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THE SELBERG TRACE FORMULA FOR GROUPS WITHOUT EISENSTEIN SERIES

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Let G be a reductive Lie group, Γ a nonuniform lattice in G. Let χ be a finite dimensional unitary representation of Γ. In order to have Eisenstein series, (G, Γ) must satisfy a certain assumption. The purpose of this note is to compute the Selberg trace formula for pairs (G, Γ) that do not possess Eisenstein series. A necessary preliminary to this, is a trace formula for $\mathrm{Ind}_\Gamma^G(\chi)$. This is also presented.

Introduction. Let G be a reductive Lie group of the Harish-Chandra class; let Γ be a nonuniform lattice in G. Let χ be a finite dimensional unitary representation of Γ. Denote by $L^2(G/\Gamma;\chi)$ the representation space of $\mathrm{Ind}_\Gamma^G(\chi)$—then G acts on $L^2(G/\Gamma;\chi)$ via the left regular representation $L_{G/\Gamma}$. Let $L_{G/\Gamma}^{\mathrm{dis}}$ be the restriction of $L_{G/\Gamma}$ to $L^2_{\mathrm{dis}}(G/\Gamma;\chi)$—the maximal completely reducible subspace. One of the central problems in the theory of automorphic forms is computing the trace of $L_{G/\Gamma}^{\mathrm{dis}}(\alpha)$ $(\alpha \in C_c^\infty(G))$; viz. the Selberg trace formula.

Let $L^2_{\mathrm{con}}(G/\Gamma;\chi)$ be the orthogonal complement of $L^2_{\mathrm{dis}}(G/\Gamma;\chi)$ in $L^2(G/\Gamma;\chi)$ and let $L_{G/\Gamma}^{\mathrm{con}}$ be the corresponding representation—then most attacks on the Selberg trace formula begin by expressing the integral kernel of $L_{G/\Gamma}^{\mathrm{con}}(\alpha)$ $(\alpha \in C_c^\infty(G))$ in terms of Eisenstein series. However, a certain assumption (cf. p. 16 of **[L2]** and p. 62 of **[OW1]**) needs to be satisfied by the pair (G, Γ) in order for a satisfactory theory of Eisenstein series to exist. The purpose of this note is to compute the Selberg trace formula for pairs (G, Γ) without Eisenstein series; i.e. that do not satisfy the assumption supra.

In order to accomplish this a trace formula needs to be given for $L^2_{\mathrm{dis}}(G/\Gamma;\chi)$, when $\chi \neq 1$. This has been done in the case $G = \mathrm{SL}_2(\mathbf{R})$ by Venkov (cf. **[V1]**). Moore has also done preliminary work for the real rank one situation (cf. **[M1]**). For the general case, Eisenstein series need to be defined with respect to χ and a spectral decomposition following Langlands needs to be given. This was accomplished by the author in his thesis (cf. **[R1]**).

When (G, Γ) does not possess Eisenstein series, the procedure to compute the trace formula is to describe $L_{G/\Gamma}$ in terms of the left regular representation of $L^2(G_n/\Gamma_n;\chi_n)$, where the pair (G_n, Γ_n) is canonically constructed from (G, Γ) and does possess Eisenstein series. It should be noted that, in general, χ_n will be non-trivial, even when $\chi = 1$. This is carried out in §2.

Section 1 is comprised of summarizing the facts needed about the spectral decomposition of $L^2(G/\Gamma;\chi)$, in order to describe the integral kernel of $L_{G/\Gamma}^{\text{con}}$ in terms of Eisenstein series.

A new type of truncation operator, due to Müller is introduced in §3 and the effect of truncating the kernels is computed (cf. **[MU1]**).

The trace formula presented in §4 follows the work of Osborne and Warner in **[OW2]** and uses the truncation operator of Müller to simplify the Dini calculus.

I would like to thank Osborne and Warner for suggesting this problem and for the substantial help they gave me in completing the spectral decomposition of $L^2(G/\Gamma;\chi)$.

1. Preliminaries. (1) Let G be a reductive Lie group of the Harish-Chandra class; let Γ be a nonuniform lattice in G. Assume that the pair satisfies the assumption spelled out on page 62 of **[OW1]** or equivalently the assumption on page 16 of **[L2]**. Let (χ, V) be a finite dimensional unitary representation of Γ. Denote by $L^2(G/\Gamma;\chi)$ the representation space of the corresponding induced representation $\text{Ind}_\Gamma^G(\chi)$. Following Langlands [cf. **[L1]**, **[L2]** and **[OW1]**], the author has obtained the spectral decomposition of the left regular representation $L_{G/\Gamma}$ acting on $L^2(G/\Gamma;\chi)$, in terms of principal series representations of G [cf. **[R1]**].

Denote by $L_{G/\Gamma}^{\text{dis}}$ the subrepresentation of $L_{G/\Gamma}$, acting on the maximal completely reducible subspace $L^2_{\text{dis}}(G/\Gamma;\chi)$. There is then an orthogonal decomposition

$$L^2(G/\Gamma;\chi) = L^2_{\text{dis}}(G/\Gamma;\chi) \oplus L^2_{\text{con}}(G/\Gamma;\chi).$$

Let $\alpha \in C_c^\infty(G)$. Let $K_\alpha(x, y)$ denote the integral kernel of $L_{G/\Gamma}(\alpha)$—then, with respect to the decomposition supra, there are integral kernels

$$\begin{cases} K_\alpha^{\text{dis}}(x, y), \\ K_\alpha^{\text{con}}(x, y), \end{cases}$$

corresponding to the representations

$$\begin{cases} L_{G/\Gamma}^{\text{dis}}(\alpha), \\ L_{G/\Gamma}^{\text{con}}(\alpha). \end{cases}$$

In order to compute the trace of $L^{\text{dis}}_{G/\Gamma}(\alpha)$, it is more convenient to work with

$$K_\alpha(x,y) - K_\alpha^{\text{con}}(x,y).$$

Hence, we shall need to recall the description of $K_\alpha^{\text{con}}(x,y)$ in terms of Eisenstein series.

(2) A maximal compact subgroup K of G has been fixed. Let δ belong to the unitary dual $\hat{K}$ of K. Let P be a (Γ-cuspidal) parabolic subgroup of G, with Langlands decomposition $P = M \cdot A \cdot N$. We shall always assume that A is stable under the Cartan involution. Denote by $\mathscr{O}$, the orbit of an infinitesimal character of M under the action of the "*Weyl group*" $W(A)$. ($W(A)$ consists of all automorphisms of A induced by an inner automorphism of G.) There is a natural representation χ_P of

$$\Gamma_M = \Gamma \cap P/\Gamma \cap N$$

on

$$V_P = \{v \in V \mid \chi(\Gamma \cap N)v = v\}.$$

Let pr_P be the orthogonal projection of V onto V_P. Define

$$\mathscr{E}_{\text{dis}}(\delta, \mathscr{O}; \chi_P)$$

to be the space of V_P-valued square integrable automorphic forms on G/AN, with K-type δ and orbit type $\mathscr{O}$, that transform on the right according to χ_P. This forms a finite dimensional subspace of

$$L^2(K \times M/\Gamma_M; \chi_P).$$

Let $\delta \in \hat{K}$—then define ξ_δ to be the normalized character of δ. Let $\mathscr{F}$ be the collection of finite subsets of $\hat{K}$ ordered by inclusion. Let $F \in \mathscr{F}$. Denote by $C_c^\infty(G;F)$ the set of $f \in C_c^\infty(G)$ such that

$$\bar{\xi}_\delta * f * \bar{\xi}_\delta = f$$

for all $\delta \in F$. Define

$$C_c^\infty(G;K) = \varinjlim_{\mathscr{F}} C_c^\infty(G;F).$$

Then $C_c^\infty(G;K)$ is an LF-space, consisting of all K-finite elements of $C_c^\infty(G)$, whose topology is finer than the subspace topology of $C_c^\infty(G)$.

Denote by $L^2_{\text{loc}}(G/\Gamma;\chi)$ the space of all measurable functions

$$\begin{cases} f\colon G \to V, \\ f(x\gamma) = \chi(\gamma^{-1})f(x) \qquad (\gamma \in \Gamma, x \in G), \end{cases}$$

such that $\|f(\cdot)\|$ is locally integrable on G/Γ. Let $f \in L^1_{\mathrm{loc}}(G/\Gamma;\chi)$—then

$$f^P(x) = \int_{N/N\cap\Gamma} \mathrm{pr}_P f(xn)\,dn$$

is called the constant term of f along P. If $f^P = 0$ for all $P \neq G$, then f is called a *cusp form* on G. Denote by

$$L^2_{\mathrm{cus}}(G/\Gamma;\chi)$$

the space of square integrable cusp forms—then there is an orthogonal decomposition

$$L^2_{\mathrm{dis}}(G/\Gamma;\chi) = L^2_{\mathrm{cus}}(G/\Gamma;\chi) \oplus L^2_{\mathrm{res}}(G/\Gamma;\chi).$$

The subspace $L^2_{\mathrm{res}}(G/\Gamma;\chi)$ is called the *residual spectrum* and is spanned by the residues of Eisenstein series associated with cusp forms [cf. §7 of **[L2]**].

Let $x \in G$—then $x = kman$, where $k \in K$, $m \in M$, $a \in A$ and $n \in N$. The factor a is uniquely determined by x and the Langlands decomposition of P. Hence, for $\Lambda \in \check{\mathfrak{a}} \otimes \mathbf{C}$, set

$$H_P(x) = \log(a)$$

and

$$\xi_\Lambda(x) = e^{\Lambda(H_P(x))}.$$

Two parabolic subgroups P and P' of G are said to be *associate*, if their split components A and A' are G-conjugate. The space of such maps from A to A' is denoted $W(A',A)$. Let $\mathscr{C}$ be a class of associate parabolic subgroups of G. If $\mathscr{C}^*$ is a subset of $\mathscr{C}$ comprised of Γ-conjugacy classes and $\mathscr{O} = \{\mathscr{O}_P\}_{P\in\mathscr{C}}$ is a collection of associate orbits, put

$$\begin{cases} \check{\mathfrak{a}}_{\mathscr{C}^*} = \left\{\Lambda \in \prod_{P\in\mathscr{C}^*} \check{\mathfrak{a}}_P \mid \Lambda_{{}^xP} = \mathrm{Ad}(x)\Lambda_P \ (x \in G)\right\}, \\ \mathfrak{a}_{\mathscr{C}^*} = \left\{\mathbf{H} \in \prod_{P\in\mathscr{C}^*} \mathfrak{a}_P \mid \mathbf{H}_P = \mathrm{Ad}(\gamma^{-1})\mathbf{H}_{{}^\gamma P} + H_P(\gamma) \ (\gamma \in \Gamma)\right\}, \\ \mathscr{E}_{\mathrm{dis}}(\delta,\mathscr{O};\mathscr{C}^*) \\ \quad = \left\{\varphi \in \prod_{P\in\mathscr{C}^*} \mathscr{E}_{\mathrm{dis}}(\delta,\mathscr{O}_P;\chi_P) \mid \varphi_{{}^\gamma P}(x) = \chi(\gamma)\varphi_P(x\gamma) \ (\gamma \in \Gamma)\right\}, \end{cases}$$

where ${}^\gamma P = \gamma P\gamma^{-1}$ and ${}^xP = xPx^{-1}$.

Fix once and for all an element $\mathbf{H} \in \mathfrak{a}_{\mathscr{C}}$. The *Eisenstein series* associated to an element φ of $\mathscr{E}_{\text{dis}}(\delta, \mathscr{O}; \mathscr{C}^*)$ is

$$\mathbf{E}(\mathscr{C}: \varphi: \Lambda: x) = \sum_{P \in \mathscr{C}^*} \varphi_P(x) \cdot e^{\langle \Lambda_P - \rho_P, H_P(x) - \mathbf{H}_P \rangle}$$

where the real part of Λ is restricted to lie in some sector of $\check{\mathfrak{a}}_{\mathscr{C}^*}$ to facilitate convergence. The Eisenstein series possesses a meromorphic continuation to all of $\check{\mathfrak{a}}_{\mathscr{C}^*} \otimes \mathbf{C}$.

The induced representations also make an appearance in this setting. One has a natural representation $(\mathscr{O}_P, \Lambda_P)$ of P on $L^2_{\text{dis}}(M/\Gamma; \mathscr{O})$:

$$\begin{cases} M \text{ operates by the left regular representation,} \\ A \text{ operates via multiplication by the quasi-character } \xi_{-\Lambda_P}, \\ N \text{ operates trivially.} \end{cases}$$

Call

$$\mathbf{Ind}_P^G(\mathscr{O}_P, \Lambda_P)$$

the associated *principal series* representation of G. Let

$$\mathbf{Ind}_{\mathscr{C}}^G(\mathscr{O}, \Lambda)$$

denote the corresponding representation on

$$\mathscr{E}_{\text{dis}}(\mathscr{O}; \mathscr{C}) = \sum_{\delta \in \hat{K}} \bigoplus \mathscr{E}_{\text{dis}}(\delta, \mathscr{O}; \mathscr{C}).$$

However, the representation space of

$$\mathbf{Ind}_{\mathscr{C}}^G(\mathscr{O}, \Lambda)$$

shall be denoted by

$$\mathscr{E}_{\text{dis}}(\mathscr{O}; \Lambda).$$

Let $\alpha \in C_c^\infty(G; K)$—then

$$\alpha * \mathbf{E}(\mathscr{C}^*: \varphi: \Lambda) = E(\mathscr{C}^*: \mathbf{Ind}_{\mathscr{C}^*}^G(\mathscr{O}, \Lambda)(\alpha)\varphi: \Lambda).$$

Let $\mathscr{C}_i$ and $\mathscr{C}_j$ be G-conjugacy classes occurring in $\mathscr{C}$—then there is a canonical intertwining operator

$$\mathbf{c}_{\text{dis}}(\mathscr{C}_j \mid \mathscr{C}_i: \mathbf{w}: \Lambda_i) \qquad (\mathbf{w} \in \mathbf{W}(\mathscr{C}_j, \mathscr{C}_i)),$$

characterized by the conditions

$$\begin{cases} \mathbf{E}(\mathscr{C}_i: \varphi_i: \Lambda_i) = \mathbf{E}(\mathscr{C}_j: \mathbf{c}_{\text{dis}}(\mathscr{C}_j \mid \mathscr{C}_i: \mathbf{w}: \Lambda_i)\varphi_i: \mathbf{w}\Lambda_i), \\ \mathbf{c}_{\text{dis}}(\mathscr{C}_j \mid \mathscr{C}_i: \mathbf{w}: \Lambda_i) \circ \mathbf{Ind}_{\mathscr{C}_i}^G(\mathscr{O}_i, \Lambda_i)(\alpha) \\ \quad = \mathbf{Ind}_{\mathscr{C}_j}^G(\mathbf{w} \cdot \mathscr{O}_i, \mathbf{w}\Lambda_i)(\alpha) \circ \mathbf{c}_{\text{dis}}(\mathscr{C}_j \mid \mathscr{C}_i: \mathbf{w}: \Lambda_i) \ (\alpha \in C_c^\infty(G; K)), \end{cases}$$

and satisfying the functional equation

$$\begin{aligned}&\mathbf{c}_{\text{dis}}(\mathscr{C}_k \mid \mathscr{C}_i\colon w_{kj}w_{ji}\colon \Lambda_i) \qquad (w_{kj} \in \mathbf{W}(\mathscr{C}_k, \mathscr{C}_j), w_{ji} \in \mathbf{W}(\mathscr{C}_j, \mathscr{C}_i))\\ &\quad = \mathbf{c}_{\text{dis}}(\mathscr{C}_k \mid \mathscr{C}_j\colon w_{kj}\colon w_{ji}\Lambda_i) \circ \mathbf{c}_{\text{dis}}(\mathscr{C}_j \mid \mathscr{C}_i\colon w_{ji}\colon \Lambda_i).\end{aligned}$$

In fact

$$\mathbf{c}_{\text{dis}}(\mathscr{C}_j \mid \mathscr{C}_i\colon \mathbf{w}\colon \Lambda_i)$$

is a linear transformation from

$$\mathscr{E}_{\text{dis}}(\delta, \mathscr{O}_i; \mathscr{C}_i) \quad \text{to} \quad \mathscr{E}_{\text{dis}}(\delta, \mathbf{w}\cdot\mathscr{O}_i; \mathscr{C}_j),$$

that is meromorphic as a function of Λ_i on $\mathfrak{a}_{\mathscr{C}_i} \otimes \mathbf{C}$. (Here $\mathscr{O}_i = \{\mathscr{O}_P\}_{P\in\mathscr{C}_i}$.) It should be mentioned that, in a suitable sense, $\mathbf{c}_{\text{dis}}$ is unitary on the imaginary axis.

(3) Denote by

$$\mathscr{L}^2(G/\Gamma; \mathscr{O}; \mathscr{C})$$

the Hilbert space of those measurable functions

$$\mathbf{F}\colon \sqrt{-1}\check{\mathfrak{a}}_{\mathscr{C}} \to \mathscr{E}_{\text{dis}}(\mathscr{O}; \mathscr{C})$$

such that components are preserved and

$$\mathbf{F}_j(\mathbf{w}\Lambda_i) = \mathbf{c}_{\text{dis}}(\mathscr{C}_j \mid \mathscr{C}_i\colon \mathbf{w}\colon \Lambda_i)\mathbf{F}_i(\Lambda_i) \qquad (\mathbf{w} \in W(\mathscr{C}_j, \mathscr{C}_i))$$

with inner product

$$(\mathbf{F}, \mathbf{G}) = \frac{1}{(2\pi)^l}\frac{1}{{}^*(\mathscr{C})}\sum_{k=1}^{r}\int_{\sqrt{-1}\check{\mathfrak{a}}_{\mathscr{C}_k}} (\mathbf{F}_k(\Lambda_k), \mathbf{G}_k(\Lambda_k)) \cdot |d\Lambda_k|,$$

where $l = \operatorname{rank}(\mathscr{C})$, r is the number of G-conjugacy classes in $\mathscr{C}$ and ${}^*(\mathscr{C})$ is the number of chambers in $\check{\mathfrak{a}}_{\mathscr{C}_k}$.

There is an isometric isomorphism

$$\begin{cases} \mathscr{L}^2(G/\Gamma; \mathscr{O}; \mathscr{C}) \to L^2(G/\Gamma; \chi), \\ \mathbf{F} \mapsto \dfrac{1}{(2\pi)^l}\dfrac{1}{{}^*(\mathscr{C})}\displaystyle\int_{\sqrt{-1}\check{\mathfrak{a}}_{\mathscr{C}}} \mathbf{E}(\mathscr{C}\colon \mathbf{F}(\Lambda)\colon \Lambda) \cdot |d\Lambda|, \end{cases}$$

whose image shall be denoted

$$L^2(G/\Gamma; \mathscr{O}; \mathscr{C}).$$

Let $\{\varphi_\mu\}_\mu$ be an orthonormal basis for $\mathscr{E}_{\text{dis}}(\mathscr{O}, \mathscr{C})$ chosen such that each φ_μ lies in some $\mathscr{E}_{\text{dis}}(\delta, \mathscr{O}, \mathscr{C})$. The inverse isomorphism

$$L^2(G/\Gamma; \mathscr{O}; \mathscr{C}) \to \mathscr{L}^2(G/\Gamma; \mathscr{O}; \mathscr{C})$$

is given by

$$f \mapsto \hat{f} = \sum_{\mu} \left\{ \int_{G/\Gamma} (f(x), \mathbf{E}(\mathscr{C}: \varphi_\mu: \Lambda: x))\, dx \right\} \varphi_\mu .$$

This is the *Eisenstein-Fourier transform* of f.

There is an important connection with the principal series representations. The *spectral decomposition* of Langlands states:

$$L^2(G/\Gamma; \chi) = \sum_{\mathscr{C}} \sum_{\mathscr{O}} \bigoplus L^2(G/\Gamma; \mathscr{O}; \mathscr{C}),$$

the spaces on the right being $L_{G/\Gamma}$-invariant. Denote by

$$\mathbf{Ind}(G/\Gamma; \mathscr{O}; \mathscr{C}),$$

the direct integral

$$\frac{1}{(2\pi)^l} \int_{C(\mathscr{C})} \bigoplus \mathbf{Ind}_{\mathscr{C}}^{G}(\mathscr{O}, \Lambda) \cdot |d\Lambda|,$$

which operates on the Hilbert space

$$\mathscr{E}(G/\Gamma; \mathscr{O}; \mathscr{C}) = \frac{1}{(2\pi)^l} \int_{C(\mathscr{C})} \bigoplus \mathscr{E}_{\mathrm{dis}}(\mathscr{O}, \Lambda) \cdot |d\Lambda|,$$

where $C(\mathscr{C})$ is the positive chamber in $\sqrt{-1}\check{\mathfrak{a}}_{\mathscr{C}}$. There is a canonical identification

$$\begin{cases} \mathscr{L}^2(G/\Gamma; \mathscr{O}; \mathscr{C}) \to \mathscr{E}(G/\Gamma; \mathscr{O}; \mathscr{C}), \\ \Phi \to \Phi \,|\, C(\mathscr{C}), \end{cases}$$

which, when composed with the Eisenstein-Fourier transform intertwines $L_{G/\Gamma}$ with **Ind**; viz.

$$(L_{G/\Gamma}(\alpha) f)^{\wedge} = \mathbf{Ind}(G/\Gamma; \mathscr{O}; \mathscr{C})(\alpha) \hat{f},$$

for all $f \in L^2(G/\Gamma; \mathscr{O}; \mathscr{C})$ and $\alpha \in C_c^\infty(G)$.

(4) The upshot of the foregoing is that

$$L^2_{\mathrm{con}}(G/\Gamma; \chi) = \sum_{\mathscr{C} \neq \{G\}} \sum_{\mathscr{O}} \bigoplus L^2(G/\Gamma; \mathscr{O}; \mathscr{C}).$$

Whence, for fixed $\mathscr{C} \neq \{G\}$, $L_{G/\Gamma}^{\mathrm{con}}$ operates on

$$L^2(G/\Gamma; \mathscr{O}; \mathscr{C})$$

according to

$$\frac{1}{(2\pi)^l} \int_{C(\mathscr{C})} \bigoplus \mathbf{Ind}_{\mathscr{C}}^{G}(\mathscr{O}, \Lambda) \cdot |d\Lambda|.$$

Let $\alpha \in C_c^\infty(G;K)$. Suppose that $f \in L^2(G/\Gamma;\mathscr{O};\mathscr{C})$ ($\mathscr{C} \neq \{G\}$)—then

$$\begin{aligned}
&\alpha * f(x) \\
&\quad = \frac{1}{(2\pi)^l}\frac{1}{*(\mathscr{C})}\int_{\mathrm{Re}(\Lambda)=0} \alpha * \mathbf{E}(\mathscr{C}\colon \hat{f}(\Lambda)\colon \Lambda\colon \cdot)(x)|d\Lambda| \\
&\quad = \frac{1}{(2\pi)^l}\frac{1}{*(\mathscr{C})}\sum_\mu \int_{\mathrm{Re}(\Lambda)=0} \mathbf{E}(\mathscr{C}\colon \mathbf{Ind}_{\mathscr{C}}^G(\mathscr{O},\Lambda)(\alpha)\varphi_\mu\colon \Lambda\colon x) \\
&\qquad \times \left\{\int_{G/\Gamma}(f(y), \mathbf{E}(\mathscr{C}\colon \varphi_\mu\colon \Lambda\colon y))dy\right\}|d\Lambda|.
\end{aligned}$$

This computation motivates the following theorem.

Set

$$\mathbf{C}_{\mu\nu}(\alpha\colon \mathscr{O},\Lambda) = (\mathbf{Ind}_{\mathscr{C}}^G(\mathscr{O},\Lambda)(\alpha)\varphi_\nu, \varphi_\mu).$$

Form

$$\begin{aligned}
&K_\alpha(x,y\colon \mathscr{O},\Lambda) \\
&\quad = \sum_{\mu,\nu}\mathbf{C}_{\mu\nu}(\alpha\colon \mathscr{O},\Lambda)\cdot \mathbf{E}(\mathscr{C}\colon \varphi_\mu\colon \Lambda\colon x)\cdot \mathbf{E}^*(\mathscr{C}\colon \varphi_\nu\colon \Lambda\colon y),
\end{aligned}$$

where

$$\begin{aligned}
&\int_{G/\Gamma} K_\alpha(x,y\colon \mathscr{O},\Lambda)f(y)\,dy \\
&\quad = \sum_{\mu,\nu}\mathbf{C}_{\mu\nu}(\alpha\colon \mathscr{O},\Lambda)\cdot \mathbf{E}(\mathscr{C}\colon \varphi_\mu\colon \Lambda\colon x) \\
&\qquad \cdot \int_{G/\Gamma}(f(y), \mathbf{E}(\mathscr{C}\colon \varphi_\nu\colon \Lambda\colon y))\,dy.
\end{aligned}$$

Write

$$K_\alpha(x,y\colon \mathscr{O};\mathscr{C})$$

in place of

$$\frac{1}{(2\pi)^l}\cdot\frac{1}{*(\mathscr{C})}\cdot\int_{\mathrm{Re}(\Lambda)=0} K_\alpha(x,y\colon \mathscr{O},\Lambda)\cdot|d\Lambda|,$$

and then put

$$K_\alpha(x,y\colon \mathscr{C}) = \sum_{\mathscr{O}} K_\alpha(x,y\colon \mathscr{O};\mathscr{C}).$$

Let $\mathscr{C}^1(G)$ denote Harish-Chandra's space of integrable rapidly decreasing functions. Let $\mathscr{C}^1(G;K)$ denote the K-finite functions in $\mathscr{C}^1(G)$, with the LF-topology.

THEOREM 1. *Let α be element of $\mathscr{C}^1(G;K)$—then $L^{\mathrm{con}}_{G/\Gamma}(\alpha)$ is an integral operator on*

$$L^2_{\mathrm{con}}(G/\Gamma;\chi)$$

with kernel

$$K^{\mathrm{con}}_\alpha(x,y)=\sum_{\mathscr{C}\neq\{G\}}K_\alpha(x,y:\mathscr{C})$$

continuous in each variable separately. □

REMARKS. The form of $K^{\mathrm{con}}_\alpha(x,y)$ follows directly from the preceding calculation. For the proof of the continuity, in slightly less generality, the reader is referred to §8 of **[OW1]**.

2. The spectral decomposition for groups without Eisenstein series.
(1) Recall that the pair (G,Γ) has been subject to a certain assumption. Let us make this assumption precise. Put

$$\begin{cases} Z= \text{ analytic subgroup of } G \text{ corresponding to the center of } \mathfrak{g}, \\ G_c= \text{ analytic subgroup of } G \text{ corresponding to the compact} \\ \quad\text{ideals of } \mathfrak{g}, \end{cases}$$

and

$$\begin{cases} \Gamma_n=\Gamma\cdot Z\cdot G_c/Z\cdot G_c, \\ G_n=G/Z\cdot G_c. \end{cases}$$

Define $E(G,\Gamma)$ to be the collection of split parabolic subgroups of G obtained by pulling back to G the percuspidal subgroups of Γ_n in G_n (cf. p. 37 of **[OW1]**).

Assumption. $E(G,\Gamma)$ comprises all Γ-percuspidal subgroups of G.

This assumption is entirely equivalent to the condition imposed by Langlands on page 16 of **[L2]** (cf. pp. 62–63 of **[OW1]**). It should be noted that an example of pair (G,Γ) that does not satisfy this assumption is constructed on pp. 63–65 of **[OW1]**.

(2) Henceforth we shall drop the assumption on (G,Γ). It is not known whether a satisfactory theory of Eisenstein series exists for the pair (G,Γ). However (G_n,Γ_n) always possesses Eisenstein series. This fact is crucial for applications to the trace formula of the spectral decomposition that follows.

Denote by G^0, the identity component of G. Set

$$\begin{cases} \Gamma^0 = \Gamma \cap G^0, \\ G_n^0 = G^0/Z \cdot G_c, \\ \Gamma_n^0 = \Gamma^0 \cdot Z \cdot G_c/Z \cdot G_c, \\ \Gamma_c = \Gamma \cap Z \cdot G_c, \\ \Gamma_Z = \Gamma_c \cdot G_c \cap Z. \end{cases}$$

Observe that G_n^0 and Γ_n^0 may be viewed as subgroups of G^0 with the property that

$$\begin{cases} G^0 = Z \cdot G_c \cdot G_n^0, \\ Z \cdot G_c \cdot \Gamma^0 = Z \cdot G_c \cdot \Gamma_n^0. \end{cases}$$

Let

$$I_c \colon L^2(G_c/G_c \cap \Gamma_c; \chi) \to L^2(G_c \cdot \Gamma_c/\Gamma_c; \chi)$$

be the canonical isomorphism. Decompose

$$L^2(G_c/G_c \cap \Gamma_c; \chi) = \sum_{U_c \in \hat{G}_c} \bigoplus m_{U_c} U_c.$$

Let χ_c be the left regular representation of Γ_Z on $I_c(E(U_c))$, where $E(U_c)$ is the representation space of $m_{U_c} U_c$. Since $G_c \cdot \Gamma_c = G_c \cdot \Gamma_Z$, it follows that

$$\begin{aligned} L^2(Z \cdot G_c/\Gamma_c; \chi) &= \mathrm{Ind}_{\Gamma_c \cdot G_c}^{Z \cdot G_c}(L^2(G_c \cdot \Gamma_c/\Gamma_c; \chi)) \\ &= \mathrm{Ind}_{\Gamma_c \cdot G_c}^{Z \cdot G_c}\left(\sum_{U_c \in \hat{G}_c} \bigoplus m_{U_c} U_c \otimes \chi_c\right) \\ &= \sum_{U_c \in \hat{G}_c} \bigoplus m_{U_c} U_c \otimes \mathrm{Ind}_{\Gamma_Z}^{Z}(\chi_c). \end{aligned}$$

Put

$$\tau = \mathrm{Ind}_{\Gamma_Z}^{Z}(\chi_c).$$

Thus

$$L^2(Z \cdot G_c/\Gamma_c; \chi) = \sum_{U_c \in \hat{G}_c} \bigoplus m_{U_c} U_c \otimes \tau.$$

The multiplicities m_{U_c} shall be computed. By the Selberg trace formula for $L^2(G_c/G_c \cap \Gamma_c; \chi)$,

$$\begin{aligned} &\sum_{U_c \in \hat{G}_c} m_{U_c} \mathrm{trace}(U_c(\alpha)) \qquad (\alpha \in C^\infty(G_c)) \\ &\quad = \sum_{\{\gamma\}} \mathrm{trace}(\chi(\gamma)) \cdot \mathrm{Vol}((G_c)_\gamma/(G_c \cap \Gamma_c)_\gamma) \cdot \int_{G_c/(G_c)_\gamma} \alpha(x\gamma x^{-1})\, dx, \end{aligned}$$

where

$$\begin{cases} (G_c)_\gamma = \text{ the centralizer of } \gamma \text{ in } G_c, \\ (G_c \cap \Gamma_c)_\gamma = \text{ the centralizer of } \gamma \text{ in } G_c \cap \Gamma_c, \end{cases}$$

and $\sum_{\{\gamma\}}$ denotes the sum over the conjugacy classes in $G_c \cap \Gamma_c$. Insert $\alpha = \operatorname{trace}(\overline{U}_c)$ into the trace formula, to obtain

$$m_{U_c} = \sum_{\{\gamma\}} \operatorname{trace}(\overline{U}_c(\gamma)) \cdot \operatorname{trace}(\chi(\gamma)) \cdot \operatorname{Vol}(G_c/(G_c \cap \Gamma)_\gamma).$$

(3) Let

$$I_0 \colon L^2(Z \cdot G_c/\Gamma_c; \chi) \to L^2(Z \cdot G_c \cdot \Gamma^0/\Gamma^0; \chi)$$

be the canonical isomorphism. Let χ_n^0 be the left regular representation of Γ_n^0 on $I_0(E(U_c \otimes \tau))$, where $E(U_c \otimes \tau)$ is the representation space of $m_{U_c} U_c \otimes \tau$. Thence

$$L^2(Z \cdot G_c \cdot \Gamma^0/\Gamma^0; \chi) = \sum_{U_c \in \hat{G}_c} \bigoplus m_{U_c} U_c \otimes \tau \otimes \chi_n^0. \tag{2.1}$$

Suppose that Γ is contained in G^0—then

$$\operatorname{Ind}_\Gamma^G(\chi) = \operatorname{Ind}_{G^0}^G(\operatorname{Ind}_\Gamma^{G^0}(\chi)).$$

Combining this observation with 2.1 yields

$$L^2(G/\Gamma; \chi) = \operatorname{Ind}_{G^0}^G \left\{ \sum_{U_c \in \hat{G}_c} \bigoplus m_{U_c} U_c \otimes \tau \otimes \operatorname{Ind}_{\Gamma_n^0}^{G_n^0}(\chi_n^0) \right\}. \tag{2.2}$$

Let $\alpha \in C_c^\infty(G)$. Put

$$\begin{aligned} &\alpha^0(U_c)(x) \\ &\quad = \int_{G/G^0} \int_Z \int_{G_c} \alpha(wxyzw^{-1}) \operatorname{trace}(\tau(z)) \operatorname{trace}(U_c(y))\, dw\, dz\, dy. \end{aligned}$$

Then

$$\operatorname{trace}(L_{G/\Gamma}^{\mathrm{dis}}(\alpha)) = \sum_{U \in \hat{G}_c} m_{U_c} \operatorname{trace}(L_{G_n^0/\Gamma_n^0}^{\mathrm{dis}, \chi_n^0}(\alpha^0(U_c)),$$

where $L_{G_n^0/\Gamma_n^0}^{\mathrm{dis}, \chi_n^0}$ denotes the left regular representation on

$$L_{\mathrm{dis}}^2(G_n^0/\Gamma_n^0; \chi_n^0).$$

However, when Γ is not contained in G^0 then

$$L^2(G/\Gamma;\chi) = \pi^{\Gamma}_{\Gamma^0}(L^2(G/\Gamma^0;\chi)),$$

where

$$\pi^{\Gamma}_{\Gamma^0}(f)(x) = \frac{1}{[\Gamma\colon\Gamma^0]} \sum_{\gamma\in\Gamma/\Gamma^0} \chi(\gamma) f(x\gamma).$$

There does not seem to be any reasonable way to incorporate $\pi^{\Gamma}_{\Gamma^0}$ into the trace formula when (G,Γ) does not possess Eisenstein series. In order to overcome this obstacle, an assumption shall be placed on G, which is satisfied by all connected groups. Assume that G_n embeds in G in such a way that

$$ZG_c \cap G_n$$

is discrete, and

$$G = Z \cdot G_c \cdot G_n.$$

More generally, assume that (G,Γ) satisfies the following

$$\begin{cases} G = G_1 \times G_2, \\ \Gamma = \Gamma_1 \times \Gamma_2, \end{cases}$$

where Γ_2 is contained in G_2^0 and G_1 is a product of groups satisfying the assumption of §2.1 and groups G' for which G'_n embeds in G' as described above.

Let

$$I\colon L^2(Z\cdot G_c/\Gamma_c;\chi) \to L^2(Z\cdot G_c\cdot\Gamma/\Gamma;\chi)$$

be the canonical isomorphism. Let χ_n be the left regular representation of Γ_n on $I(E(U_c\otimes\chi_n))$—then

$$L^2(Z\cdot G_c\cdot\Gamma/\Gamma;\chi) = \sum_{U_c\in\hat{G}_c} \bigoplus m_{U_c}\tau\otimes U_c\otimes\chi_n.$$

Ergo

$$L^2(G/\Gamma;\chi) = \mathrm{Ind}^G_{Z\cdot G_c\cdot\Gamma}(L^2(Z\cdot G_c\cdot\Gamma/\Gamma;\chi)). \tag{2.3}$$

(4) We shall now explicate the trace formula that arises from the decomposition (2.3), the situation in the case of (2.2) being entirely analogous.

Let $\alpha\in C_c^\infty(G)$. Put

$$\alpha(U_c\colon x) = \int_Z\int_{G_c} \mathrm{trace}(\tau(z))\cdot\mathrm{trace}(U_c(y))\cdot\alpha(xyz)\,dz\,dy.$$

Then $\alpha(U_c)$ belongs to $C_c^\infty(G_n)$, by the Schwarz kernel theorem (cf. Appendix 2.2 to Vol. I of [**W1**]). Let $L_{G_n/\Gamma_n}^{\mathrm{dis},\chi_n}$ denote the left regular representation of G_n on

$$L^2_{\mathrm{dis}}(G_n/\Gamma_n;\chi_n).$$

Whence, on the assumption that $L_{G/\Gamma}^{\mathrm{dis}}(\alpha)$ is of the trace class,

$$\begin{aligned}&\mathrm{trace}(L_{G/\Gamma}^{\mathrm{dis}}(\alpha))\\ &= \sum_{U_c\in\hat{G}_c} \bigoplus \sum_{\{\gamma\}} \mathrm{trace}(\overline{U}_c(\gamma))\cdot\mathrm{trace}(\chi(\gamma))\cdot\mathrm{Vol}(G_c/(G_c\cap\Gamma)_\gamma)\\ &\qquad\cdot\mathrm{trace}(L_{G_n/\Gamma_n}^{\mathrm{dis},\chi_n}(\alpha(U_c))).\end{aligned}$$

The function

$$\alpha(U_c\colon x)$$

can be further described by the Selberg trace formula for

$$\tau = \mathrm{Ind}_{\Gamma_Z}^{Z}(\chi_c).$$

Perhaps it is better to call it the Poisson summation formula since Z is abelian. Indeed

$$\alpha(U_c\colon x)$$

equals

$$\sum_{\delta\in\Gamma_Z}\mathrm{trace}(\chi_c(\delta))\cdot\mathrm{Vol}(Z/\Gamma_Z)\cdot\int_{G_c}\mathrm{trace}(U_c(y))\cdot\alpha(xy\delta)\,dy.$$

REMARK. Let $\alpha\in C_c^\infty(G;K)$—then $\alpha(U_c)=0$ for all but finitely many $U_c\in\hat{G}_c$. Therefore, whenever $L_{G_n/\Gamma_n}^{\mathrm{dis},\chi_n}(\alpha)$ is of the trace class, so is $L_{G/\Gamma}^{\mathrm{dis},\chi}(\alpha)$.

3. Truncating the kernels. In this section the basic properties of the truncation operator $Q^{\mathbf{H}}$ are reviewed. In addition, a partial truncation operator Q_N, due to Müller (cf. [**MU1**]), is introduced. The effect of truncating the kernels introduced in §1 is then investigated.

(1) Assume that (G,Γ) satisfies the assumption of §2.1.

Let $\mathscr{C}(\Gamma)$ be the set of all Γ-cuspidal subgroups of G. Give $\mathfrak{a}_{\mathscr{C}(\Gamma)}$ the obvious definition. There is a natural order "$\leq$" on $\mathfrak{a}_{\mathscr{C}(\Gamma)}$ that need not be specified until the applications. Let $\mathbf{H}\in\mathfrak{a}_{\mathscr{C}(\Gamma)}$ and $f\in L^1_{\mathrm{loc}}(G/\Gamma;\chi)$—then the truncation operator $Q^{\mathbf{H}}$ is defined by

$$Q^{\mathbf{H}}f(x)=\sum_{P\in\mathscr{C}(\Gamma)}(-1)^{\mathrm{rank}(P)}\chi_{P:\,]}(\mathbf{H}_P-H_P(x))\cdot f^P(x),$$

where $\chi_{P:\,]}$ is the characteristic function of the positive cone of P.

In order to state the salient properties of Q^{H}, a few facts need to be recalled. Let P be a fixed Γ-percuspidal subgroup of G. Let

$$\mathfrak{S}_{t,\omega} = K \cdot A[t] \cdot \omega,$$

where ω is a compact neighborhood of 1 in $M \cdot N$ and

$$A[t] = \bigcap_{\alpha \in \Sigma_P^0} \{a \in A \mid \xi_\alpha(a) \le t\}.$$

Here, Σ_P^0 is the set of simple roots determined by P. It follows from Lemma 2.11 of **[OW1]** that $t_0, \omega, {}_0t$ and $\mathfrak{s} = \{\kappa_i \colon 1 \le i \le r\} \subset K$ can be chosen so that

$$(3.1) \qquad \begin{cases} G = \mathfrak{S}_{t_0,\omega} \cdot \mathfrak{s} \cdot \Gamma, \\ \#\{\gamma \in \Gamma \mid \mathfrak{S}_{t_0,\omega} \cdot \mathfrak{s} \cdot \gamma \cap \mathfrak{S}_{t_0,\omega} \cdot \mathfrak{s} \ne \varnothing\} < \infty, \\ \mathfrak{S}_{t_0,\omega} \cdot \kappa_i \cdot \gamma \cap \mathfrak{S}_{{}_0t,\omega} \cdot \kappa_j = \varnothing \qquad (i \ne j), \\ \mathfrak{S}_{t_0,\omega} \cdot \kappa_i \cdot \gamma \cap \mathfrak{S}_{{}_0t,\omega} \cdot \kappa_i \ne \varnothing \Rightarrow \gamma \in \Gamma \cap P_i, \end{cases}$$

where $P_i = \kappa_i^{-1} P \cdot \kappa_i$. In addition, we shall assume that $(\kappa_i^{-1}\omega\kappa_i) \cdot (\Gamma \cap P_i) = M_i \cdot N_i$.

Put

$$\Xi_P(x) = \inf_{\alpha \in \Sigma_P^0} \xi_\alpha(x).$$

Let $f \in L^1_{\mathrm{loc}}(G/\Gamma; \chi)$—then f is said to be *slowly increasing* with *exponent of growth* r ($r \in \mathbf{R}$) if there is a constant $c > 0$ such that

$$|f(x\kappa_i)| \le c\,\Xi_P^r(x) \qquad (x \in \mathfrak{S}_{t_0,\omega}, 1 \le i \le r).$$

Let $S_r^\infty(G/\Gamma; \chi)$ be comprised of all smooth $f \in L^1_{\mathrm{loc}}(G/\Gamma; \chi)$ such that for every right invariant differential operator D, Df is slowly increasing with exponent of growth r—then the seminorms

$$|f|_{r,D} = \sup_{1 \le i \le r} \sup_{x \in \mathfrak{S}_{t_0,\omega}} \Xi_P^{-r}(x) |Df(x\kappa_i)|$$

endow $S_r^\infty(G/\Gamma; \chi)$ with the structure of a Fréchet space. Denote by $R(G/\Gamma; \chi)$, the space of functions on G that are slowly increasing with exponent of growth r, for every real number r. The seminorms $|\cdot|_{r,1}$ ($r \in \mathbf{R}$) also provide a Fréchet space topology. The functions in $R(G/\Gamma; \chi)$ are said to be *rapidly decreasing*.

Let us summarize the properties of the truncation operator that are of the most use.

THEOREM 2.

(i) $\lim_{H \to -\infty} Q^{\mathbf{H}} f = f$ *uniformly on compact subsets of* G.

(ii) $Q^{\mathbf{H}} = ID$ *on cusp forms.*

(iii) $Q^{\mathbf{H}}: S_r^{\infty}(G/\Gamma; \chi) \to R(G/\Gamma; \chi)$ *is continuous.*

(iv) $Q^{\mathbf{H}}$ *is a bounded linear operator on* $L^2(G/\Gamma; \chi)$.

In fact, there exists $\mathbf{H}_0 \in \mathfrak{a}_{\mathscr{C}(\Gamma)}$ *such that for all* $\mathbf{H} \leq \mathbf{H}_0$, $Q^{\mathbf{H}}$ *is an orthogonal projection on* $L^2(G/\Gamma; \chi)$ *and as such*

$$\lim_{H \to -\infty} Q^{\mathbf{H}} = ID$$

in the strong operator topology on $L^2(G/\Gamma; \chi)$. □

(2) Let P^* be an element of $\mathscr{C}(\Gamma)$ different from G. Denote by $\mathscr{C}^*$ the association class containing P^*. Order the orbits $\mathscr{O}_1^*, \mathscr{O}_2^*, \ldots$. Choose an orthonormal basis $\{\varphi_\mu^i\}_{\mu=1}^{\infty}$ for $\mathscr{E}_{\mathrm{dis}}(\mathscr{O}_i^*; \mathscr{C}^*)$, such that each φ_μ^i lies in some $\mathscr{E}_{\mathrm{dis}}(\delta, \mathscr{O}_i^*; \mathscr{C}^*)$. Let $f \in L^1_{\mathrm{loc}}(G/\Gamma; \chi)$ and let $k \in K$, $m \in M^*$, $a \in A^*$ and $n \in N^*$. Define the function

$$\pi_{P^*,N}(f)(kman)$$

to be

$$\sum_{i=N+1}^{\infty} \sum_{\mu} \left\{ \int_K \int_{M^*/\Gamma_{M^*}} (f^{P^*}(k^* m^* a), \varphi^i_{\mu,P^*}(k^* m^*)) dk^* dm^* \right\} \varphi^i_{\mu,P^*}(km).$$

Set

$$\pi_{G,N}(f) = f.$$

The *partial truncation operator* Q_N is defined by

$$(Q_N f)(x) = \sum_{P^* \in \mathscr{C}(\Gamma)} (-1)^{\mathrm{rank}(P^*)} \chi_{P^*:\,]}(\mathbf{H}_0 - H_{P^*}(x)) \cdot \pi_{P^*,N}(f)(x),$$

where $\mathbf{H}_0$ is a fixed element of $\mathfrak{a}_{\mathscr{C}(\Gamma)}$ that is sufficiently negative in a sense yet to be made precise.

(3) Specialize now to the case that G is of Γ-rank 1 (i.e. the Γ-percuspidal subgroups of G are of rank one). $\mathfrak{a}_{\mathscr{C}(\Gamma)}$ can be identified with $\prod_{i=1}^{r} \mathfrak{a}_i$. We shall restrict to the diagonal determined by $\mathfrak{s}$ and identify it with $\mathfrak{a}$. For $H \in \mathfrak{a}$, set $t_H = e^{\alpha(H)}$ ($\Sigma_P^0 = \{\alpha\}$). Choose $H_0 \in \mathfrak{a}$ such that $t_{H_0} < {}_0 t$—then it follows from (3.1) that for $t_H < {}_0 t$ and $x \in \mathfrak{S}_{t_0,\omega} \cdot \mathfrak{s}$,

$$Q^H f(x) = \begin{cases} f(x) - f^{P_i}(x): & x \in \mathfrak{S}_{t_H,\omega} \cdot \kappa_i \ (\exists i), \\ f(x): & x \notin \mathfrak{S}_{t_H,\omega} \cdot \mathfrak{s} \end{cases}$$

and

$$Q_N f(x) = \begin{cases} f(x) - \pi_{P_i,N}(f)(x): & x \in \mathfrak{S}_{t_{H_0},\omega} \cdot \mathscr{H}_i \ (\exists i), \\ f(x): & x \notin \mathfrak{S}_{t_{H_0},\omega} \cdot \mathfrak{s}. \end{cases}$$

Using this formulation the following theorem is easily seen to be true.

THEOREM 3.
(i) *If $f \in L^1_{\text{loc}}(G/\Gamma; \chi)$ and $x \in \mathfrak{S}_{t_0,\omega} \cdot \mathfrak{s}$, then*

$$Q^H \circ Q_N f(x) = Q_N \circ Q^H f(x) = \begin{cases} Q^H f(x): & x \in \mathfrak{S}_{t_H,\omega} \cdot \mathfrak{s}, \\ Q_N f(x): & x \notin \mathfrak{S}_{t_H,\omega} \cdot \mathfrak{s}. \end{cases}$$

(ii) $\lim_{N\to\infty} Q_N f = f$ *uniformly on compact subsets of* G.

(iii) $Q_N = ID$ *on cusp forms and on Eisenstein series associated with an automorphic form* φ^i_μ *for* $i \leq N$.

(iv) Q_N *is an orthogonal projection on* $L^2(G/\Gamma; \chi)$ *and as such*

$$\lim_{N\to\infty} Q_N = ID$$

in the strong operator topology on $L^2(G/\Gamma; \chi)$. □

REMARK. I do not know whether Theorem 2 and Theorem 3 are valid without the assumption on (G, Γ) specified in §2.

(4) Return now to the situation that the pair (G, Γ) is of arbitrary rank.

Let α belong to $\mathscr{C}^1(G; K)$. Define $\widetilde{K}_\alpha(x, y)$ to be any one of

$$\begin{cases} K_\alpha(x, y), \\ K_\alpha^{\text{dis}}(x, y), \\ K_\alpha^{\text{con}}(x, y). \end{cases}$$

Let Q_1^H, Q_N^1 (resp., Q_2^H, Q_N^2) denote truncation in the first (resp., second) variable of $\widetilde{K}_\alpha(x, y)$. Lemma 8.1 of **[OW1]**, combined with Theorems 1, 2 and 3 imply that

$$\begin{cases} Q_1^H \widetilde{K}_\alpha(x, y), \\ Q_2^H \widetilde{K}_\alpha(x, y), \\ Q_1^H Q_2^H \widetilde{K}_\alpha(x, y) \end{cases}$$

are separately continuous (off a set of measure zero) and locally norm bounded on $G/\Gamma \times G/\Gamma$. Moreover, the functions

$$\begin{cases} \text{trace}(Q_1^H \widetilde{K}_\alpha(x, x)), \\ \text{trace}(Q_2^H \widetilde{K}_\alpha(x, x)), \\ \text{trace}(Q_1^H Q_2^H \widetilde{K}_\alpha(x, x)) \end{cases}$$

are integrable on G/Γ. It follows from the Theorem in the appendix to §8 of **[OW1]** that

$$Q^H \circ L_{G/\Gamma}(\alpha * \alpha^*) \circ Q^H \qquad (\alpha^*(x) = \overline{\alpha(x^{-1})})$$

is of the trace class. (Of course, the same is true when $L_{G/\Gamma}$ is replaced by $L^{\mathrm{dis}}_{G/\Gamma}$ or $L^{\mathrm{con}}_{G/\Gamma}$.) In fact,

$$\operatorname{trace}(Q^H \circ L_{G/\Gamma}(\alpha * \alpha^*) \circ Q^H) = \int_{G/\Gamma} \operatorname{trace}(Q_1^H Q_2^H K_{\alpha*\alpha^*}(x,x))\, dx.$$

Thus

$$Q^H \circ L_{G/\Gamma}(\alpha)$$

is Hilbert-Schmidt (when H is sufficiently negative). The theory of paramatrix (cf. Theorem 4.4 of **[W2]**) implies that for every integer $p \geq 1$, there exists an integer $N \geq 1$ and $\mu \in C_c^p(G)$, $\nu \in C_c^\infty(G)$ such that

$$\Delta^N \cdot \mu = \delta + \nu,$$

where δ is the dirac distribution at $1 \in G$ and Δ is the Laplacian on G. Ergo

$$\alpha = (\Delta^N \cdot \alpha) * \mu - \alpha * \nu.$$

Thence

$$Q^H \circ L_{G/\Gamma}(\alpha) \circ Q^H \qquad (\alpha \in \mathscr{C}^1(G))$$

is of the trace class.

Observe

$$\lim_{H' \to -\infty} \int_{G/\Gamma} |Q_1^H Q_2^{H'} \widetilde{K}_\alpha(x,x)|\, dx = \int_{G/\Gamma} |Q_1^H \widetilde{K}_\alpha(x,x)|\, dx,$$

and if $H' \leq H \ll 0$, then

$$\begin{aligned}\operatorname{trace}(Q^H \circ L_{G/\Gamma}(\alpha) \circ Q^{H'}) &= \operatorname{trace}(Q^H \circ (Q^H \circ L_{G/\Gamma}(\alpha) \circ Q^{H'}))\\ &= \operatorname{trace}((Q^H \circ L_{G/\Gamma}(\alpha) \circ Q^{H'}) \circ Q^H)\\ &= \operatorname{trace}(Q^H \circ L_{G/\Gamma}(\alpha) \circ Q^H).\end{aligned}$$

Therefore $\operatorname{trace}(Q_1^H \widetilde{K}_\alpha(x,x))$ and $\operatorname{trace}(Q_1^H Q_2^H \widetilde{K}_\alpha(x,x))$ have the same integral, which, by a similar argument, coincides with the integral of $\operatorname{trace}(Q_2^H \widetilde{K}_\alpha(x,x))$. Proceeding in the same manner, it is easily seen that

$$\operatorname{trace}(Q_N \circ L_{G/\Gamma}(\alpha) \circ Q_N \circ Q^H) \tag{3.2}$$

is equal to the sum of

$$\int_{G/\Gamma} \operatorname{trace}(Q_2^{H_0} K_\alpha(x,x))\, dx \qquad (H < H_0)$$

and

$$\int_{G/\Gamma} \operatorname{trace}((Q_2^H \circ Q_N^2 - Q_2^{H_0})K_\alpha(x,x))\,dx.$$

REMARK. If $L_{G/\Gamma}^{\text{cus}}$ denotes the restriction of $L_{G/\Gamma}$ to $L_{\text{cus}}^2(G/\Gamma;\chi)$, then since $K_\alpha^{\text{cus}}(x,y)$ is represented by cusp forms, the preceding results imply that $L_{G/\Gamma}^{\text{cus}}(\alpha)$ $(\alpha \in \mathscr{C}^1(G))$ is of the trace class.

Let α be an element of $\mathscr{C}^1(G;K)$—then, in general, it is not known whether $L_{G/\Gamma}^{\text{dis}}(\alpha)$ is of the trace class (cf. **[OW6]** and **[W3]**). However, if G is Γ-rank 1, then Donnelly has answered this question in the affirmative. If G is real rank 1 and $\delta \in \hat{K}$, then it follows from the spectral decomposition of Langlands that the δ-isotypic component of

$$L_{\text{res}}^2(G/\Gamma;\chi)$$

is finite dimensional. (Here the assumption of §2 is needed.) This observation combined with the remark supra implies the traceability of $L_{G/\Gamma}^{\text{dis}}(\alpha)$ directly. If G is Γ-rank 0, i.e. Γ is cocompact in G, then

$$L_{\text{cus}}^2(G/\Gamma;\chi) = L^2(G/\Gamma;\chi),$$

so that $L_{G/\Gamma}(\alpha)$ $(\alpha \in \mathscr{C}^1(G))$ itself is of the trace class.

(5) For the remainder of the paper the pair (G,Γ) shall be of Γ-rank 1 and satisfy the assumption of §2.1. Observe that there are only two G-conjugacy classes of Γ-cuspidal parabolic subgroups of G; viz., $\{G\}$ and $\mathscr{C}$.

Let α be an element of $C_c^\infty(G;K)$. Then the results of Donnelly (cf. **[D1]**) imply that

$$L_{G/\Gamma}^{\text{dis}}(\alpha)$$

is of the trace class. Since

$$Q_N \to ID$$

in the strong operator topology on $L^2(G/\Gamma;\chi)$, it follows immediately that

$$\lim_{N\to\infty} \operatorname{trace}(Q_N \circ L_{G/\Gamma}(\alpha) \circ Q_N) = \operatorname{trace}(L_{G/\Gamma}(\alpha)).$$

Given a positive integer N, let

$$L_{\text{con}}^{2,N}(G/\Gamma;\chi) = \sum_{i=1}^{N} \bigoplus L^2(G/\Gamma;\mathscr{O}_i;\mathscr{C}),$$

and let $L_{\text{dis}}^{2,N}(G/\Gamma;\chi)$ be the complement of $L_{\text{con}}^{2,N}(G/\Gamma;\chi)$ in

$$Q_N(L^2(G/\Gamma;\chi)).$$

Denote by

$$\begin{cases} L_{G/\Gamma}^{\mathrm{dis},N}, \\ L_{G/\Gamma}^{\mathrm{con},N}, \end{cases}$$

the restriction of $Q_N \circ L_{G/\Gamma} \circ Q_N$ to

$$\begin{cases} L_{\mathrm{dis}}^{2,N}(G/\Gamma;\chi), \\ L_{\mathrm{con}}^{2,N}(G/\Gamma;\chi). \end{cases}$$

It is easily seen that

$$L_{G/\Gamma}^{\mathrm{dis},N} = Q_N \circ L_{G/\Gamma}^{\mathrm{dis}} \circ Q_N + T_N^{\mathrm{con}},$$

where T_N^{con} is the restriction of

$$Q^{H_0} \circ L_{G/\Gamma}^{\mathrm{con}} \circ Q^{H_0}$$

to

$$Q^{H_0}\left(\sum_{i=N+1}^{\infty} \bigoplus L^2(G/\Gamma;\mathscr{O}_i;\mathscr{C})\right).$$

Therefore $L_{G/\Gamma}^{\mathrm{dis},N}(\alpha)$ is of the trace class, with

$$\mathrm{trace}(L_{G/\Gamma}^{\mathrm{dis},N}(\alpha)) = \int_{G/\Gamma} \mathrm{trace}(K_\alpha^{\mathrm{dis},N}(x,x))\,dx,$$

where

$$K_\alpha^{\mathrm{dis},N}(x,y) = Q_N^1 Q_N^2 K_\alpha(x,y) - \sum_{i=1}^{N} K_\alpha(x,y : \mathscr{O}_i;\mathscr{C}),$$

(cf. §1.4). Here, we have implicitly used (an obvious variant of) Theorem 2 on page 23 of **[O1]**. Furthermore,

$$\lim_{N\to\infty} \mathrm{trace}(L_{G/\Gamma}^{\mathrm{dis},N}(\alpha)) = \mathrm{trace}(L_{G/\Gamma}^{\mathrm{dis}}(\alpha)).$$

4. The Selbert trace formula. In this section the pair (G,Γ) shall satisfy the assumption of §2.1, and be of Γ-rank 1.

On the basis of the work of Donnelly, the closed graph theorem implies

$$\alpha \mapsto \mathrm{trace}(L_{G/\Gamma}^{\mathrm{dis}}(\alpha))$$

is continuous in the topology of $C_c^\infty(G;K)$ (or even in the topology of $\mathscr{C}^1(G;K)$). The remainder of this paper will be devoted to an explicit realization of this distribution. The techniques used are based on the work of Arthur, Müller, Osborne and Warner (cf. **[A1]**, **[MU1]** and **[OW2]**).

(1) Fix an element α of $C_c^\infty(G;K)$—then

$$\lim_{H\to-\infty} \operatorname{trace}(L_{G/\Gamma}^{\mathrm{dis},N}(\alpha)\circ Q^H) = \operatorname{trace}(L_{G/\Gamma}^{\mathrm{dis},N}(\alpha)).$$

Write

$$\operatorname{trace}(L_{G/\Gamma}^{\mathrm{dis},N}(\alpha)\circ Q^H) = \int_{G/\Gamma} \operatorname{trace}(Q_2^H K_\alpha^{\mathrm{dis},N}(x,x))\,dx \qquad (H<H_0)$$

$$= \int_{G/\Gamma} \operatorname{trace}(Q_2^{H_0} K_\alpha(x,x))\,dx \tag{4.1}$$

$$+ \int_{G/\Gamma} \operatorname{trace}((Q_2^H\circ Q_N^2 - Q_2^{H_0})K_\alpha(x,x))\,dx \tag{4.2}$$

$$- \sum_{i=1}^N \int_{G/\Gamma} \operatorname{trace}(Q_2^H K_\alpha(x,x:\mathscr{O}_i;\mathscr{C}))\,dx. \tag{4.3}$$

The plan of attack is to send $H\to-\infty$ first and send $N\to\infty$ second.

We shall need the following fact from reduction theory. Let C be a compact subset of G. Assume, without loss of generality, that

$$C\subset \mathfrak{S}_{t_0,\omega}.$$

Parametrize A by $\xi_\alpha(a(t))=t$. Let $\gamma\in\Gamma$—then

$$a(t)\gamma a(-t)\in C \Rightarrow a(t)\gamma\in K\cdot A[t_0]\cdot a(t)M\cdot N.$$

Thus, if $0<\varepsilon<{}_0t$ is chosen small enough

$$A[t_0]a(t)\subset A[{}_0t] \qquad (t<\varepsilon).$$

Hence, for all $0<t<\varepsilon$,

$$a(t)\in \mathfrak{S}_{{}_0t,\omega}\cdot(\Gamma\cap P)\gamma^{-1}\cap \mathfrak{S}_{{}_0t,\omega}\cdot(\Gamma\cap P),$$

which, in view of (3.1), implies $\gamma\in\Gamma\cap P$.

Let $x \in \mathfrak{S}_{t_{H_0},\omega}\,(H_0 \ll 0)$—then a consequence of the calculation supra is the following.

$$
\begin{aligned}
&\int_{N/N\cap\Gamma} \mathrm{pr}_P K_\alpha(x, xn)\, dn \\
&= \int_{N/N\cap\Gamma} \mathrm{pr}_P \left\{ \sum_{\gamma\in\Gamma} \alpha(x\gamma n^{-1}x^{-1})\chi(\gamma) \right\} dn \\
&= \int_{N/N\cap\Gamma} \mathrm{pr}_P \left\{ \sum_{\gamma\in\Gamma/\Gamma\cap N} \sum_{\delta\in\Gamma\cap N} \alpha(x\gamma\delta^{-1}n^{-1}x^{-1})\chi(\gamma)\chi(\delta^{-1}) \right\} dn \\
&= \int_N \left\{ \sum_{\gamma\in\Gamma/\Gamma\cap N} \mathrm{pr}_P\alpha(x\gamma n x^{-1})\chi(\gamma) \right\} dn \\
&= \int_N \left\{ \sum_{\gamma\in\Gamma/\Gamma\cap P} \sum_{\delta\in\Gamma_M} \mathrm{pr}_P\alpha(x\gamma\delta n x^{-1})\chi(\gamma)\chi(\delta) \right\} dn \\
&= \int_N \left\{ \sum_{\gamma\in\Gamma\cap P/\Gamma\cap P} \sum_{\delta\in\Gamma_M} \mathrm{pr}_P\alpha(x\gamma\delta n x^{-1})\chi(\gamma)\chi(\delta) \right\} dn \\
&= \int_N \left\{ \sum_{\delta\in\Gamma_M} [\mathrm{pr}_P\alpha(x\delta n x^{-1})\chi(\delta) \right\} dn.
\end{aligned}
$$

Observe that if $x \in \mathfrak{S}_{t_0,\omega} \cdot \mathfrak{s}$ and $f \in L^1_{\mathrm{loc}}(G/\Gamma;\chi)$,

$$
\begin{aligned}
&(Q^H \circ Q_N - Q^{H_0})(f)(x) \\
&\quad = \begin{Bmatrix} f^{P_i} - \pi_{P_i,N}(f)\colon & t_H < \xi_\alpha(x\kappa_i^{-1}) \le t_{H_0}\ (\exists i) \\ 0\colon & \text{otherwise} \end{Bmatrix}.
\end{aligned}
$$

Moreover, if

$$\alpha_P^K(m) = \int_K \int_N \alpha(k^{-1}mnk)\, dk\, dn,$$

then the Schwarz kernel theorem implies that α_P^K belongs to $C_c^\infty(M)$. Whence

$$K_{\alpha_P^K}(m_1, m_2) = \sum_{\delta\in\Gamma_M} \mathrm{pr}_P\{\alpha_P^K(m_1\delta m_2^{-1})\chi(\delta)\}$$

is the integral kernel of the trace class operator $L_{M/\Gamma_M}(\alpha_P^K)$. (Recall that M/Γ_M is compact.) An elementary calculation now shows that

$$\int_{G/\Gamma} \mathrm{trace}((Q_2^H \circ Q_N^2 - Q_2^{H_0})K_\alpha(x,x))\, dx$$

is equal to

$$\text{(4.4)} \qquad \frac{\alpha(H_0) - \alpha(H)}{\|\alpha\|} \cdot \operatorname{trace}(L_{M/\Gamma_M}(\alpha_P^K) \cdot \tau_N),$$

where

$$\tau_N \colon \sum_{i=1}^{\infty} \bigoplus \mathscr{E}_{\text{dis}}(\mathscr{O}_i; \mathscr{C}) \to \sum_{i=1}^{N} \bigoplus \mathscr{E}_{\text{dis}}(\mathscr{O}_i; \mathscr{C})$$

is the orthogonal projection. The notation is poor because α is being used to denote both a simple root and a function. There should be no ambiguity.

(2) Let

$$\mathbf{c}(\Lambda) = \mathbf{c}_{\text{dis}}(\mathscr{C} \mid \mathscr{C} \colon -\mathbf{1} \colon \Lambda),$$

where $-\mathbf{1}$ is the unique nonidentity element of $\mathbf{W}(\mathscr{C}, \mathscr{C})$. As a function on

$$\mathscr{E}_{\text{dis}}(\delta, \mathscr{O}; \mathscr{C}),$$

$\mathbf{c}(\Lambda)$ is unitary on the imaginary axis and

$$\mathbf{c}(\Lambda)^* = \mathbf{c}(\overline{\Lambda}).$$

The following functional equations are satisfied

$$\begin{cases} \mathbf{c}(\Lambda)\mathbf{c}(-\Lambda) = ID, \\ \mathbf{E}(\mathscr{C} \colon \mathbf{c}(\Lambda)\varphi \colon -\Lambda) = \mathbf{E}(\mathscr{C} \colon \varphi \colon \Lambda). \end{cases}$$

In view of the identifications, write

$$\pi_\Lambda(\alpha) = \mathbf{Ind}_{\mathscr{C}}^{G}(\mathscr{O}_i, \Lambda)(\alpha) \qquad (\Lambda \in \check{\mathfrak{a}} \otimes \mathbf{C}).$$

It follows that

$$K_\alpha(x, y \colon \mathscr{O}_i, \Lambda) = \sum_{\mu} \mathbf{E}(\mathscr{C} \colon \pi_\Lambda(\alpha)\varphi_\mu^i \colon \Lambda \colon x) \cdot \mathbf{E}^*(\mathscr{C} \colon \varphi_\mu^i \colon \Lambda \colon y).$$

Let $\Lambda \in \sqrt{-1}\check{\mathfrak{a}}$ and $\zeta \in \mathfrak{a}$ with $\Lambda \neq 0$—then the L^2 inner product formula of Langlands (cf. p. 135 of **[L2]** and **[R2]**) shows that

$$(Q^H \mathbf{E}(\mathscr{C} \colon \pi_\Lambda(\alpha)\varphi \colon \Lambda + \zeta), Q^H \mathbf{E}(\mathscr{C} \colon \varphi \colon \Lambda + \zeta))_{G/\Gamma}$$

is equal to

$$\begin{aligned} &\frac{1}{2\zeta(H_\alpha)} \left\{ \mathbf{e}^{-2\zeta(H)}(\mathbf{c}(\Lambda + \zeta)\pi_\Lambda(\alpha)\varphi, \mathbf{c}(\Lambda + \zeta)\varphi) - \mathbf{e}^{2\zeta(H)}(\pi_\Lambda(\alpha)\varphi, \varphi) \right\} \\ &\quad + \frac{1}{2\Lambda(H_\alpha)} \left\{ \mathbf{e}^{-2\Lambda(H)}(\mathbf{c}(\Lambda + \zeta)\pi_\Lambda(\alpha)\varphi, \varphi) \right. \\ &\qquad\qquad\qquad \left. - e^{2\Lambda(H)}(\pi_\Lambda(\alpha)\varphi, \mathbf{c}(\Lambda + \zeta)\varphi) \right\}, \end{aligned}$$

where $\alpha(H_\alpha) = \|\alpha\|$. Letting $\zeta \to 0$, obtains

$$-\frac{2\alpha(H)}{\|\alpha\|}(\pi_\Lambda(\alpha)\varphi, \varphi) + (\mathbf{c}'(\Lambda)\pi_\Lambda(\alpha)\varphi, \mathbf{c}(\Lambda)\varphi)$$
$$+\frac{1}{2\Lambda(H_\alpha)}\{e^{-2\Lambda(H)}(\mathbf{c}(\Lambda)\pi_\Lambda(\alpha)\varphi, \varphi) - e^{2\Lambda(H)}(\pi_\Lambda(\alpha)\varphi, \mathbf{c}(\Lambda)\varphi)\}.$$

Therefore

$$\begin{aligned}\operatorname{trace}(L^{\mathrm{con},N}_{G/\Gamma}(\alpha)\circ Q^H) &= \sum_{i=1}^{N}\int_{G/\Gamma}\operatorname{trace}(Q_2^H K_\alpha(x,x\colon \mathscr{O}_i;\mathscr{C}))\,dx\\ &= \frac{1}{4\pi}\cdot\sum_{i=1}^{N}\int_{\mathrm{Re}(\Lambda)=0}\int_{G/\Gamma}\operatorname{trace}(Q_2^H K_\alpha(x,x\colon \mathscr{O}_i,\Lambda))\cdot|d\Lambda|\,dx\\ &= \frac{1}{4\pi}\cdot\sum_{i=1}^{N}\sum_{\mu}\int_{\mathrm{Re}(\Lambda)=0}(Q^H\mathbf{E}(\mathscr{C}\colon \pi_\Lambda(\alpha)\varphi_\mu\colon \Lambda),\\ &\qquad\qquad Q^H\mathbf{E}(\mathscr{C}\colon \varphi_\mu\colon \Lambda))_{G/\Gamma}\cdot|d\Lambda|\end{aligned}$$

which is equal to $1/4\pi$ times the sum over $\sum_{i=1}^N$ of the integral over $\mathrm{Re}(\Lambda) = 0$ of the sum of the following four terms:

$$-2\frac{\alpha(H)}{\|\alpha\|}\operatorname{trace}(\mathbf{Ind}^G_{\mathscr{C}}(\mathscr{O}_i,\Lambda)(\alpha)), \tag{4.5}$$

$$\operatorname{trace}(\mathbf{Ind}^G_{\mathscr{C}}(\mathscr{O}_i,\Lambda)(\alpha)\cdot\mathbf{c}(\overline{\Lambda})\cdot\mathbf{c}'(\Lambda)), \tag{4.6}$$

$$\frac{1}{2\Lambda(H_\alpha)}e^{-2\Lambda(H)}\operatorname{trace}(\mathbf{Ind}^G_{\mathscr{C}}(\mathscr{O}_i,\Lambda)(\alpha)\cdot\mathbf{c}(\Lambda)), \tag{4.7}$$

and

$$-\frac{1}{2\Lambda(H_\alpha)}e^{2\Lambda(H)}\operatorname{trace}(\mathbf{Ind}^G_{\mathscr{C}}(\mathscr{O}_i,\Lambda)(\alpha)\cdot\mathbf{c}(\overline{\Lambda})). \tag{4.8}$$

Consider the term (4.5). It is readily computed that

$$\operatorname{trace}(\mathbf{Ind}^G_{\mathscr{C}}(\mathscr{O}_i,\Lambda)(\alpha))$$

is given by

$$\int_A\int_{M/\Gamma_M}\operatorname{trace}\left\{\sum_{\delta\in\Gamma_M}\mathrm{pr}_P\alpha^K_P(m\delta m^{-1}a)\chi(\delta)\right\}\xi_{-(\Lambda-\rho)}\,da\,dm.$$

The Schwartz kernel theorem implies that

$$a\mapsto\operatorname{trace}(L_{M/\Gamma_M}(\alpha^K_P(\cdot\colon a)))$$

belongs to $C_c^\infty(A)$; thus

$$\mathrm{trace}(\mathbf{Ind}_{\mathscr{C}}^G(\mathscr{O}_i, \Lambda)(\alpha))$$

is rapidly decreasing. Moreover

$$\sum_{i=1}^N \frac{1}{4\pi} \int_{\mathrm{Re}(\Lambda)=0} \mathrm{trace}(\mathbf{Ind}_{\mathscr{C}}^G(\mathscr{O}_i, \Lambda)(\alpha))|d\Lambda|$$

is equal to

$$\frac{1}{2}\,\mathrm{trace}(L_{M/\Gamma_M}(\alpha_P^K)\cdot\tau_N),$$

by Fourier inversion. Hence the contribution of terms of the form (4.5) to (4.3) cancels the part of (4.4) depending on H.

Consider the terms (4.7) and (4.8). Parametrize $\sqrt{-1}\,\breve{\mathfrak{a}}$ and $\mathfrak{a}$ by $\Lambda = \sqrt{-1}\zeta\,\alpha/\|\alpha\|$ and $2H = -\xi H_\alpha$—then $\Lambda(H_\alpha) = \sqrt{-1}\,\zeta$. Write

$$\int_{\mathrm{Re}(\Lambda)=0} (4.7) + (4.8)\,|d\Lambda|$$

as the sum of

$$\int_{|\zeta|>\varepsilon} (4.7)\,d\zeta + \int_{|\zeta|>\varepsilon} (4.8)\,d\zeta \tag{4.9}$$

and

$$\int_{-\varepsilon}^{\varepsilon} (4.7) + (4.8)\,d\zeta. \tag{4.10}$$

The Riemann-Lebesgue lemma implies that both integrals in (4.9) are $o(H)$. Express (4.10) as the sum of

$$\int_{-\varepsilon}^{\varepsilon} \cos(\xi\zeta)\cdot\mathrm{trace}\left(\mathbf{Ind}_{\mathscr{C}}^G(\mathscr{O}_i,\zeta)(\alpha)\cdot\left(\frac{\mathbf{c}(\zeta)-\mathbf{c}(-\zeta)}{2\sqrt{-1}\,\zeta}\right)\right)d\zeta \tag{4.11}$$

and

$$\int_{-\varepsilon}^{\varepsilon} \frac{\sin(\xi\zeta)}{\zeta}\cdot\mathrm{trace}\left(\mathbf{Ind}_{\mathscr{C}}^G(\mathscr{O}_i,\zeta)(\alpha)\cdot\left(\frac{\mathbf{c}(\zeta)+\mathbf{c}(-\zeta)}{2}\right)\right)d\zeta. \tag{4.12}$$

Another application of the Riemann-Lebesgue lemma shows that (4.11) is $o(H)$. On the other hand, suppose $g \in L^1(\mathbf{R})$ is differentiable at 0—then by writing

$$g(\zeta) = g(0) + \zeta\left\{\frac{g(\zeta)-g(0)}{\zeta}\right\},$$

it follows that

$$\lim_{\xi\to\infty}\int_{-\varepsilon}^{\varepsilon}\frac{\sin(\xi\zeta)}{\zeta}g(\zeta)d\zeta = \pi g(0).$$

Then the limit as $H \to -\infty$ of (4.12) is

$$\pi \operatorname{trace}(\mathbf{Ind}_{\mathscr{C}}^{G}(\mathscr{O}_i, 0)(\alpha) \cdot \mathbf{c}(0)).$$

(3) Let us summarize what has been shown so far.

$$\operatorname{trace}(L_{G/\Gamma}^{\mathrm{dis},N}(\alpha) \circ Q^H) \qquad \operatorname{mod}(o(H))$$

is equal to the sum of

$$\int_{G/\Gamma} \operatorname{trace}(Q_2^{H_0} K_\alpha(x,x))\, dx,$$

$$\frac{\alpha(H_0)}{\|\alpha\|} \cdot \operatorname{trace}(L_{M/\Gamma_M}(\alpha_P^K) \cdot \tau_N),$$

$$-\frac{1}{4\pi} \cdot \sum_{i=1}^{N} \int_{\operatorname{Re}(\Lambda)=0} \operatorname{trace}(\mathbf{Ind}_{\mathscr{C}}^{G}(\mathscr{O}_i, \Lambda)(\alpha) \cdot \mathbf{c}(\overline{\Lambda}) \cdot \mathbf{c}'(\Lambda)) \cdot |d\Lambda|,$$

and

$$-\frac{1}{4} \cdot \sum_{i=1}^{N} \operatorname{trace}(\mathbf{Ind}_{\mathscr{C}}^{G}(\mathscr{O}_i, 0)(\alpha) \cdot \mathbf{c}(0)).$$

Send $H \to -\infty$—then send $N \to \infty$. Hence

$$\operatorname{trace}(L_{G/\Gamma}^{\mathrm{dis}}(\alpha))$$

is equal to the sum of

$$\int_{G/\Gamma} \operatorname{trace}(Q_2^{H_0} K_\alpha(x,x))\, dx, \tag{4.13}$$

$$\frac{\alpha(H_0)}{\|\alpha\|} \cdot \operatorname{trace}(L_{M/\Gamma_M}(\alpha_P^K)), \tag{4.14}$$

$$-\frac{1}{4\pi} \cdot \sum_{\mathscr{O}} \int_{\operatorname{Re}(\Lambda)=0} \operatorname{trace}(\mathbf{Ind}_{\mathscr{C}}^{G}(\mathscr{O}, \Lambda)(\alpha) \cdot \mathbf{c}(\overline{\Lambda}) \cdot \mathbf{c}'(\Lambda)) \cdot |d\Lambda|, \tag{4.15}$$

and

$$-\frac{1}{4} \sum_{\mathscr{O}} \operatorname{trace}(\mathbf{Ind}_{\mathscr{C}}^{G}(\mathscr{O}, 0)(\alpha) \cdot \mathbf{c}(0)). \tag{4.16}$$

Observe that $\mathbf{c}(0)$ extends to a bounded operator on

$$\sum_{\mathscr{O}} \bigoplus \mathscr{E}_{\mathrm{dis}}(\mathscr{O}; \mathscr{C}).$$

Let $\hat{\alpha}_P^K(m\colon \Lambda)$ denote the Fourier transform of the function

$$a \mapsto \alpha_P^K(ma).$$

Then (4.16) is quickly seen to equal

$$-\frac{1}{4}\operatorname{trace}(L_{M/\Gamma_M}(\hat{\alpha}_P^K(\cdot : 0)) \cdot \mathbf{c}(0)).$$

Whence, Weierstrass' theorem on conditional convergence implies that the sum in (4.15) converges absolutely.

(4) Denote by $T_{H_0}(\alpha)$ the sum of (4.13) and (4.14)—then T_{H_0} and (4.16) extend to distributions on $C_c^\infty(G)$. Thus (4.15) must be a distribution on $C_c^\infty(G; K)$. If it could be shown that there are constants C and L ($C > 0$), independent of δ and $\mathscr{O}$, such that

$$\|\mathbf{c}'(\Lambda)\|_{OP} \leq C(1 + \|\delta\| + \|\mathscr{O}\| + \|\Lambda\|)^L \qquad (\Lambda \in \sqrt{-1}\,\mathfrak{a}),$$

then the integral series in (4.15) is absolutely convergent and (4.15) extends to a distribution on $C_c^\infty(G)$. Here

$$\begin{cases} \|\delta\| = |\langle \delta, \omega_K \rangle|, \\ \|\mathscr{O}\| = |\langle \mathscr{O}, \omega \rangle|, \end{cases}$$

where

$$\begin{cases} \omega_K = \text{ the Casimir of } K, \\ \omega = \text{ the Casimir of } G, \end{cases}$$

and $\|\cdot\|_{OP}$ is the operator norm on

$$\mathscr{E}_{\text{dis}}(\delta, \mathscr{O}; \mathscr{E}).$$

This, of course, would imply that

$$\alpha \mapsto L_{G/\Gamma}^{\text{dis}}(\alpha)$$

is also a distribution on $C_c^\infty(G)$ (or even on $\mathscr{C}^1(G)$). In particular, $L_{G/\Gamma}^{\text{dis}}(\alpha)$ is of the trace class for all α in $C_c^\infty(G)$.

The term $T_{H_0}(\alpha)$ is now unraveled $\operatorname{mod} o(H_0)$ into orbital integrals corresponding to the semisimple elements of Γ and a term

$$\lim_{s \to 0}(s \text{Ч}_\alpha(\delta : s)), \tag{4.17}$$

corresponding to the non-semisimple elements of Γ. This is done by Osborne and Warner in **[OW2]**, pp. 56–92. In particular see the formula on page 93 of **[OW2]** for the complete trace formula. Just recently, the non-semisimple term (4.17) has been completely explicated by Hoffman (cf. **[H1]**), in terms of zeta functions attached to prehomogeneous vector spaces. The argument is quite analogous to the **R**-rank 1 situation (cf. **[W2]**).

Added in proof. By utilizing a result of Arthur [cf. Theorem 8.1; Amer. J. Math., Vol. 104, No. 6, pp. 1289–1336], it can be shown that the integral series in (4.15) is absolutely convergent and hence each of its terms are distributions on $C_c^\infty(G;K)$.

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