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RICCI CURVATURE AND VOLUME GROWTH

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We give an example of a complete manifold M^m of nonnegative Ricci curvature for which the volume of distance tubes around a totally geodesic submanifold L^l divided by the corresponding volume in $L \times \mathbf{R}^{m-l}$ goes to infinity. Recall that in the case of nonnegative sectional curvature, this quotient is nonincreasing and bounded by 1.

1. Introduction. One of the fundamental tools in the study of Ricci curvature is the Bishop-Gromov volume inequality, which states that in a complete manifold M^m of Ricci curvature $\geq (m-1)\kappa$, the map

$$r \mapsto \frac{\operatorname{vol} B_r(p)}{\operatorname{vol}(D_r, \hat{g}_\kappa)}$$

is monotonically nonincreasing. Here, $B_r(p)$ is the ball of radius r around $p \in M$, and $(D_r, \hat{g}_\kappa)$ is a ball of same radius in the simply connected space of constant sectional curvature κ. Under somewhat different assumptions, this inequality still holds when p is replaced by a compact, totally geodesic submanifold L^l of M: The comparison space now becomes $(L \times D_r, g_\kappa)$, where for $x = (x_0, x_1)$ in the tangent space of $L \times D_r$ at (p, u), $g_\kappa(x, x) = c_\kappa^2(|u|)\,\check{g}(x_0, x_0) + \hat{g}_\kappa(x_1, x_1)$. (Here $\check{g}$ is the metric on L induced by the imbedding $L \hookrightarrow M$, and c_κ is the solution of the equation $c_\kappa'' + \kappa c_\kappa = 0$, with $c_\kappa(0) = 1$, $c_\kappa'(0) = 0$.) The volume inequality now reads (cf. [4], [3], [6]):

(∗) If the radial sectional curvatures of M are $\geq \kappa$, then

$$q_L(r) \stackrel{\text{def}}{=} \frac{\operatorname{vol} B_r(L)}{\operatorname{vol}(L \times D_r, g_\kappa)}$$

is a nonincreasing function of r, with $q_L(0) = 1$. (A 2-plane $\sigma \subset M_q$ is said to be radial if it contains the tangent vector of some minimal geodesic from q to L.)

(∗∗) If all sectional curvatures of M are $\geq \kappa$, then $q_L(r') = q_L(r)$ for some $0 < r' < r$ only if the normal bundle of $L \hookrightarrow M$ is flat with respect to the induced connection, and $B_r(L)$ is (locally) isometric to $(L \times D_r, g_\kappa)$.

In this note, we show that $(*)$ no longer holds in general if one only assumes $\mathrm{Ric}_M \geq (m-1)\kappa$ (see also **[1]** for a related result): In fact, the quotient $q_L(r)$ may go to infinity as $r \to \infty$. Moreover, even if the radial sectional curvatures are $\geq \kappa$—so that $(*)$ must hold—$(**)$ is no longer true if one replaces $K_M \geq \kappa$ by $\mathrm{Ric}_M \geq (m-1)\kappa$. More precisely, we have:

1.1. THEOREM. *Let $L = \mathbf{C}P^1$, and $M = \mathbf{C}P^2$. Then*

(a) *The normal bundle E of $L \hookrightarrow M$ admits a complete metric of nonnegative Ricci curvature such that*

$$q_L(r) \overset{\text{def}}{=} \frac{\operatorname{vol} B_r(L)}{\operatorname{vol}(L \times D_r, g_0)}$$

goes monotonically to infinity as $r \to \infty$.

(b) *There is a complete metric on M with the following properties:*

(1) *L is totally geodesically imbedded in M.*
(2) *$\mathrm{Ric}_M \geq 3$, and the radial sectional curvatures are ≥ 1.*
(3) *$q_L(r) \overset{\text{def}}{=} \frac{\operatorname{vol} B_r(L)}{\operatorname{vol}(L\times D_r, g_1)} \equiv 1$ for $r \leq \varepsilon$, provided ε is sufficiently small.*

2. Ricci curvature for connection metrics. Let $L = \mathbf{C}P^1 \hookrightarrow \mathbf{C}P^2$ with the standard metric of curvature $1 \leq K \leq 4$. As in **[5]**, we identify a distance tube $B_r(L)$ around L with $[0, r] \times S^3/\sim$, where all the Hopf fibers are collapsed to a point at $\{0\} \times S^3$. Consider the class $d\sigma_r^2$ of metrics on S^3 obtained by multiplying the standard metric by $f^2(r)$ in the Hopf fiber direction, and by $h^2(r)$ on its orthogonal complement. If f is an odd smooth function with $f'(0) = 1$, and h is even and positive, then the metric $dr^2 + d\sigma_r^2$ on $(0, r] \times S^3$ extends to $B_r(L)$. The standard metric corresponds to $f(r) = (1/2)\sin 2r$ and $h(r) = \cos r$. Using the same vector fields X_i, $0 \leq i \leq 3$, as in **[5]** (where X_0 is radial, X_1 is tangent to the Hopf fiber, and X_2, X_3 are orthogonal to it), we obtain for $R_{ij} := \mathrm{Ric}(X_i/|X_i|, X_j/|X_j|)$:

$$R_{00} = -\frac{f''}{f} - 2\frac{h''}{h}, \tag{2-1}$$

$$R_{11} = -\frac{f''}{f} - 2\frac{f'h'}{fh} + 2\frac{f^2}{h^4}, \tag{2-2}$$

$$R_{22} = R_{33} = -\frac{h''}{h} - \frac{f'h'}{fh} + \frac{4h^2 - 2f^2 - h'^2h^2}{h^4}, \tag{2-3}$$

$$R_{ij} = 0, \qquad i \neq j. \tag{2-4}$$

The proof is straightforward and will be omitted.

This class of metrics is actually a special case of the following construction: Let $(L^l, \check{g})$ be a Riemannian manifold, and $\mathbf{R}^k \to E \xrightarrow{\pi} L$ a vector bundle with inner product $\langle\ ,\ \rangle$ and Riemannian connection ∇. Fix $0 < r_0 \le \infty$, and consider the disk bundle $E^{r_0} = \{u \in E \mid \langle u, u\rangle < r_0\}$. If $\mathscr{V}$ denotes the vertical distribution defined by π, and $\mathscr{H}$ the horizontal distribution determined by the connection, define

$$g(x, x) = h^2(|u|)\, \check{g}(\pi_* x, \pi_* x) \qquad (x \in \mathscr{H} \cap T_u E),$$

where h is an even, smooth, positive function on $(-r_0, r_0)$. The fibers of E^{r_0} are endowed with a metric given in polar coordinates by

$$dr^2 + f^2(r)\, d\sigma^2,$$

where $d\sigma^2$ is the standard metric on the sphere, and f is an odd, smooth function with $f'(0) = 1$. We then obtain a metric g on E^{r_0} by declaring $\mathscr{H}$ and $\mathscr{V}$ to be mutually orthogonal. The fibers of the bundle are totally geodesic submanifolds in this metric, and the projection π restricted to a sphere bundle of radius r becomes a Riemannian submersion with base $(L, h^2(r)\, \check{g})$. One can easily compute the Ricci curvatures by using O'Neill's formula for Riemannian submersions and the Gauss equations (cf. also **[2]**): If ∂_r denotes the unit radial vector field (dual to dr), v a unit vertical vector orthogonal to ∂_r, and x a unit horizontal vector, then

$$\operatorname{Ric}(\partial_r, \partial_r) = -l\,\frac{h''}{h} - (k-1)\frac{f''}{f}, \tag{2-5}$$

$$\operatorname{Ric}(\partial_r, x) = \operatorname{Ric}(\partial_r, v) = 0, \tag{2-6}$$

$$\begin{aligned}\operatorname{Ric}(v, v) = &-\frac{f''}{f} + (k-2)\frac{1-f'^2}{f^2} - l\,\frac{f'h'}{fh}\\ &+ \sum_{i=1}^{l}\langle A_{x_i} v, A_{x_i} v\rangle,\end{aligned} \tag{2-7}$$

$$\begin{aligned}\operatorname{Ric}(x, x) = &-\frac{h''}{h} - (l-1)\frac{h'^2}{h^2} - (k-1)\frac{h'f'}{hf}\\ &+ \operatorname{Ric}^{\vee}(\pi_* x, \pi_* x) - 2\sum_{i=1}^{l}\langle A_x x_i, A_x x_i\rangle,\end{aligned} \tag{2-8}$$

$$\operatorname{Ric}(v, x) = \langle(\check{\delta} A)x, v\rangle. \tag{2-9}$$

Here, $\{x_i\}$ is an orthonormal basis of $\mathscr{H}$, A is the O'Neill tensor of the submersion with divergence $\check{\delta} A = \sum_{i=1}^{l} D_{x_i} A(x_i, \cdot)$ (D is the Levi-Civita connection of (E^{r_0}, g)), and $\mathrm{Ric}^{\vee}$ is the Ricci tensor of $(L, h^2(r)\check{g})$.

Moreover, if ∇ is a Yang-Mills connection, then (cf. **[2]**, p. 243):

$$\mathrm{Ric}(v, x) = 0. \tag{2-9'}$$

In the special case when E is the normal bundle of $\mathbf{C}P^1 \hookrightarrow \mathbf{C}P^2$, let ∇ denote the connection on E induced by the Levi-Civita connection of the symmetric space $\mathbf{C}P^2$. Then ∇ is Yang-Mills since the curvature tensor R^{∇} is parallel. In particular, (2-9') holds, and it is straightforward to check that (2-5)–(2-9) reduce to (2-1)–(2-4). Notice that the A-tensor can be expressed in terms of R^{∇}, cf. **[6]**.

3. Proof.

Proof of 1.1(a). The volume of a distance tube $B_r(L)$ with respect to the class of metrics described in §2 is given by:

$$\begin{aligned}\mathrm{vol}\, B_r(L) &= \int_0^r \mathrm{vol}\, S_t(L)\, dt \\ &= C \cdot \mathrm{vol}\,(L) \cdot h^{-l}(0) \cdot \int_0^r h^l(t) f^{k-1}(t)\, dt,\end{aligned}$$

where $S_t(L)$ is a distance sphere around L, $\mathrm{vol}\,(L) := \mathrm{vol}\,(L, h^2(0)\check{g})$, and C is the volume of the standard sphere $S^{k-1} \subset \mathbf{R}^k$. It thus suffices to find functions f and h such that (2-1)–(2-3) yield $\mathrm{Ric} \geq 0$, and $h^l(r) f^{k-1}(r)/r^{k-1} = h^2(r) f(r)/r \to \infty$ as $r \to \infty$. Let $f(r) := r/(1+r^2)^{1/2}$, and $h(r) := (r/f(r))^{\alpha}$, where α is any constant in the interval $[1/2, 1]$. Notice that $q_L(r) \to \infty$ as $r \to \infty$ if $\alpha > 1/2$, and $q_L(r) \equiv 1$ for $\alpha = 1/2$.

A straightforward calculation shows that (2-1)–(2-3) become:

$$\begin{aligned}R_{0,0} &= \frac{-3(2\alpha - 1)}{(1+r^2)^2} + \frac{2\alpha}{1+r^2}\left(2 - (\alpha+1)\frac{r^2}{1+r^2}\right) \\ &= \frac{\alpha}{1+r^2}(4 - \varphi_\alpha(r)),\end{aligned} \tag{3-1}$$

where $\varphi_\alpha(r) = \left(3(2\alpha - 1) + 2\alpha(\alpha+1)r^2\right)/\alpha(1+r^2)$. Since φ_α is an increasing function on $[0, \infty)$ with $\lim_{r\to\infty} \varphi_\alpha(r) = 2(\alpha+1) \leq 4$, we conclude that $R_{0,0} \geq 0$.

$$R_{1,1} = \frac{3 - 2\alpha}{(1+r^2)^2} + 2\frac{f^2}{h^4} \geq 0. \tag{3-2}$$

$$(3\text{-}3)\, R_{2,2} = R_{3,3} = \frac{-3\alpha}{(1+r^2)^2} + \frac{\alpha}{1+r^2}\left(1 - \alpha\frac{r^2}{1+r^2}\right)$$
$$+ 4\left(\frac{f(r)}{r}\right)^{2\alpha} - 2r^2\left(\frac{f(r)}{r}\right)^{2+4\alpha} - \frac{\alpha^2 r^2}{(1+r^2)^2}$$
$$\geq (1+r^2)^{-\alpha}(4 - (\psi_\alpha(r) + \theta_\alpha(r))),$$

where $\psi_\alpha(r) := 2r^2/(1+r^2)^{1+\alpha}$, and $\theta_\alpha(r) := (3\alpha + \alpha^2 r^2)/(1+r^2)^{2-\alpha}$.

One easily checks that the maximum of ψ_α equals

$$\eta(\alpha) = 2/\alpha(1+1/\alpha)^{1+\alpha} \leq \eta(1/2) = 4/3\sqrt{3},$$

for $\alpha \geq 1/2$. Moreover, θ_α is a decreasing function if $\alpha \leq 1$, with $\theta_\alpha(0) = 3\alpha$. Thus:

$$R_{2,2} = R_{3,3} \geq (1+r^2)^{-\alpha}(4 - (3 + 4/3\sqrt{3})) > 0,$$

thereby completing the proof of 1.1(a).

Proof of 1.1(b). When $h \equiv \cos$, (2-1)–(2-3) become:

(i) $$R_{0,0} = 2 - \frac{f''}{f},$$

(ii) $$R_{1,1} = -\frac{f''}{f} + 2\frac{f'\sin}{f\cos} + 2\frac{f^2}{\cos^4},$$

(iii) $$R_{2,2} = R_{3,3} = 1 + \frac{f'\sin}{f\cos} + \frac{4\cos^2 - 2f^2 - \sin^2\cos^2}{\cos^4}.$$

We will choose f so that $f(r) = \sin r$ for $r \leq \varepsilon$, $f(r) = \sin r \cos r$ for $r \geq \pi/4$, and $R_{i,i} \geq 3$. Define $k := f/\sin$. (i) and (ii) transform into:

(i') $$R_{0,0} = 3 - \frac{k''}{k} - 2\frac{k'\cos}{k\sin},$$

(ii') $$R_{1,1} = 3 - \frac{k''}{k} - 2\frac{k'}{k}\left(\frac{\cos}{\sin} - \frac{\sin}{\cos}\right) + 2k^2\frac{\sin^2}{\cos^4}.$$

If $\varepsilon > 0$ is sufficiently small, there exists a function k such that $k \equiv 1$ on $[0, \varepsilon]$, $k \equiv \cos$ on $[\pi/4, \pi/2]$, and $k'' \leq 0$. Then $R_{0,0}, R_{1,1} \geq 3$. To show that $R_{2,2} \geq 3$, observe that, since $f \leq \sin$,

$$F \stackrel{\text{def}}{=} (4\cos^2 - 2f^2 - \sin^2\cos^2)/\cos^4$$
$$\geq (4\cos^2 - 2\sin^2 - \sin^2\cos^2)/\cos^4 \stackrel{\text{def}}{=} G.$$

Now, the minimum value of $G = (5/\cos^2) - (2/\cos^4) + 1$ on the interval $[0, \pi/4]$ is $G(\pi/4) = 3$. Since $R_{2,2} - F = 2 + (k' \sin)/(k \cos) \geq 1$, the result follows.

We now proceed to show that the radial sectional curvatures are ≥ 1: Let $x \in T_pL$, and consider a unit-speed geodesic γ originating at p and orthogonal to L. If E denotes the parallel field along γ with $E(0) = x$, then $J := hE$ is a Jacobi field along γ, cf. [**3**]. Therefore, $R(E, \dot\gamma)\dot\gamma = -(h''/h)E$, so that $\langle R(E, \dot\gamma)\dot\gamma, E\rangle \equiv 1$. On the other hand, if v is orthogonal to both $\dot\gamma(0)$ and T_pL, and if F denotes the parallel field along γ with $F(0) = v$, then $R(F, \dot\gamma)\dot\gamma = -(f''/f)F$, and

$$\langle R(F, \dot\gamma)\dot\gamma, F\rangle = -f''/f = 1 - (k''/k) - 2(k'/k)(\cos/\sin).$$

This last expression is ≥ 1 and identically 1 on $[0, \varepsilon]$. The same is therefore true for all radial curvatures.

Finally, observe that the comparison space in [**4**] or [**3**] has the same volume growth as $(L \times D_r, g_\kappa)$. It follows that $q_L(r) \equiv 1$ for our choices of f and h when $r \leq \varepsilon$.

4. Remarks.

4.1. In 1.1(a), the maximal growth rate for the volume of $B_r(L)$ obtained by our method is of order r^3.

4.2. The maximal distance from L with respect to the metric g from 1.1(b) is $\pi/(2\sqrt{\kappa}) = \pi/2$, where κ is the infimum of the radial sectional curvatures and the Ricci curvature. Nevertheless, (M, g) is not symmetric, cf. the remark on p. 322 in [**3**].

4.3. As the general formulas of §2 show, one can produce similar examples on other vector bundles. It is, however, essential to have some information about the divergence of the A-tensor, cf. (2-9), (2-9').

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