ $(z^2 = \lambda)$ which is holomorphic for $|z| > r$, then $u_0 \in$ LP.

We use this observation to find $u_0 \in$ LP for which the spectrum Σ of L has a Cantor-like part, i.e. $\Sigma \cap (-\infty, r^2)$ is a Cantor set for some $r \in \mathbf{R}$. We then show how to "explicitly" construct a large subclass of LP. To do so, we use the Floquet exponent $w = w(\lambda)$ $(\operatorname{Im}\lambda \geq 0)$ introduced by Johnson-Moser [**15**] and studied by Kotani [**16, 17**], De Concini-Johnson [**10**], Giachetti-Johnson [**13**], and others. The construction goes as follows. Let $h(\lambda)$ be a function holomorphic in the upper half-plane $U = \{\lambda | \operatorname{Im}\lambda > 0\}$ with positive imaginary part and with certain additional properties; in particular it is supposed that the boundary value $\hat{h}(\lambda) = \lim_{\varepsilon \to 0^+} \hat{h}(\lambda + i\varepsilon)$ $(\lambda \in \mathbf{R})$ satisfies $\operatorname{Re}\hat{h}(\lambda) = 0$ for large real λ. In [**17**], Kotani shows how to find a stationary stochastic process $(\Omega, \mathscr{B}, \mu)$ which (with slight abuse of terminology; see §3) has Floquet exponent $w(\lambda) = h(\lambda)$. By Kotani's construction, Ω is a subset of a certain Hilbert space of potentials u_0. It turns out that μ-a.a. potentials are in LP.

Our results may be summarized as follows. On the one hand, potentials in the class LP $\subset \mathscr{C}^{(2)}$ are quite special: the restriction on the behavior of the m-functions is very strong. On the other hand, it will be clear from §3 that LP contains much more than the solitons and the algebro-geometric potentials.

2. The m-functions. We begin with a brief outline of the Segal-Wilson construction of the class $\mathscr{C}^{(2)}$. The formulas below differ slightly from those of [**25**], because we use $L = -d^2/dx^2 + u_0(x)$ instead of $L = +d^2/dx^2 + u_0(x)$.

Let $\mathbf{K}$ be the unit circle, and let $H_+ \subset L^2(\mathbf{K})$ be the set of boundary values in $L^2(\mathbf{K})$ of holomorphic functions on the unit disc $\{z \,|\, |z| < 1\}$. Thus $H_+ = \operatorname{cls\,span}\{1, z, z^2, \dots\}$. One considers subspaces $W \subset L^2(\mathbf{K})$ which are comparable with H_+ in the sense that: (i) the orthogonal projection $\operatorname{pr} = \operatorname{pr}(W) : W \to H_+$ is Fredholm of index zero; (ii) the orthogonal projection from W onto $H_- = (H_+)^\perp = \text{cls span}\{z^{-1}, z^{-2}, \ldots\}$ is compact. The group Γ_+ of exponential power series

$$\exp(xz + t_2 z^2 + t_3 z^3 + \cdots) \qquad (x, t_i \in \mathbf{C})$$

acts on the Grassmannian Gr of all such subspaces W by pointwise multiplication of functions. One constructs a determinant bundle Det over Gr, which in turn can be used to define the determinant of pr(W) when $W \in$ Gr. The τ-function τ_W of W is now defined as follows:

$$\tau_W(x, t_2, t_3, \ldots) = \det \text{pr}(W)/\det \text{pr}[\exp(-xz - t_2 z^2 - \cdots) \cdot W].$$

Then τ_W is meromorphic in all variables. Moreover if $\det \text{pr}(W) \neq 0$, then $\tau_W(x, t_2, t_3, \ldots) = \infty$ exactly when $\det \text{pr}[\exp(-xz - t_2 z^2 - \cdots) \cdot W] = 0$, and this occurs exactly when $\exp(-xz - t_2 z^2 - \cdots) \cdot W$ intersects H_- nontrivially.

One says that a subspace $W \in$ Gr is *transverse* if $W \cap H_- = \{0\}$; thus W is transverse iff $\det \text{pr}(W) \neq 0$. The poles of τ_W are in 1-1 correspondence with non-transverse subspaces $\exp(-xz - t_2 z^2 - \cdots) \cdot W$ if W itself is transverse.

Let us now restrict attention to the subset $\text{Gr}^{(2)}$ of Gr consisting of subspaces $W \subset L^2(\mathbf{K})$ which are invariant under $z^2 : z^2 W \subset W$. The subset $\{\exp \sum_{i=1}^{\infty} t_{2i} z^{2i}\}$ of Γ_+ leaves such a W fixed. Let $\tau_W(x, t_3, t_5, \ldots)$ be the corresponding τ-function. Define

$$u_W(x, t) = -2 \frac{d^2}{dx^2} \log \tau_W(ix, -it, 0, 0, \ldots);$$

i.e., $t_3 = it$ and all other t_is equal zero. Then $u_W(x, t)$ is the solution to the K-dV equation (2) with initial condition $u_0(x) = u_W(x, 0)$.

An important intermediate step in showing that $u_W(x, t)$ solves the K-dV equation is the construction of the Baker function $\psi_W(x, z)$. For our purposes, the following description of ψ_W will suffice; a more general discussion is given in [**25**, §5].

Let $W \in \text{Gr}^{(2)}$ be a transverse space, and suppose that $(\exp{-ixz}) \cdot W$ is transverse for all real x. Then there is a unique function

$$\psi_W(x, z) = e^{ixz}\left(1 + \sum_{i=1}^{\infty} a_i(x) z^{-i}\right)$$

in the space W; in fact $\exp(-ixz)\psi_W(x, z)$ is the inverse image of 1 under the orthogonal projection of $\exp(-ixz) \cdot W$ onto H_+. The series in parentheses converges for $|z| > 1$. Moreover

$$\left(\frac{-d^2}{dx^2} + u_0(x)\right)\psi_W(x, z) = z^2 \psi_W(x, z) \qquad (x \in \mathbf{R}, |z| > 1),$$

where $u_0(x) = -2(d^2/dx^2)\log\tau_W(ix, 0, 0, \ldots)$. One calls $\psi_W(x,z)$ the *Baker function* of W, or of $u_0(x)$.

Note that any differential operator $L = (-d^2/dx^2) + u_0(x)$ with C^∞ potential $u_0(x)$ gives rise to a *formal* Baker function

$$\tilde{\psi}(x,z) = e^{ixz}\left(1 + \sum_{i=1}^{\infty} \tilde{a}_i(x) z^{-i}\right) \tag{3}$$

which *formally* satisfies (i) $L\tilde{\psi} = z^2\tilde{\psi}$, and (ii) $\tilde{\psi}(0,z) = 1$. In fact, the coefficients $\tilde{a}_i(x)$ are C^∞ functions which are determined recursively by $a_0 \equiv 1$, $a'_{i+1} = (-i/2)La_i$, $a_i(0) = 0$ ($i \geq 1$). The quantity $e^{-ixz}\tilde{\psi}(x,z)$ is the only element of the ring $\mathscr{L}$ of formal Laurent series $s(x,z) = \sum_{i=1}^{\infty} b_i(x)z^{-i}$ with C^∞ coefficients $b_i(x)$ such that $e^{ixz}s(x,z)$ satisfies (i) and (ii).

Define $\mathscr{C}^{(2)}$ to be the class of (real or complex) potentials $u_0(x)$ such that, for some complex $\lambda \neq 0$, there exists $W \in \mathrm{Gr}^{(2)}$ such that $\lambda^2 u_0(\lambda x) = -2(d^2/dx^2)\log\tau_W(x,0,0,\ldots)$. Thus $\mathscr{C}^{(2)}$ contains those potentials obtained directly from $W \in \mathrm{Gr}^{(2)}$ by differentiating $\log\tau_W$, and also scalings of those potentials. Every $u_0 \in \mathscr{C}^{(2)}$ is a meromorphic function of x [**25**, §5].

2.1. Definition. Let $\mathrm{LP} \subset \mathscr{C}^{(2)}$ be the set of Sato-Segal-Wilson potentials u_0 which satisfy the following additional properties: (i) $u_0(x)$ is real and finite (i.e., no poles) for all real x; (ii) $L = -d^2/dx^2 + u_0(x)$ is in the limit-point case at $x = \pm\infty$.

Fix $u_0 \in \mathrm{LP}$, and let $m_\pm(\lambda)$ be the corresponding Weyl m-functions. Thus

$$m_\pm(\lambda) = \phi'_\pm(0)/\phi_\pm(0) \qquad (\operatorname{Im}\lambda \neq 0),$$

where $\phi_\pm$ are non-zero solutions of $L\phi_\pm = \lambda\phi_\pm$ which are in $L^2(0, \pm\infty)$. Since these solutions are unique up to constant multiple for $\operatorname{Im}\lambda \neq 0$, the m-functions are well-defined. They are holomorphic, and satisfy $\operatorname{sgn}[\operatorname{Im} m_\pm(\lambda)\cdot\operatorname{Im}\lambda] = \pm 1$.

Note that, with $\phi_\pm(x)$ as above, the quantities $m_\pm(s,\lambda) = \phi'_\pm(s)/\phi_\pm(s)$ are the m-functions for the translated potential $x \to u_0(x+s)$ ($s \in \mathbf{R}$).

Define

$$\hat{\psi}(x,z) = \begin{cases} \exp\int_0^x m_+(s,z^2)\,ds, & \operatorname{Im} z > 0, \operatorname{Re} z \neq 0, \\ \exp\int_0^x m_-(s,z^2)\,ds, & \operatorname{Im} z < 0, \operatorname{Re} z \neq 0. \end{cases}$$

Then $\hat{\psi}$ is defined for all real x and for all $z \in Q = \{z \in \mathbf{C} \mid \operatorname{Re} z \neq 0, \operatorname{Im} z \neq 0\}$. Clearly $L\hat{\psi} = z^2\hat{\psi}$ for all $z \in Q$, and $\hat{\psi}(0, z) = 1$, $\hat{\psi}'(0, z) = m_{\pm}(z^2)$ with the appropriate choices of sign.

It is well-known (e.g., [**14**, Ch. 10]) that $|m_{\pm}(x, \lambda) \pm \sqrt{-\lambda}| = O(|\lambda|^{-1/2})$ as $|\lambda| \to \infty$ in closed subsectors of $\{\lambda \in \mathbf{C} \mid \operatorname{Im} \lambda \neq 0\}$. Moreover the estimate on the right is uniform (in closed subsectors) if x is restricted to a compact interval. It follows that $\hat{\psi}(x, z) = e^{ixz}(1 + O(|\lambda|^{-1/2}))$ as $|z| \to \infty$ in each closed subsector of Q, if x is in a compact interval.

Now u_0 is C^∞, so by, e.g. [**20**, pp. 37–48], $\hat{\psi}(x, z)$ has an asymptotic expansion

$$\hat{\psi}(x, z) \sim e^{ixz}\left(1 + \frac{\hat{a}_1(x)}{z} + \frac{\hat{a}_2(x)}{z^2} + \cdots\right),$$

valid in Q. Moreover the $\hat{a}_i(x)$ are smooth functions which can be determined recursively by substituting $\hat{\psi}$ into $L\phi = z^2\phi$. Since $\hat{\psi}(0, z) = 1$, we see that $\hat{a}_i(x) = \tilde{a}_i(x)$, where the $\tilde{a}_i$ are the coefficients of the formal Baker function (see (3)).

Since $u_0 \in \mathscr{C}^{(2)}$, there is a true Baker function

$$\psi(x, z) = e^{ixz}(1 + a_1(x)/z + \cdots)$$

which converges for large $|z|$, and which satisfies $L\psi = z^2\psi$. Write

$$\psi(x, z)/\psi(0, z) = e^{ixz}(1 + b_1(x)z + \cdots).$$

Using the uniqueness of $\tilde{\psi}$ in the ring $\mathscr{L}$, we see that $b_i(x) = \tilde{a}_i(x) = \hat{a}_i(x)$ for all i and x. Thus in each sector of Q, the asymptotic series $1 + \hat{a}_1(x)/z + \cdots$ coincides with a series which converges for, say, $|z| > r$. We conclude that $\hat{\psi}(x, z) = \psi(x, z)/\psi(0, z)$ for $|z| > r$.

2.2. THEOREM. *Let $u_0(x)$ be a real, locally-integrable function of $x \in \mathbf{R}$ such that $L = -d^2/dx^2 + u_0(x)$ is in the limit-point case at $x = \pm\infty$. Then $u_0 \in$ LP if and only if the Weyl m-functions $m_{\pm}(\lambda)$ have the property that*

$$\mathscr{M}(z) = \begin{cases} m_+(z^2), & \operatorname{Im} z > 0, \operatorname{Re} z \neq 0, \\ m_-(z^2), & \operatorname{Im} z < 0, \operatorname{Re} z \neq 0 \end{cases} \tag{4}$$

extends holomorphically to the region $|z| > r$ for some $r > 0$. If $\mathscr{M}(z)$ admits such an extension, then $\mathscr{M}(z)$ has a simple pole at $z = \infty$ with residue i.

Proof. We first complete the proof of the "only if" statement. If $z \in Q$, then $\mathscr{M}(z) = \hat{\psi}'(0, z)$ by definition of $\hat{\psi}$. Since $\hat{\psi}(x, z)$ is holomorphic in $|z| > r$ and smooth in x (because $L\hat{\psi} = z^2\hat{\psi}$), we see that

$\mathscr{M}(z)$ is holomorphic for $|z| > r$. Simple division shows that $\mathscr{M}(z) = iz + \cdots$ for large $|z|$.

Let us consider the "if" statement. Suppose that $\mathscr{M}(z)$ admits an extension as described. Let $m_{\pm}(s,z)$ correspond to $u_0(s+x)$, and let $\mathscr{M}(s,z)$ be defined by (4) with $m_{\pm}(s,z^2)$ in place of $m_{\pm}(z^2)$. Then $\mathscr{M}(s,z)$ is holomorphic in $|z| > r_1$ for each $s \in \mathbf{R}$, and is jointly continuous in $s \in \mathbf{R}$ and $|z| > r_1$. Here $r_1 \geq r$ is independent of s.

We prove the last statement. First recall that $\operatorname{sgn}[\operatorname{Im} m_{\pm}(s,\lambda) \cdot \operatorname{Im}\lambda] = \pm 1$ if $\operatorname{Im}\lambda \neq 0$. Note also that $\mathscr{M}(s,z)$ is meromorphic in $|z| > r$. These facts imply that $\mathscr{M}(s,z)$ takes values in $\mathbf{R} \cup \{\infty\}$ if and only if z is pure imaginary, i.e., if and only if $\lambda = z^2 \leq -r^2$.

Next note that, for fixed s, $m_-(s,\lambda)$ increases and $m_+(s,\lambda)$ decreases as $\lambda \downarrow -\infty$ (unless λ is a pole, of course). Now, $\mathscr{M}(z)$ has no poles for $|z| > r$. Thus we can find $r_1 \geq r$ such that, if $\lambda \leq -r_1^2$, then $m_-(0,\lambda)$ and $m_+(0,\lambda)$ are never equal. It follows that, if $s \in \mathbf{R}$ and $\lambda \leq -r_1^2$, then $m_-(s,\lambda)$ and $m_+(s,\lambda)$ are never equal. This implies that $\mathscr{M}(s,z)$ omits some interval of real values on $|z| > r_1$. By the Picard theorem **[2]**, $\mathscr{M}(s,z)$ is meromorphic at $z = \infty$. By the preceding paragraph, $\mathscr{M}(s,z)$ has at most a simple pole at $z = \infty$, and by the relations $|m_{\pm}(s,\lambda) \pm \sqrt{-\lambda}| \to 0$ if $|\lambda| \to \infty$ with $\delta < |\arg\lambda| < \pi - \delta$(**[14]**), we see that $\mathscr{M}(s,z) = iz + \cdots$. It follows from this and the first sentence of the present paragraph that $\mathscr{M}(s,z)$ is holomorphic for $|z| > r_1$. The continuity statement is clear.

Define

$$\hat{\psi}(x,z) = \exp\int_0^x \mathscr{M}(s,z)\,ds \qquad (|z| > r_1).$$

We can write

$$\hat{\psi}(x,z) = e^{ixz}\left(1 + \frac{\hat{a}_1(x)}{z} + \frac{\hat{a}_2(x)}{z^2} + \cdots\right) \qquad (x \in \mathbf{R}),$$

where the series converges for $|z| > r_1$ and the coefficients are continuously differentiable for x. In fact they are obtained by integrating the coefficients of $\mathscr{M}(s,z)$ and combining powers of $1/z$ in the exponential; this can be proved using the Montel theorem **[2]**.

We now follow Segal-Wilson **[25**, Prop. 5.22 and the preceding discussion]. First of all, we scale u_0 (i.e., replace u_0 by $\delta^2 u_0(\delta x)$ for sufficiently small $\delta > 0$) so as to make $\mathscr{M}(z)$ holomorphic in $|z| > 1 - \varepsilon$ for some $\varepsilon > 0$. Consider the closed subspace $W \subset L^2(\mathbf{K})$ which contains $1 = \hat{\psi}(0,z)$, $\mathscr{M}(z) = \hat{\psi}'(0,z)$, and is invariant under multiplication by z^2. Then $W \in \mathrm{Gr}^{(2)}$ **[25]**, and W is transverse by its very definition, i.e., contains no function whose Laurent expansion about $z = 0$ consists entirely of negative powers of z.

Next let $\phi_i(x, z^2)$ be the solutions of $L\phi = z^2\phi$ satisfying $D^j\phi_i(0, z^2) = \delta_{ij}(i, j = 1, 2)$. Then the ϕ_i are entire in z^2 for each $x \in \mathbf{R}$. Also, $\hat{\psi}(x, z)$ and $\phi_1(x, z)\hat{\psi}(0, z) + \phi_2(x, z)\hat{\psi}'(0, z)$ are both solutions of $L\phi = z^2\phi$ with the same initial conditions, hence are equal for all $x \in \mathbf{R}$. Since W is z^2-invariant, it follows that $\hat{\psi}(x, z) \in W$ for all $x \in \mathbf{R}$. Moreover $\hat{\psi}(x, z) = e^{ixz}(1 + \text{lower order terms in } z)$ for each x. However, these two properties characterize the Baker function $\psi_W(x, z)$, at least if $\exp(-ixz) \cdot W$ is transverse; see the beginning of this section and [**25**, Prop. 5.1]. Let $u_W(x)$ be the potential in $\mathscr{C}^{(2)}$ defined by W. Then u_W is meromorphic in x [**25**, §5]. Thus $\exp(-ixz) \cdot W$ is transverse except for isolated points (the poles of u_W), and we conclude that $\hat{\psi}(x, z) = \psi_W(x, z)$ except perhaps at these poles. But since u_0 is locally integrable, there are no poles. Thus $u_0 = u_W \in \text{LP}$, which is what we wanted to prove. This completes the proof of Theorem 2.2.

We finish the section by using a simple limit procedure to construct potentials in LP. First consider a quasi-periodic potential u of algebro-geometric type [**11, 18, 21**]. Thus the spectrum Σ of $L = -d^2/dx^2 + u(x)$ (viewed as a self-adjoint operator on $L^2(-\infty, \infty)$) is a finite union of intervals: $\Sigma = [\lambda_0, \lambda_1] \cup [\lambda_2, \lambda_3] \cup \cdots \cup [\lambda_{2g}, \infty)$. Moreover one has

$$u(x) = \sum_{i=0}^{2g} \lambda_i - 2\sum_{j=1}^{g} P_j(x), \tag{5}$$

where $P_j(x) \in [\lambda_{2j-1}, \lambda_{2j}]$ $(1 \le j \le g)$ and the motion of P_j is determined by

$$P_j' = \left. \frac{\pm\sqrt{(\lambda - \lambda_0)(\lambda - \lambda_1) \cdots (\lambda - \lambda_{2g})}}{\prod_{s \ne j}(P_j - P_s)} \right|_{\lambda = P_j} \qquad (1 \le j \le g). \tag{6}$$

See [**18, 21**].

Let us now choose a sequence $\{u_n\}_{n=1}^{\infty}$ of such potentials in the following way. Let Σ_n be the spectrum of $L_n = -d^2/dx^2 + u_n(x)$ as a self-adjoint operator on $L^2(-\infty, \infty)$. We suppose that $-r^2 < \lambda_0^{(n)} < \lambda_{2g}^{(n)} = r^2$ for some $r > 0$ independent of n. Further we suppose that $\Sigma_{n+1} \subset \Sigma_n$, that $C = (-\infty, r^2) \cap \bigcap_{n=1}^{\infty} \Sigma_n$ is a Cantor set, and that $u_n(x)$ converges to a limit function $u_0(x)$, uniformly on compact subsets of $\mathbf{R}$. It is clear from (5) and (6) that such a sequence can be found. Note that $|u_n(x)| \le 2r^2$ $(x \in \mathbf{R}, n = 0, 1, 2, \ldots)$.

It is easy to check that the spectrum Σ_0 of $L_0 = -d^2/dx^2 + u_0(x)$ equals $C \cup [r^2, \infty)$ (this uses the fact that Σ_n decreases with n). That is, Σ_0 has a "Cantor-like part".

It must be shown that $u_0 \in \mathrm{LP}$. Let $m_{\pm}^{(n)}(\lambda)$ be the m-functions for L_n, and let $\mathscr{M}_n(z)$ be the function defined by (4) ($n = 0, 1, 2, \dots$). It follows from **[11]** (see also **[10]**) that $\mathscr{M}_n(z)$ extends holomorphically to $|z| > r$ ($n \geq 1$). It can also be shown that there is a fixed interval $I \subset \mathbf{R}$ such that $\{m_{+}^{(n)}(\lambda) \mid \lambda \leq -4r^2\} \cup \{m_{-}^{(n)}(\lambda) \mid \lambda \leq -4r^2\}$ does not intersect I for larger n. This assertion follows from the convergence $u_n \to u_0$ and the bound $\|u_n\|_\infty \leq 2r^2$ ($n \geq 0$); we omit the proof.

We conclude that each $\mathscr{M}_n(z)$ omits the set I of values for $|z| > 2r$ ($n = 1, 2, \dots$). By the Montel theorem **[2]**, $\{\mathscr{M}_n\}_{n=1}^{\infty}$ is a normal family of holomorphic functions on $\{z \mid |z| > 2r\}$. One checks that $m_{\pm}^{(n)}(\lambda) \to m_{\pm}^{(0)}(\lambda)$ if $\operatorname{Im}\lambda \neq 0$. Hence $\tilde{\mathscr{M}}_0(z) = \lim_{n\to\infty} \mathscr{M}_n(z)$ is well-defined and equals $\mathscr{M}_0(z)$ for $z \in Q$, $|z| > 2r$. By Theorem 2.2, $u_0 \in \mathrm{LP}$.

2.3. Remarks (a). It seems unlikely that the above procedure will always produce an almost periodic u_0. However, using the more detailed construction of Chulaevsky **[4]** one can obtain limit-periodic potentials which are in LP.

(b) Neither the construction above nor that of **[4]** make it clear that the resulting potential is meromorphic in the complex x-plane. This is a remarkable consequence of the Segal-Wilson theory.

3. The Floquet exponent. In this section we will describe a method for finding potentials in the class LP which generalizes the one given at the end of §2. We will use the Floquet exponent $w = w(\lambda)$ of $-d^2/dx^2 + u_0(x)$ **[10, 15, 16]**. This quantity is defined with respect to a "stationary ergodic process" of potentials, and not just with respect to a single u_0. For our purposes, it is convenient to adopt the following definitions **[17]**.

3.1. Definitions. Let $\Omega = L^2_{\mathrm{real}}(\mathbf{R}, (1 + |x|^3)^{-1}\,dx)$ with the Borel field $\mathscr{B}$ defined by the weak topology. Let $\{\tau_s \mid s \in \mathbf{R}\}$ be the shift operators defined by $(\tau_s u)(x) = u(s + x)$ ($u \in \Omega, s \in \mathbf{R}$). Let μ be a probability measure on $(\Omega, \mathscr{B})$ such that μ restricted to each ball $\{u \mid \|u\|_\Omega \leq R\}$ is Radon, and such that

(i) $$\mu(\tau_x(A)) = \mu(A) \quad \text{for all } x \in \mathbf{R},\ A \in \mathscr{B};$$

(ii) $$\int_\Omega \left(\int_0^1 |u(s)|^2\, ds \right) d\mu(u) < \infty.$$

Then $(\Omega, \mathscr{B}, \mu)$ is a *stationary stochastic process*, and μ is *invariant*. If in addition:

(iii) $$\mu(\tau_x(A) \Delta A) = 0 \quad \text{for all } x \in \mathbf{R} \to \mu(A) = 0 \text{ or } 1$$

for each $A \in \mathscr{B}$, then $(\Omega, \mathscr{B}, \mu)$ is a *stationary ergodic process*, and μ is *ergodic*.

Kotani [17] shows that any $u \in \Omega$ is in the limit-point case at $x = \pm\infty$. Let $m_{\pm}(\lambda) \equiv m_{\pm}(u, \lambda)$ be the Weyl m-functions; they are holomorphic in λ for $\operatorname{Im}\lambda \neq 0$, and jointly continuous in (u, λ) when Ω has the weak topology.

Let $(\Omega, \mathscr{B}, \mu)$ be a stationary stochastic process. Define

$$w(\lambda) = w_\mu(\lambda) = \int_\Omega m_+(u, \lambda)\, d\mu(u).$$

Since $u \to m_+(u, \lambda)$ is μ-integrable [17], this definition makes sense. One can show that $w(\lambda)$ is holomorphic in the upper half-plane $U = \{\lambda \in \mathbf{C} \mid \operatorname{Im}\lambda > 0\}$. Moreover $\operatorname{Im} w > 0$, $\operatorname{Re} w < 0$, and $\operatorname{Im} dw/d\lambda > 0$ for $\lambda \in U$. If μ is ergodic, then w has additional properties which justify the name "Floquet exponent". Especially, the boundary value

$$\hat{w}(\lambda) = \beta(\lambda) + i\alpha(\lambda) = \lim_{\varepsilon \to 0^+} w(\lambda + i\varepsilon) \qquad (\lambda \in \mathbf{R})$$

satisfies the following conditions. (i) The *rotation number* $\lambda \to \alpha(\lambda) = \lim_{x \to \infty} 1/x \ \arg(\phi'(x) + i\phi(x))$ is continuous, monotone increasing, and increases exactly on the spectrum Σ_u of $L_u = -d^2/dx^2 + u(x)$ for μ − a.a. u ([15]; see also [16]). (ii) The *Lyapunov number* $\beta(\lambda) = \lim_{x \to \infty} (1/2x) \ln[\phi^2(x) + \phi'^2(x)]$ determines the absolutely continuous spectrum Σ_u^{ac} of L_u for μ − a.e. u; in fact the essential support of Σ_u^{ac} is $\{\lambda \in \mathbf{R} \mid \beta(\lambda) = 0\}$ [16].

Kotani proves the following result [17].

3.2. THEOREM. *Suppose $w = w(\lambda)$ is a holomorphic function on U such that $\operatorname{Im} w > 0$, $\operatorname{Re} w < 0$, and $\operatorname{Im}(dw/d\lambda) > 0$ for $\lambda \in U$. Suppose in addition that $\lim_{\lambda \to -\infty} w(\lambda)/\sqrt{-\lambda} = 1$, and that there exists $r^2 > 0$ such that $\beta(\lambda) < 0$ for $\lambda \le 0$ and $\beta(\lambda) = 0$ for $\lambda \ge r^2$. Then there is a stationary stochastic process $(\Omega, \mathscr{B}, \mu)$ such that:* (i) $w = w_\mu$; (ii) $\mu\{u \in \Omega \mid \langle L_u\phi, \psi\rangle$ *is non-negative definite as a bilinear form on* $C^\infty_{\text{compact}}(\mathbf{R})\} = 1$.

We will also use the following theorem of De Concini-Johnson [10]. Though their result is stated for a slightly different space Ω, the proof works in the case at hand.

3.3. THEOREM. *Let $(\Omega, \mathscr{B}, \mu)$ be a stationary ergodic process such that Ω is (weakly) compact, and such that the topological support of μ equals Ω. Let $w = w_\mu$ be the corresponding Floquet exponent.*

(a) *Suppose that* $\beta(\lambda) = 0$ *for a.a.* λ *in an open interval* $I \subset \mathbf{R}$. *Then for each* $u \in \Omega$: *the function* $\lambda \to m_+(u, \lambda)$ *extends holomorphically from* U *through* I, *and the extended function equals* $m_-(u, \lambda)$ *for* $\operatorname{Im} \lambda < 0$. *The same statement holds with* $+$ *and* $-$ *interchanged.*

(b) *Suppose the spectrum* $\Sigma = \Sigma_u$ *of* L_u *is a finite union of intervals for* μ*-a.a.* $u \in \Omega$, *and that* $\beta(\lambda) = 0$ *for a.a.* $\lambda \in \Sigma$. *Then each* $u \in \Omega$ *is an algebro-geometric potential* (*see* §2).

We now turn to the main result of this section.

3.4. THEOREM. *Let* $w = w(\lambda)$ *satisfy the conditions of Theorem* 3.2. *Then there is a stationary ergodic process* $(\Omega, \mathscr{B}, \mu)$ *which satisfies* (i) *and* (ii) *of* 3.2 *such that* $u \in \mathrm{LP}$ *for* μ*-a.a.* $u \in \Omega$.

Our proof of 3.4 repeats a good share of Kotani's proof of 3.2.

Proof. Following Kotani, we construct potentials u_k $(k \geq 1)$ with the following properties. (i) The function $u_k(x)$ is T_k-periodic and belongs to Ω (i.e., is in $L^2[0, T_k]$). (ii) The Floquet exponent w_k (defined by normalized Haar measure μ_k on the circle $C_k = \{\tau_s u_k \mid 0 \leq s \leq T_k\} \subset \Omega$) satisfies $\beta_k(\lambda) = \operatorname{Re} w_k(\lambda) = 0$ for $\lambda \geq r_k^2$, where $r_k \to r$ as $k \to \infty$. (iii) $\beta_k(\lambda) > 0$ for $\lambda \leq 0$. (iv) $w_k(\lambda) \to w(\lambda)$, uniformly on compact subsets of U.

Condition (ii) implies that the spectrum Σ_k of $L_k = -d^2/dx^2 + u_k(x)$ contains $[r_k^2, \infty)$; also, (iii) implies that $\Sigma_k \subset (0, \infty)$, since u_k is periodic (see, e.g., Moser [**19**, Ch. 3]). Again by periodicity of u_ι, Σ_k is a finite union of intervals, and $\beta_k(\lambda) = 0$ for all $\lambda \in \Sigma_k$. By Theorem 3.3, $u_k(x)$ is an algebro-geometric potential. Thus from (5) in §2,

$$u_k(x) = \sum_{i=0}^{2g_k} \lambda_i^{(k)} - 2 \sum_{j=1}^{g_k} P_j^{(k)}(x),$$

where

$$P_j^{(k)}(x) \in \left[\lambda_{2j-1}^{(k)}, \lambda_{2j}^{(k)}\right] \quad \text{and} \quad 0 < \lambda_0^{(k)} < \cdots < \lambda_{2g_k}^{(k)} \leq r_k^2.$$

We conclude that $|u_k(x)| \leq 2r_k^2 < 2(r^2 + 1)$ for all large k.

The circles C_k are thus all contained in the weakly compact and translation-invariant subset $\Omega_1 = \operatorname{cls}\{u \mid \|u\|_\infty \leq 2(r^2 + 1)\} \subset \Omega$. The measures μ_k define Radon measures on Ω_1, hence there is a weak limit point μ of $\{\mu_k\}_{k=1}^\infty$. The topological support Ω_μ of μ is contained in Ω_1. Since the translations $\{\tau_x \mid x \in \mathbf{R}\}$ are weakly continuous on Ω_1, μ is invariant. Also $w = w_\mu$ by weak continuity of $u \to m_+(u, \lambda)$.

Next introduce an ergodic decomposition [**22**] $\{\mu_\gamma \mid \gamma \in \Gamma\}$ of μ. Thus Γ is a measure space with probability measure σ, each μ_γ is an ergodic measure on $\Omega_\mu \subset \Omega$, and for all continuous functions $h\colon \Omega \to \mathbf{R}$ one has

$$\int_\Omega h\,d\mu = \int_\Gamma \left(\int_\Omega h\,d\mu_\gamma\right) d\sigma(\gamma).$$

In particular, letting $w_\gamma(\lambda)$ be the Floquet exponent with respect to μ_γ, one has

$$w_\mu(\lambda) = \int_\Gamma w_\gamma(\lambda)\,d\sigma(\gamma) \qquad (\operatorname{Im}\lambda > 0). \tag{7}$$

Let $K \subset U$ be precompact in cls U (i.e., K is a bounded subset of U). Then there is a constant c_K depending only on K such that $|\operatorname{Re} w_\gamma(\lambda)| \le c_K$ for all $\gamma \in \Gamma$ and $\lambda \in K$. This follows from the description of $\beta_\gamma(\lambda)$ as a Lyapunov number, together with the estimates of [**17**, Lemma 2.8]. Let $R = r^2$, and let $n \ge 2$. By bounded convergence we have

$$\begin{aligned} 0 &= \int_R^{nR} \operatorname{Re} w_\mu(\lambda)\,d\lambda = \lim_{\varepsilon\to 0^+} \int_R^{nR} \operatorname{Re} w_\mu(\lambda + i\varepsilon)\,d\lambda \\ &= \lim_{\varepsilon\to 0^+} \int_R^{nR} \int_\Gamma \operatorname{Re} w_\gamma(\lambda + i\varepsilon)\,d\sigma(\gamma)\,d\lambda \\ &= \int_\Gamma \lim_{\varepsilon\to 0^+} \int_R^{nR} \operatorname{Re} w_\gamma(\lambda + i\varepsilon)\,d\lambda. \end{aligned}$$

We conclude that, for σ-a.a. γ, $\beta_\gamma(\lambda) = \operatorname{Re} w_\gamma(\lambda) = 0$ for a.a. $\lambda \ge R = r^2$.

Now use Theorem 3.3(a): for each u in the support of μ_γ, $\lambda \to m_\pm(u,\lambda)$ extends holomorphically from the upper half-plane U through (r^2, ∞), and the extension equals $m_\mp(u,\lambda)$ in the lower half-plane.

Next consider $L_u = -d^2/dx^2 + u(x)$ with domain $\mathscr{D} = C^\infty_{\text{compact}}(\mathbf{R}) \subset L^2(\mathbf{R})$. Since L_u is in the limit-point case at $x = \pm\infty$, it has deficiency indices zero, hence has a unique self-adjoint extension (its closure), which moreover is associated to the non-negative bilinear form $\langle L_u\phi, \psi\rangle$ on $\mathscr{D}$ [**12**]. Therefore this self-adjoint extension has no spectrum in $(-\infty, 0)$. One now proves in a standard way that $m_\pm(u,\lambda)$ are meromorphic on $\operatorname{Re}\lambda < 0$, and that $m_-(u,\lambda) \ne m_+(u,\lambda)$ there. Since $m_+(u,\lambda)$ decreases and $m_-(u,\lambda)$ increases as $\lambda \downarrow -\infty$, we can find $r_1 \ge r$ such that $\mathscr{M}(z) = \mathscr{M}(u,z)$ has no poles on $|z| > r_1$, i.e., is holomorphic there. By Theorem 2.2, $u \in$ LP. Note that $\mathscr{M}(z) = iz + \cdots$ for large $|z|$; therefore $\mathscr{M}(z)$ is holomorphic for $\operatorname{Re} z^2 = \operatorname{Re}\lambda < 0$. Hence $\mathscr{M}(z)$ is holomorphic on $|z| > r$.

Finally, let $u \in \Omega_\mu$. We can find u_n in Ω_μ such that $u_n \to u$ weakly and such that each u_n is in the support of some μ_{γ_n}. The m-functions

$m_{\pm}(u_n, \lambda)$ are meromorphic on $\operatorname{Re}\lambda < 0$, and $m_+(u_n, \lambda) < m_-(u_n, \lambda)$ for negative real λ. Furthermore $m_+(u_n, \lambda)$ decreases and $m_-(u_n, \lambda)$ increases as $\lambda \downarrow -\infty$. Choosing a subsequence if necessary, we can assume that $m_{\pm}(u_n, -r^2)$ are convergent sequences in $\mathbf{R} \cup \{\infty\}$. Then for large n, $\{m_+(u_n, \lambda) \mid \operatorname{Re}\lambda < -r^2\}$ and $\{m_-(u_n, \lambda) \mid \operatorname{Re}\lambda < -r^2\}$ omit intervals $I_{\pm}$ of real values. Using the Montel theorem once again, we see that $\{m_+(u_n, \cdot) \mid n \geq 1\}$ and $\{m_-(u_n, \cdot) \mid n \geq 1\}$ are normal families of meromorphic functions for $\operatorname{Re}\lambda < -r^2$. Using the weak continuity in u of $m_{\pm}(u, \lambda)$ for $\operatorname{Im}\lambda \neq 0$, we conclude easily that $\mathscr{M}(u_n, z) \to \mathscr{M}(u, z)$ for $|z| > r$, and that $\mathscr{M}(z) = iz + \cdots$. Thus $\mathscr{M}(z)$ is holomorphic on $|z| > r$, and so $u \in \mathrm{LP}$ by Theorem 2.2.

3.5. Remarks (a). We have actually shown that $u \in \mathrm{LP}$ for all u in the topological support Ω_μ of Ω.

(b) One can replace the assumption $\operatorname{Re} w(\lambda) < 0$ for $\lambda \leq 0$ by $\operatorname{Re} w(\lambda) < 0$ for $\operatorname{Re}\lambda \leq c$, for any constant $c < r^2$.

(c) Let $(\Omega, \mathscr{B}, \mu)$ be a stationary ergodic process such that the topological support Ω_μ of μ is compact. Suppose further that there is a fixed constant r such that: (i) the operators L_u satisfy $\langle L_u\phi, \phi\rangle \geq -r^2\langle \phi, \phi\rangle$ for all smooth ϕ with compact support; (ii) $\operatorname{Re} w(\lambda) = 0$ for $\lambda \geq r^2$. Then from the proof of 3.4 one sees that $u \in \mathrm{LP}$ for each $u \in \Omega_\mu$.

(d) The point of 3.2 is that the function $w(\lambda)$ is quite general. One can, for example, choose $w(\lambda)$ so that $\lim_{\varepsilon \to 0^+} \operatorname{Re} w(\lambda) = \beta(\lambda) < 0$ for all $\lambda < r^2$. Then either Ω contains only the constant function $u(x) \equiv r^2$, or μ-a.a. $u \in \Omega$ have spectrum in $(-\infty, r^2)$ (**[16]**; also **[10]**). Only the latter possibility is of interest. It indicates (but does not prove) that there exist $u \in \mathrm{LP}$ with at least some point spectrum.

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