

BORSUK-ULAM THEOREM, FIXED POINT INDEX AND CHAIN APPROXIMATIONS FOR MAPS WITH MULTIPLICITY

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In this article we consider m -acyclic maps with respect to a field \mathbb{F} and prove the existence of chain approximation for such maps. Furthermore we provide a unified approach to the Borsuk-Ulam theorem and the Bourgin-Yang generalization. Finally we prove the existence of A -systems for certain m -acyclic maps and define a fixed point index.

There are essentially two ways to handle multivalued fixed point or coincidence problems. The first one is based on homological arguments (homological method) and the second on the homotopical method where the multivalued map is approximated with single valued maps. For a survey of both methods we recommend [3], for single valued maps we refer to [4, 8].

The homological method also splits in two directions—the considerations on the level of homology groups and chain approximation techniques. For the first one see [3], the second one—chain approximations of multivalued maps—has roots in the early work of L. Vietoris (see [1] where the Vietoris-Begle mapping theorem is proved). The chain approximation technique is used by S. Eilenberg and D. Montgomery [10] to prove a Lefschetz fixed point theorem for acyclic maps on compact ANR's. B. O'Neil constructed chain approximations for a more general mapping class, i.e. $(1, n)$ -mappings, and also proved a Lefschetz fixed point theorem for such mappings on polyhedra [22]. The same technique was the main tool for developing the fixed point index with all properties (including commutativity and mod- p -property; multiplicity is proved in [26]) for multivalued maps of ANR's ([9, 11, 25]). The main result in [25] may be stated as follows: If a class of multivalued maps has arbitrarily close chain approximations, then there is a fixed point index with all properties for this class.

In this paper we consider so called m -acyclic maps with respect to a given field \mathbb{F} and prove that for such mappings there exist chain

approximations. m -maps are also called acyclic carriers and are investigated in [5, 6]. Therefore from [25], for m -acyclic maps there is also a fixed point index with all properties. The class of m -acyclic maps includes \mathbb{F} -acyclic maps, $(1, n)$ -maps [22], single valued maps with values in symmetric products of a space [20], m -maps [16], weighted maps [7], and others.

We also provide a unified approach to Borsuk-Ulam theorems for single valued mappings as well as for m -acyclic maps with respect to \mathbb{Z}_2 via chain approximations. The Borsuk-Ulam theorem or antipodal theorem can be stated in several ways; we prefer the following formulation: If $f: S^n \rightarrow \mathbb{R}^n$ is a continuous map, then there exists a point $x \in S^n$ such that $f(x) = f(-x)$, see [30]. We prove that this theorem holds for every multivalued map $F: S^n \rightarrow \mathbb{R}^n$ which has arbitrarily close chain approximations with odd Kronecker index. Therefore the Borsuk-Ulam theorem is true for m -acyclic maps w.r.t. \mathbb{Z}_2 , especially for $(1, 2k + 1)$ -maps. This is a partial answer to a question of H. Schirmer in [23]; see also [15] for acyclic maps.

There are various generalizations and applications of Borsuk-Ulam theorems (cf. [30] for a good review). We shall consider also the Bourgin-Yang generalization of the Borsuk-Ulam theorem, see [12, 30]. If $f: S^{n+k} \rightarrow \mathbb{R}^n$, $k \geq 0$, is a continuous map and $B(f) = \{x \in S^{n+k} | f(x) = f(-x)\}$ then $\dim B(f) \geq k$, or, more generally $g(B(f)) \geq k$, where $g(B(f))$ is the genus of the space $B(f)$ with the antipodal involution $x \mapsto -x$, cf. [17, 30]. The Bourgin-Yang theorem is proved for acyclic (or admissible maps) in [12].

We shall show that the Bourgin-Yang theorem holds for all multivalued maps $F: S^{n+k} \rightarrow \mathbb{R}^n$, $k \geq 0$, having arbitrarily closed chain approximations with odd Kronecker index. Therefore the Bourgin-Yang theorem is true for m -acyclic maps w.r.t. \mathbb{Z}_2 .

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I. Preliminaries.

1. *Multivalued maps with multiplicity.* We consider multivalued maps $F: K \rightarrow L$ [2] for which $F(x)$ is a nonempty compact subset of L for each $x \in K$. Two points $(x_i, y_i) \in \Gamma(F) = \{(x, y) | x \in K, y \in F(x)\} \subset K \times L$, $i = 1, 2$, are equivalent $((x_1, y_1) \sim (x_2, y_2))$ if and only if $x_1 = x_2$ and y_1, y_2 are in the same connected component of $F(x)$. This defines a new set $\tilde{\Gamma}(F) = \Gamma(F) / \sim$, with elements denoted by $(x, C(x))$; $C(x)$ denotes also a connected component of $F(x)$ as a subset of L .

By \mathbb{A} we denote a ring without zero divisors, and by \mathbb{F} we denote a field.

DEFINITION 1. Let $F: K \rightarrow L$ be a multivalued map and \mathbb{A} a ring without zero divisors. A map $m: \tilde{\Gamma}(F) \rightarrow \mathbb{A}$ is called a *multiplicity function*.

DEFINITION 2 ([5, 6]). Let $F: K \rightarrow L$ be a multivalued map with multiplicity-function $m: \tilde{\Gamma}(F) \rightarrow \mathbb{A}$. F is an *m-map* (w.r.t. \mathbb{A}) if the following two conditions are satisfied:

1. $F(x)$ consists of finitely many connected components for each $x \in K$.
2. For all $x_0 \in K$ with $F(x_0) = C_1(x_0) \cup \dots \cup C_s(x_0)$, $s = s(x_0)$, and disjoint neighbourhoods U_i of $C_i(x_0)$ there exists a neighbourhood U of x such that (a) $F(U) \subset \bigcup_{i=1}^s U_i$, and (b) $m(x_0, C_i(x_0)) = \sum_{C(x) \subset U_i} m(x, C(x))$ for all $x \in U$ and $i = 1, \dots, s$.

REMARK 1. If K is connected and $x_1, x_2 \in K$ then

$$\sum_{C(x_1)} m(x_1, C(x_1)) = \sum_{C(x_2)} m(x_2, C(x_2)),$$

where the summation is taken over the connected components of $F(x_i)$, $i = 1, 2$ (cf. Lemma 2.3 in [16]). Therefore, for K connected it makes sense to speak of the *multiplicity* of the *m-map* F , $m(F) = \sum_{C(x)} m(x, C(x))$, which is well defined.

In all that follows we suppose K to be connected.

DEFINITION 3. Let $F: K \rightarrow L$ be an *m-map* with multiplicity $m(F) = a \neq 0$. Then F is called an *m-acyclic map* (w.r.t. \mathbb{A}) if for each $x \in K$ the connected components $C(x)$ of $F(x)$ are acyclic compact sets with respect to Čech homology with coefficients in \mathbb{A} , i.e., $\tilde{H}_i(C(x), \mathbb{A}) = 0, i \geq 0$.

If for each $x \in K$ the connected components of $F(x)$ are points we call F an *m-point map* (w.r.t. \mathbb{A}).

REMARK 2. If F is an *m-acyclic map* with multiplicity $m(F) \neq 0$ w.r.t. the field \mathbb{F} , then we may consider F as an *m-acyclic map* with multiplicity 1 and multiplicity-function

$$m_1(x, C(x)) = m(x, C(x))m(F)^{-1}.$$

We assume that *m-acyclic maps* (w.r.t. a field \mathbb{F}) have multiplicity 1.

EXAMPLES. 1. Acyclic [10] or single valued maps are m -acyclic maps, m -point maps resp. with multiplicity-function $m(x, C(x)) \equiv 1$.

2. $(1, n)$ -maps F [9, 22], with $n \equiv 1 \pmod{2}$ are m -acyclic maps w.r.t. \mathbb{Z}_2 and multiplicity-function $m(x, y) \equiv 1$ for all $x, y \in \tilde{\Gamma}(F)$.

3. Let $R: \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}, \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, be a rational map; then the map $R^{-1}(z) = \{w \in \hat{\mathbb{C}} | R(w) = z\}$ is an m -point map w.r.t. \mathbb{Z} and multiplicity-function $m(z, w) = \text{local degree of } R \text{ at } w$. Furthermore, $m(R^{-1}) = d$, the degree of R . For polynomials cf. [14].

4. Let \tilde{X}, X be finite polyhedra and $\pi: \tilde{X} \rightarrow X$ a ramified covering with multiplicity map $\tilde{\mu}: \tilde{X} \rightarrow \mathbb{N}$, see [28]. The map $\pi^{-1}: X \rightarrow \tilde{X}$ is an m -point map (w.r.t. \mathbb{Z}) with multiplicity-function $\mu: \tilde{\Gamma}(\pi^{-1}) \rightarrow \mathbb{Z}, \mu(x, \tilde{x}) = \tilde{\mu}(\tilde{x})$.

5. Let X be a G -space and $|G| < \infty, G_x$ denotes the stability subgroup of $x, G_x = \{g \in G | gx = x\}$. Then the inverse map, π^{-1} , of the projection $\pi: X \rightarrow X/G$ is an m -point map w.r.t. \mathbb{Z} . If $\pi^{-1}(y) = \{x_1, \dots, x_s\}, s = s(y)$ the multiplicity-function is defined by

$$m(y, x_i) = |G_{x_i}|.$$

6. Let $f: K \rightarrow SP^n L, n \in \mathbb{N}$, be a single valued map; $SP^n L$ denotes the n -th symmetric product of L [20]. Then f induces an m -point map F w.r.t. \mathbb{Z} which is defined by

$$F(x) = \pi(f(x)),$$

with $\pi: SP^n L \rightarrow L, \pi(x_1^{k_1} \cdots x_s^{k_s}) = \{x_1, \dots, x_s\}, \sum k_i = n$. The multiplicity-function is given by $m(x, x_i) = k_i$ and F has multiplicity n .

On the other hand, if $F: K \rightarrow L$ is an m -point map w.r.t. \mathbb{Z} with multiplicity $m(F) = n > 0$ and $m(x, C(x)) \geq 0$ for all $x \in K$, then this induces a single valued map $\hat{F}: K \rightarrow SP^n L$ defined by $\hat{F}(x) = x_1^{k_1} \cdots x_s^{k_s}, s = s(x)$, and $F(x) = \{x_1, \dots, x_s\}, k_i = m(x, x_i)$.

More examples may be found in [5], [21], [28].

2. *Block complexes.* We shall use a special kind of CW-complexes, namely block-complexes, cf. [19, p. 134] (it is possible to work with regular CW-complexes [29, p. 60]).

DEFINITION 4. Let K be a finite simplicial complex with fixed triangulation $\bar{\tau}$. An n -block (w.r.t. \mathbb{A}) in K is a pair of subcomplexes (e, \dot{e}) in $\bar{\tau}$ such that $\dot{e} \subset e, \dim e = n, \dim \dot{e} = n-1$ and $H_s(e, \dot{e}) = H_s(B^n, \dot{B}^n)$ for every $s \geq 0$ (B^n denotes the n -dimensional ball and

\dot{B}^n its boundary, homology with coefficients in \mathbb{A}). The subcomplex \dot{e} is called boundary of the block e , and $\text{Int}(e) = e \setminus \dot{e}$ is called interior of e .

DEFINITION 5. A *block dissection* (b.d.) τ of the triangulation $\bar{\tau}$ is a set of blocks (w.r.t. \mathbb{A}) of $\bar{\tau}$ such that

- (a) every simplex of the triangulation $\bar{\tau}$ is in the interior of just one block of τ , and
- (b) the boundary \dot{e} of each block e of τ is a union of $(\dim e - 1)$ -blocks.

If in the simplicial complex K with given triangulation $\bar{\tau}$ a b.d. τ is fixed, we call K a block complex with block structure τ , and write (K, τ) .

REMARK 3. The block complex K is a regular CW-complex, [29, p. 60]. The mesh of the block complex (K, τ) is defined as $\text{mesh}(K, \tau) = \max\{\text{diam}(e) \mid e \in \tau\}$ (we consider the metric in K defined by barycentric coordinates of the triangulation $\bar{\tau}$ of K).

We shall consider block complexes w.r.t. a ring \mathbb{A} or a field \mathbb{F} .

DEFINITION 6. Let (K, τ) be a block complex of the simplicial complex with triangulation $\bar{\tau}$ and let τ_1 be a block structure of a subdivision $\bar{\tau}_1$ of the triangulation $\bar{\tau}$. The block structure τ_1 is called a *subdivision* of the block structure τ if every block of τ_1 is in the interior of just one block of τ . In this case we write $\tau_1 > \tau$.

DEFINITION 7. Let τ_1 be a b.d. of the triangulation $\bar{\tau}_1$. The sequence $\hat{\tau} = \{\tau_i\}$ of block subdivisions of τ_1 is called a *fundamental sequence of block dissections* (f.s.b.d.) with respect to τ_1 if:

- (a) τ_{i+1} is a subdivision of τ_i , $i \geq 0$, and
- (b) $\lim_{i \rightarrow \infty} \text{mesh}(K, \tau_i) = 0$.

Let τ be a b.d. (w.r.t. \mathbb{A}) of the triangulation $\bar{\tau}$. We denote by $C_*(K, \tau)$ or $C_*(\tau)$ the chain complex of the block complex τ with coefficients in a given ring \mathbb{A} (the chain complex of the block-complexes (K, τ) , [19]).

For $M \subset K$ we define $St(M, \tau) = \bigcup\{e \in \tau \mid e \cap M \neq \emptyset\}$ and $St^{k+1}(M, \tau) = St(St^k(M, \tau), \tau)$ for $k \in \mathbb{N}$.

For a given b.d. τ_1 and its block subdivision τ_2 there is a chain map (chain subdivision)

$$b(\tau_1, \tau_2): C_*(\tau_1) \rightarrow C_*(\tau_2)$$

such that (a) $b(\tau_1, \tau_2)(e) = e$ for every 0-block, and (b) $|b(\tau_1, \tau_2)(e)| \subset e$ for every block $e \in \tau$, [26]. Here $|c|$ denotes the support of the

chain $c \in C_*(\tau)$, i.e., the smallest block subcomplex $K' \subset K$ of the block structure τ such that $c \in C_*(K', \tau)$.

There exists a chain map (chain approximation of the identity)

$$\chi(\tau_2, \tau_1): C_*(\tau_2) \rightarrow C_*(\tau_1)$$

such that (a) $\chi(\tau_2, \tau_1)(e)$ is a 0-block in τ_1 for every 0-block $e \in \tau_2$, and (b) $|\chi(\tau_2, \tau_1)(e)| \subset |e|$ for all blocks $e \in \tau_2$, where $|e|$ is the support of the block e in τ_1 , i.e., $|e|$ is the unique block in τ_1 such that $e \subset \text{Int}|e|$.

The chain map $\chi(\tau_2, \tau_1)$ is not unique but every two such maps are chain homotopic with a chain homotopy D such that $|D(e)| \subset |e|$, [26].

If $\hat{\tau} = \{\tau_i\}$ is a f.s.b.d. of K we shall consider

$$C_*(\hat{\tau}) = \{C_*(\tau_i), b(\tau_i, \tau_{i+1}), \chi(\tau_{i+1}, \tau_i), i = 0, 1, \dots\}$$

with fixed chain maps $\chi(\tau_{i+1}, \tau_i)$. If the f.s.b.d. is clear from the context we use the notation $b(i, i+1)$ and $\chi(i+1, i)$, and for $k < l$ we denote the chain subdivision map from $C_*(\tau_k)$ in $C_*(\tau_l)$ by $b(k, l) = b(l-1, l)b(l-2, l-1) \cdots b(k, k+1)$ and by $\chi(\tau_l, \tau_k) = \chi(k-1, k) \cdots \chi(l, l-1)$ the chain map from $C_*(\tau_l)$ in $C_*(\tau_k)$.

3. Chain approximation of upper semicontinuous maps.

DEFINITION 8. Let K and L be finite simplicial complexes with given block structures $\tau_1, \tau_2, \tau_2 > \tau_1$ on K , and μ on L (w.r.t. \mathbb{A}). Let $F: K \rightarrow L$ be an upper semicontinuous (u.s.c.) map. The chain map

$$\varphi: C_*(\tau_2) \rightarrow C_*(\mu)$$

is called a (τ_1, μ) -chain approximation (or chain approximation) of the map F (with coefficients in a ring \mathbb{A}) if for every block $e \in \tau_2$ there is a point $s(e) \in K$ such that (a) $e \subset \text{St}(s(e), \tau_1)$, and (b) $|\varphi(e)| \subset \text{St}(F(s(e)), \mu)$.

REMARK 4. 1. For upper semicontinuous maps see [2].

2. In [25], [26] it is proved that every \mathbb{F} -acyclic map F has arbitrarily close chain approximations (with coefficients in \mathbb{F} , i.e., for every block structure τ_1 and μ there is a block structure $\tau_2, \tau_2 > \tau_1$, and a chain map $\varphi: C_*(\tau_2) \rightarrow C_*(\mu)$ which is a (τ_1, μ) -chain approximation of F . If $\tau_1 \in \hat{\tau}$ and $\mu \in \hat{\mu}$ ($\hat{\tau}$ and $\hat{\mu}$ are f.s.b.d. of K and L respectively) then $\tau_2 \in \hat{\tau}$. We say F has a chain approximation with respect to $\hat{\tau}$ and $\hat{\mu}$.

3. If $F: K \rightarrow L$ is an u.s.c. map which has chain approximations with respect to a given f.s.b.d. $\hat{\tau}$ and $\hat{\mu}$ of K and L , respectively, then F has chain approximation with respect to any f.s.b.d. $\hat{\tau}_1$ and $\hat{\mu}_1$ of K and L , respectively (follows from [26]).

4. In [9, 22] the existence of arbitrary close chain approximations (with coefficients in \mathbb{F}) for $(1, n)$ -maps $F: K \rightarrow L$, $n \geq 2$, is proved.

DEFINITION 9. Let K, L be finite simplicial complexes with triangulations $\bar{\tau}, \bar{\mu}$, resp., and let $\hat{\tau} = \{\tau_i\}$, $\hat{\mu} = \{\mu_i\}$ be f.s.b.d. with respect to $\bar{\tau}$ and $\bar{\mu}$, resp. Let $F: K \rightarrow L$ be an u.s.c. map. The graded set

$$A(F) = \{A(F)_i | i \in \mathbb{N}\}$$

where $A(F)_i$ is a nonempty set of chain maps from $C_*(\tau_i)$ to $C_*(\mu_i)$ with coefficients in a ring \mathbb{A} is called an *approximation system* (A-system) of F with respect to $\hat{\tau}$ and $\hat{\mu}$ if

(a) for every $\varphi \in A(F)_i$ there is $k > i$, $k \in \mathbb{N}$, such that $\varphi = \bar{\varphi}b(i, k)$ and $\bar{\varphi}: C_*(\tau_k) \rightarrow C_*(\mu_i)$ is a (τ_i, μ_i) -chain approximation of F , where $b(i, k): C_*(\tau_i) \rightarrow C_*(\tau_k)$ is a barycentric subdivision, and

(b) for every $i \in \mathbb{N}$ there is $i_1 > i$, $i_1 \in \mathbb{N}$, such that for $m \geq l \geq i_1$, $\varphi = \bar{\varphi}b(l, l_1) \in A(F)_l$, $\psi = \bar{\psi}b(m, m_1) \in A(F)_m$ with $m_1 \geq l_1$, the chain maps $\bar{\varphi}\chi(m_1, l_1)$ and $\chi(m, l)\bar{\psi}$ are homotopic with a chain homotopy D satisfying the following condition: for every block $e \in \tau_{m_1}$ there is a point $b(e) \in K$ such that $e \subset St(b(e), \tau_i)$ and $|D(e)| \subset St(F(b(e)), \mu_i)$.

REMARK 5. 1. In [26] it is proved that every \mathbb{F} -acyclic map has an A-system (coefficients in the field \mathbb{F}) with respect to a given f.s.b.d. $\hat{\tau}$ and $\hat{\mu}$ with Kronecker index, $KIA(F) = 1$, i.e., for every $\varphi \in A(F)$, $KI\varphi = 1$. The Kronecker index $KI\varphi$ of a chain map $\varphi: C_*(\tau_i) \rightarrow C_*(\mu_i)$ is defined as $KI\varphi = \varepsilon(\varphi(e))$, where $\varepsilon: C_0(\mu_i) \rightarrow \mathbb{A}$ is the augmentation of the chain complex $C_*(\mu_i)$ and e is a 0-dimensional block, see [27, p. 167]. Since we consider connected polyhedra the Kronecker index $KI\varphi$ of the chain map φ is well defined.

2. In [9] the existence of an A-system $A(F)$ for every $(1, n)$ -map with $KIA(F) = n$ with coefficients in a fixed field \mathbb{F} is proved. In particular, if one chooses \mathbb{F} to be \mathbb{Z}_2 and n odd there exists an A-system of F with Kronecker index different from zero. We consider only A-systems with nonzero Kronecker index.

4. *Equivariant maps, block complexes and A-systems.* Let X be a compact space. The single valued continuous map $\gamma: X \rightarrow X$ is called

an involution if $\gamma^2(x) = x$ for all $x \in X$. The involution is called free if $\gamma(x) \neq x$ for all $x \in X$. If $\gamma(x) = x$ for all $x \in X$ then γ is called the trivial involution. By α we denote the antipodal involution on \mathbb{R}^n , $\alpha(x) = -x$, $x \in \mathbb{R}^n$, or some α -invariant subset of \mathbb{R}^n .

Let X_i , $i = 1, 2$, be spaces with involution $\gamma_i: X_i \rightarrow X_i$, resp. and $F: X_1 \rightarrow X_2$ an m -acyclic map (w.r.t. \mathbb{A}) and multiplicity function $m: \tilde{\Gamma}(F) \rightarrow \mathbb{A}$. Suppose that F is (γ_1, γ_2) -equivariant, i.e., $F(\gamma_1(x)) = \gamma_2(F(x))$. Then there is an involution γ on the graph $\Gamma(F) = \{(x, y) \in X_1 \times X_2 \mid y \in F(x)\}$ defined by $\gamma(x, y) = (\gamma_1(x), \gamma_2(y))$ with $(x, y) \in X_1 \times X_2$. The involution γ induces an involution on $\tilde{\Gamma}(F)$ which we also denote by γ .

DEFINITION 10. Let X_i , $i = 1, 2$, be spaces with involution γ_i , respectively. The m -acyclic map $F: X_1 \rightarrow X_2$ w.r.t. \mathbb{A} with multiplicity function $m: \tilde{\Gamma}(F) \rightarrow \mathbb{A}$ is called (γ_1, γ_2) -equivariant if

- (a) $F(\gamma_1(x)) = \gamma_2(F(x))$ for all $x \in X_1$, and
- (b) $m(\gamma(x, C(x))) = m(x, C(x))$ for all $(x, C(x)) \in \tilde{\Gamma}(F)$.

Let K be a simplicial complex with a block structure τ_1 and involution $\gamma: K \rightarrow K$. The block structure τ_1 is γ -equivariant if $\gamma(e)$ is a block in τ_1 for every block $e \in \tau_1$.

REMARK 6. 1. The map $F: X_1 \rightarrow X_2$ is called (γ_1, γ_2) -equivariant if the condition (a) of Definition 10 is fulfilled. If $\hat{\tau}$ is a f.s.b.d. on K , $\hat{\tau}$ is called γ -equivariant if each τ_i is γ -equivariant.

2. Let $\mathbb{R}^m = \{(x_1, \dots, x_m) \mid x_i \in \mathbb{R}\}$ be the m -dimensional Euclidean space. If $n > m$ we consider \mathbb{R}^m as a linear subspace of \mathbb{R}^n consisting of the points $(x_1, \dots, x_m, 0, \dots, 0)$.

Let $a \in \mathbb{R}$, $a > 0$, then D_a^m denotes the cube

$$D_a^m = \{(x_1, \dots, x_m) \mid |x_i| \leq a\}.$$

The inclusion $\iota: \mathbb{R}^m \rightarrow \mathbb{R}^n$ ($n > m$) induces the inclusion $D_a^m \subset D_a^n$. The boundary of D_a^{n+1} is denoted by S_a^n , or simply S^n , again $S^m \subset S^n$ for $n > m$. The antipodal involution α acts on both spaces D_a^n and S_a^n .

We consider the following block structure $\mu(a)$ (cubic block subdivision of \mathbb{R}^n) which consists of all translations of D_a^n with vectors $\sum_{i=1}^n 2ak_i e_i$, where e_i denotes the i -th unit vector and $k_i \in \mathbb{Z}$. If $a \in \mathbb{N}$ then $\mu_k = \mu(1/k)$, $k \in \mathbb{N}$, is a block structure on D_a^n , and S_a^n is a block subcomplex. S_a^m is a block subcomplex in S_a^n , $n > m$, also. The block structure on S_a^n induced by μ_k is denoted by τ_k . Then $\hat{\mu} = \{\mu_k\}$ and $\hat{\tau} = \{\tau_k\}$ are α -equivariant f.s.b.d. of D_a^n and S_a^n , respectively.

3. Let K be a block complex with a given f.s.b.d. $\hat{\tau} = \{\tau_i\}$. The simplicial complex $K \times K$ has a natural block structure $\tau_i \times \tau_i$ induced by τ_i for every $i \in \mathbb{N}$:

$$\tau_i \times \tau_i = \{e_1 \times e_2 | e_i \in \tau_i, i = 1, 2\}.$$

Then $\hat{\tau} \times \hat{\tau}$ is a f.s.b.d. for $K \times K$. We call $\hat{\tau} \times \hat{\tau}$ the product block structure. E.g., the block structure $\hat{\mu} = \{\mu_k\}$ of D_a^n is a product of block structures on the interval D_a^1 .

4. On the space $K \times K$ acts the involution $T: K \times K \rightarrow K \times K$, $T(x, y) = (y, x)$. The f.s.b.d. $\hat{\tau} \times \hat{\tau}$ is T -equivariant.

Let K_s , $s = 1, 2$, be finite simplicial complexes with involutions γ_s and γ_s -equivariant f.s.b.d. $\hat{\tau}_s = \{\tau_i^s\}$. Since the block structures τ_i^s are γ_s -equivariant each γ_s induces a chain map which we denote also by γ_s :

$$\gamma_s: C_*(\tau_i^s) \rightarrow C_*(\tau_i^s).$$

A chain map (or homomorphism) $\varphi: C_*(\tau_i^1) \rightarrow C_*(\tau_i^2)$ is (γ_1, γ_2) -equivariant if $\varphi\gamma_1 = \gamma_2\varphi$. If $K_1 = K_2$, $\gamma_1 = \gamma_2 = \gamma$ we simply say γ -equivariant.

Since the f.s.b.d. $\hat{\tau}_s$ are γ_s -equivariant the chain subdivision maps $b(k+1, k) = b(\tau_k^s, \tau_{k+1}^s)$ are γ_s -equivariant too. In this situation we choose chain approximations of the identity $\chi(k+1, k) = \chi(\tau_{k+1}^s, \tau_k^s)$ to be γ_s -equivariant also, and we consider $C_*(\tau^s)$ with these chain maps.

Let $F: K_1 \rightarrow K_2$ be a (γ_1, γ_2) -equivariant u.s.c. map. The A-system, $A(F)$, with respect to a γ_s -equivariant f.s.b.d. $\hat{\tau}_s$, $s = 1, 2$, is called (γ_1, γ_2) -equivariant if every chain map $\varphi \in A(F)$ is (γ_1, γ_2) -equivariant. Again, if $K_1 = K_2$, $\gamma_1 = \gamma_2 = \gamma$ the A-system $A(F)$ is called γ -equivariant.

DEFINITION 11 ([29, p. 60]). Let (K_1, τ_1) and (K_2, τ_2) be block complexes. A carrier from (K_1, τ_1) to (K_2, τ_2) is a function \mathcal{C} which assigns to each block $e \in \tau_1$ a block subcomplex $\mathcal{C}(e)$ in τ_2 such that for $e_1 \in e$, $\mathcal{C}(e_1) \subset \mathcal{C}(e)$. An n -acyclic (resp. acyclic) carrier \mathcal{C} satisfies $\tilde{H}_i(\mathcal{C}(e)) = 0$ for $0 \leq i \leq n$ (resp. for all i) and every block $e \in \tau_1$ (coefficients in \mathbb{A}).

If γ_i is an involution on K_i such that the block structure τ_i is γ_i -equivariant, $i = 1, 2$. Then the carrier \mathcal{C} is called (γ_1, γ_2) -equivariant (or equivariant) if $\mathcal{C}(\gamma_1(e)) = \gamma_2(\mathcal{C}(e))$ for all blocks $e \in \tau_1$. A chain map $\varphi: C_*(\tau_1) \rightarrow C_*(\tau_2)$ is carried by the carrier \mathcal{C} if $\varphi(e)$ is a chain in the complex $\mathcal{C}(e)$ for every block $e \in \tau_1$.

A map $f: K_1 \rightarrow K_2$ is carried by the carrier \mathcal{E} if $f(e) \subset \mathcal{E}(e)$ for every block $e \in \tau_1$.

REMARK 7. 1. The block structure $\hat{\tau} = \{\tau_k\}$ on S_a^n is α -equivariant and the block complex S_a^n is α -free cell complex, i.e., $\alpha(e) \cap e = \emptyset$ for every $e \in \tau_k$

2. For later use we note the following. The covering $w_1 = \{A_1, \dots, A_N\}$ of a given space is finer than the covering $w_2 = \{B_1, \dots, B_M\}$ of the same space if every A_i is contained in some B_j . If the covering w_1 is finer than the covering w_2 we denote this by $w_1 > w_2$.

II. Borsuk, Borsuk-Ulam and Bourgin-Yang theorems for chain mappings. In this section we state some lemmata which will be important for the following. In this section we consider chains, A-systems, and homology with \mathbb{Z}_2 coefficients.

For a block complex (K, τ) we denote by $K^{(i)}$ the i -dimensional skeleton of K w.r.t. τ .

LEMMA 1. *Let S^n , $n \geq 1$, be the n -dimensional sphere with the α -equivariant block structure τ_k . Then there exists an α -equivariant chain map*

$$\beta: C_*((S^n)^{(n-1)}, \tau_k) \rightarrow C_*(S^{n-1}, \tau_k)$$

with Kronecker index $KI\beta \equiv 1 \pmod{2}$ and such that

(a) *the chain map β with values in $C_*(S^n, \tau_k)$ is chain homotopic to the identity inclusion $(S^n)^{n-1} \subset S^n$ with an α -equivariant chain homotopy D , and*

(b) *for each block e in S^{n-1} one has $\beta(e) = e$ and $D(e) = 0$.*

Proof. Consider the carrier $\mathcal{E}: ((S^n)^{(n-1)}, \tau_k) \rightarrow (S^{n-1}, \tau_k)$, defined by $\mathcal{E}(e) = S^{n-1}$ for every block $e \in (S^n)^{(n-1)}$, cf. Definition I.11. \mathcal{E} is an α -equivariant $(n-2)$ -acyclic carrier. Since $((S^n)^{n-1}, \tau_k)$ is a free α -complex, Lemma 2.2 in [29] implies the existence of an α -equivariant extension β of the identity chain map on S^{n-1} over $(S^n)^{(n-1)}$. Furthermore, β considered as a chain map with values in $C_*(S^n, \tau_k)$ and the chain map induced by the identity inclusion of $(S^n)^{(n-1)} \subset S^n$ are chain homotopic with equivariant chain homotopy D satisfying (b). □

REMARK 1. Lemma 2.2 in [29] is proved for an acyclic carrier, but for our purposes it suffices to assume that \mathcal{E} is an $(n - 2)$ -acyclic carrier.

The following lemma shows that each α -equivariant chain map $\kappa: C^*(S^n, \tau_k) \rightarrow C^*(S^n, \tau_k)$ is homotopic to an α -equivariant map $\bar{\kappa}$, which is a chain map on (S^{n-1}, τ_k) .

LEMMA 2. *Let $\kappa: C^*(S^n, \tau_k) \rightarrow C^*(S^n, \tau_k)$ be an α -equivariant chain map. Then there exists an α -equivariant chain map $\bar{\kappa}: C^*(S^n, \tau_k) \rightarrow C^*(S^n, \tau_k)$ such that*

- (a) κ and $\bar{\kappa}$ are chain homotopic, and
- (b) for all blocks $e \in S^{n-1}$, $\bar{\kappa}(e)$ is a chain in S^{n-1} .

Proof. Let $\beta = \{\beta_i\}$, $D = \{D_i\}$, $i = 1, 2$, be as in Lemma 1 and let e be an i -dimensional block in S^n with $\partial e = \sum_j e(j)$. Define

$$\bar{\kappa}_i(e) = \begin{cases} \beta_i(\kappa(e)) & \text{for } e \in S^{n-1}, \\ \kappa(e) + D_{i-1}(\kappa(e(j))) & \text{if } e \cap S^{n-1} = e(j), \\ \kappa(e) & \text{otherwise.} \end{cases}$$

Then $\bar{\kappa} = \{\bar{\kappa}_i | i = 0, \dots, n\}$ is an α -equivariant chain map satisfying (b) and $KI\kappa = KI\bar{\kappa} \equiv 1 \pmod{2}$. Consider the carrier $\mathcal{C}: (S^n, \tau_k) \rightarrow (S^n, \tau_k)$, $\mathcal{C}(e) = S^n$ for every block $e \in \tau_k$. It is an $(n - 1)$ -acyclic carrier. The chain maps κ and $\bar{\kappa}$ are carried by \mathcal{C} (cf. Definition I.11). Since τ_k is an α -free complex and the Kronecker indices of these chain maps are equal, $KI\kappa = KI\bar{\kappa} \equiv 1 \pmod{2}$, using Remark 1 following Lemma 2.2 in [29], we conclude that the chain maps κ and $\bar{\kappa}$ are homotopic. □

LEMMA 3. *Let $\kappa: C_*(S^n, \tau_k) \rightarrow C_*(S^n, \tau_k)$ be an α -equivariant chain map with Kronecker index $KI\kappa \equiv 1 \pmod{2}$. Then $\text{deg } \kappa \equiv 1 \pmod{2}$, where deg denotes the degree of κ .*

Proof. Due to Lemma 2 we may assume that $\kappa: C_*(S^{n-1}, \tau_k) \rightarrow C_*(S^{n-1}, \tau_k)$. Because chain homotopy does not change either the Kronecker index or the degree this is no restriction. The proof proceeds by induction on n . We start with $n = 1$.

Due to our assumption $\kappa: C_*(S^0, \tau_k) \rightarrow C_*(S^0, \tau_k)$ and since $KI\kappa \equiv 1 \pmod{2}$ we obtain $\kappa(1) = 1$ or $\kappa(1) = \bar{1}$, $S^0 = \{1, \bar{1}\}$. Let $D_+ = \{(x_1, x_2) \in S^1 | x_2 \geq 0\}$, $D_- = \{(x_1, x_2) \in S^1 | x_2 \leq 0\}$ be subcomplexes of S^1 which we consider as 1-dimensional chains in τ_k , such that $\partial D_+ = \bar{1} + 1 = \partial D_-$ (coefficients in \mathbb{Z}_2). The chain $\kappa(D_+)$ may be written as $\kappa(D_+) = uD_+ + vD_-$ and since $\kappa(1) = 1$ or $\bar{1}$, then $\partial\kappa(D_+) = (u + v)(1 + \bar{1}) = 1 + \bar{1} \pmod{2}$. Now, $(u + v) \equiv 1 \pmod{2}$ yields

$$\deg \kappa(D_+ + D_-) = \kappa(D_+ + D_-) \equiv D_+ + D_- \pmod{2},$$

therefore $\deg \kappa \equiv 1 \pmod{2}$.

Now let us assume that the assumption is true up to $n - 1$. Then $\kappa|_{S^{n-1}}: C_*(S^{n-1}, \tau_k) \rightarrow C_*(S^{n-1}, \tau_k)$ is α -equivariant and $\deg(\kappa|_{S^{n-1}}) \equiv 1 \pmod{2}$. Let

$$D_+ = \{(x_1, \dots, x_{n+1}) | x_{n+1} \geq 0\}, \quad D_- = \{(x_1, \dots, x_{n+1}) | x_{n+1} \leq 0\}$$

be considered as chains in S^n . If we consider S^{n-1} as an $(n - 1)$ -dimensional chain in τ_k we obtain the relation $\partial D_+ = \partial D_- = S^{n-1}$ (w.r.t. \mathbb{Z}_2 coefficients).

Now we calculate $\kappa(D_+ + D_-)$ using $\kappa(D_+) = uD_+ + vD_-$, $\deg \kappa|_{S^{n-1}}(S^{n-1}) = \kappa(\partial D_+) = (u + v)(S^{n-1}) = S^{n-1}$ and $u + v = \deg(\kappa|_{S^{n-1}}) \equiv 1 \pmod{2}$. Then

$$\deg \kappa(D_+ + D_-) = \kappa(D_+ + D_-) = (u + v)(D_+ + D_-) = D_+ + D_-.$$

Therefore $\deg \kappa \equiv 1 \pmod{2}$. □

An immediate consequence is

COROLLARY 4 (Borsuk theorem for chain maps). *Let $\kappa: C_*(S^n \tau_k) \rightarrow C_*(S^m, \tau_k)$ be an α -equivariant chain map with Kronecker index $KI\kappa \equiv 1 \pmod{2}$. Then $m \geq n$.*

LEMMA 5 (Borsuk-Ulam theorem for chain maps). *Let $\kappa: C_*(S^n, \tau) \rightarrow C_*(D_S^n, \mu_k)$ (cf. Remark I.6.2) be a chain map with Kronecker index $KI\kappa \equiv 1 \pmod{2}$. Then there exists a block $e \in \tau$ such that $|\kappa(e)| \cap |\kappa(\alpha(e))| \neq \emptyset$.*

Proof. Assume that

$$(1) \quad |\kappa(e)| \cap |\kappa(\alpha(e))| = \emptyset \quad \text{for all blocks } e \in \tau.$$

Consider the map $\bar{\Delta}: S^n \rightarrow S^n \times S^n$ defined by $\bar{\Delta}(x) = (x, -x)$ and the block structure on $S^n \times S^n$ given by $\tau \times \tau$. The map $\bar{\Delta}$ is (α, T) -equivariant (cf. I.4, and Remark I.6.2). The carrier $\mathcal{E}: (S^n, \tau) \rightarrow (S^n \times S^n, \tau \times \tau)$, with $\mathcal{E}(e) = e \times \alpha(e)$ is (α, T) -invariant and acyclic ($\bar{\Delta}$ is carried by \mathcal{E}). Since (S^n, τ) is an α -free block complex, Lemma 2.2 in [29] implies the existence of an (α, T) -equivariant chain map $\bar{\Delta}_\#: C_*(S^n, \tau) \rightarrow C_*(S^n \times S^n, \tau \times \tau)$ which is carried by the carrier \mathcal{E} , i.e., $\bar{\Delta}_\#(e)$ is a chain in $e \times \alpha(e)$ for all blocks in τ and $KI\bar{\Delta}_\# \equiv 1 \pmod{2}$.

Consider the chain map $(\kappa \otimes \kappa)\bar{\Delta}_\#: C_*(\tau) \rightarrow C_*(\tau \times \tau)$. Since $|\bar{\Delta}_\#(e)| \subset e \times \alpha(e)$ for every block $e \in \tau$ then

$$|(\kappa \otimes \kappa)\bar{\Delta}_\#(e)| \subset |\kappa(e)| \times |\kappa(\alpha(e))|$$

and from (1) we obtain

$$(2) \quad |(\kappa \otimes \kappa)\bar{\Delta}_\#(e)| \cap |(\kappa \otimes \kappa)\bar{\Delta}_\#(\alpha(e))| = \emptyset$$

for every block $e \in \tau$.

The chain map $(\kappa \otimes \kappa)\bar{\Delta}_\#$ is (α, T) -equivariant and from (2) it follows that

$$|(\kappa \otimes \kappa)\bar{\Delta}_\#(e)| \cap M = \emptyset$$

for every block $e \in \tau$ and $M = \{(x, y) \in D_s^n \times D_s^n | x = y\}$. Then there exists a block subdivision μ_l ($l > k$) of the block structure μ_k (cf. Definition I.6) such that $|(\kappa \otimes \kappa)\bar{\Delta}_\#(e)| \cap M' = \emptyset$ for every block e in τ , and $M' = St(M, \mu_l \times \mu_l)$. Let N be the closure of $D_s^n \times D_s^n \setminus M'$. Then N is a subcomplex in $D_s^n \times D_s^n$ with respect to the block structure $\mu_l \times \mu_l$ and N is T -equivariant. Since $|(\kappa \otimes \kappa)\bar{\Delta}_\#(e)|$ is in N then $b(\mu_k \times \mu_k, \mu_l \times \mu_l)(\kappa \otimes \kappa)\bar{\Delta}_\#(e)$ is a chain in N and therefore $b(\mu_k \times \mu_k, \mu_l \times \mu_l)(\kappa \otimes \kappa)\bar{\Delta}_\#$ is a chain map from $C_*(S^n, \tau)$ to $C_*(N, \mu_l \times \mu_l)$ which is (α, T) -equivariant (cf. I.2 for the chain subdivision b).

Let $d: N \rightarrow S^{n-1}$ be defined by $d(u, v) = (u - v) / \|u - v\|$. Obviously d is (α, T) -equivariant. Let $d_\#: C_*(N, \mu_l \times \mu_l) \rightarrow C_*(S^{n-1}, \tau)$ be the (α, T) -equivariant chain map induced by d with Kronecker index $KId_\# \equiv 1 \pmod{2}$ for l sufficiently large.

Finally, consider the chain map

$$\Xi = d_\#b(\mu_k \times \mu_k, \mu_l \times \mu_l)(\kappa \otimes \kappa)\bar{\Delta}: C_*(S^n, \tau) \rightarrow C_*(S^{n-1}, \tau),$$

which is α -equivariant with Kronecker index $KI\Xi \equiv 1 \pmod{2}$. This contradicts Corollary 4. □

Let (K, ν) be a block complex with γ -equivariant block structure ν and γ a free involution on K .

DEFINITION 6. The *chain genus* of a γ -free block complex (K, ν) is defined by

$$g_c(K, \gamma) = \min \left\{ l \left| \begin{array}{l} \text{there exists a } (\gamma, \alpha)\text{-equivariant} \\ \text{chain map } \eta: C_*(K, \nu) \rightarrow C_*(S^l, \tau_k) \\ \text{with } KI\eta \equiv 1 \pmod{2} \text{ and } |\eta(e)| \cap \\ |\eta(\alpha(e))| = \emptyset \text{ for all } e \in \nu. \end{array} \right. \right\}.$$

REMARK 2. The genus $g(K, \gamma)$ of a free involution γ on K is defined as

$$g(K, \gamma) = \min \left\{ l \mid \begin{array}{l} \text{there is a } (\gamma, \alpha)\text{-equivariant, continuous,} \\ \text{single valued map } f: K \rightarrow S^l; \end{array} \right\}$$

it is easy to see that $g(K, \gamma) \geq g_c(K, \gamma)$. Furthermore, recall that $\dim K \geq g(K, \gamma)$, cf. [30]. Therefore $\dim K \geq g_c(K, \gamma)$.

COROLLARY 7. Let $\kappa: C_*(S^{n+k}, \tau_l) \rightarrow C_*(D^n, \mu_l)$, $k \geq 0$, be a chain map with Kronecker index $KI\kappa \equiv 1 \pmod{2}$. Then $g_c(B(\kappa), \gamma) \geq k$, with $B(\kappa) = \bigcup \{e \in \tau_k \mid |\kappa(e)| \cap |\kappa(\alpha(e))| \neq \emptyset\}$.

Proof. The case $k = 0$ is an immediate consequence of Lemma 5. We assume $k \geq 1$ and use the idea in [12, 2.6]. Assume that $g_c(B(\kappa), \alpha) = m < k$. Then we consider the chain map $\eta: C_*(B(\kappa), \tau_l) \rightarrow C_*(S^m, \tau_l)$ (cf. Definition 6) as a chain map in $C_*(D^{m+1}, \mu_l)$ (S^m is a boundary of D^{m+1}). Let \mathcal{E} be the carrier from $B(\kappa)$ in D^{m+1} defined by $\mathcal{E}(e) = D^{m+1}$ for all blocks. This is an α -equivariant acyclic carrier. Since $KI\eta \equiv 1 \pmod{2}$ there is an (α -equivariant) chain map

$$\tilde{\eta}: C_*(S^{n+k}, \tau_l) \rightarrow C_*(D^{m+1}, \mu_l)$$

with Kronecker index $KI\tilde{\eta} \equiv 1 \pmod{2}$ and which is an extension of the chain map η (Lemma 2.2, [29]).

Consider the carrier \mathcal{E}_1 from (S^{n+k}, τ_l) in $(S^{n+k} \times S^{n+k}, \tau_l \times \tau_l)$ defined by $\mathcal{E}_1(e) = e \times e$. This is an $(\alpha, \bar{\alpha})$ -equivariant acyclic carrier, with $\bar{\alpha}(x, y) = (\alpha(x), \alpha(y))$ for all $(x, y) \in S^{n+k} \times S^{n+k}$. Lemma 2.2 in [29] guarantees the existence of an $(\alpha, \bar{\alpha})$ -equivariant chain map $\Delta_\#: C_*(\tau_l) \rightarrow C_*(\tau_l \times \tau_l)$ with $KI\Delta_\# \equiv 1 \pmod{2}$ and $|\Delta_\#(e)| \subset \mathcal{E}_1(e)$ for all blocks $e \in \tau_l$.

Since $C_*(D^n, \mu_l) \otimes C_*(D^{m+1}, \mu_l) \approx C_*(D^n \times D^{m+1}, \mu_l \times \mu_l) = C_*(D^{m+n+1}, \mu_l)$ we consider the chain map $\Xi = (\kappa \otimes \tilde{\eta})\Delta_\#$ as a chain map

$$\Xi: C_*(S^{n+k}, \tau_l) \rightarrow C_*(D^{m+n+1}, \mu_l)$$

with Kronecker index $KI\Xi \equiv 1 \pmod{2}$. Since $m < k$, then $(D^{m+n+1}, \mu_l) \subset (D^{n+k}, \mu_l)$ and we consider Ξ as a chain map in $C_*(D^{n+k}, \mu_l)$.

Then from Lemma 5 there is a block $e \in \tau_l$ such that

$$|(\kappa \otimes \tilde{\eta})\Delta_\#(e)| \cap |(\kappa \otimes \tilde{\eta})\Delta_\#(\alpha(e))| \neq \emptyset$$

and therefore also

$$(|\kappa(e)| \times |\tilde{\eta}(e)|) \cap (|\kappa(\alpha(e))| \times |\tilde{\eta}(\alpha(e))|) \neq \emptyset,$$

i.e., $|\kappa(e)| \cap |\kappa(\alpha(e))| \neq \emptyset$ and $|\tilde{\eta}(e)| \cap |\tilde{\eta}(\alpha(e))| \neq \emptyset$. The first implies that $e \in B(\kappa)$, therefore $\tilde{\eta}(e) = \eta(e)$ and $\tilde{\eta}(\alpha(e)) = \eta(\alpha(e))$. This is impossible since $|\eta(e)| \cap |\eta(\alpha(e))| = \emptyset$, cf. Definition 6. \square

COROLLARY 8 (Bourgin-Yang theorem for chain maps). *Let $\kappa: C_*(S^{n+k}, \tau_l) \rightarrow C_*(D^n, \mu_l)$, $k \geq 0$, be a chain map with Kronecker index $KI\kappa \equiv 1 \pmod{2}$, then $\dim B(\kappa) \geq k$.*

The proof is a direct consequence from Corollary 7 and Remark 2.

III. Borsuk, Borsuk-Ulam, and Bourgin-Yang theorems for maps with A-systems with coefficients in \mathbb{Z}_2 .

1. *Borsuk theorem.*

PROPOSITION 1. *Let $F: (S^n, \alpha) \rightarrow (S^m, \alpha)$ be an u.s.c. α -equivariant map. If F has an α -equivariant A-system $A(F)$ with coefficients in \mathbb{Z}_2 then $m \geq n$.*

Proof. Let $\varphi \in A(F)_k$. Then φ is α -equivariant and $KI\varphi \equiv 1 \pmod{2}$. From Corollary II.4 follows $m \geq n$. \square

2. *Existence of A-systems for products of two maps with A-system.*
To prove the Borsuk-Ulam theorem for u.s.c. maps $F: S^n \rightarrow \mathbb{R}^n$ with A-systems (with coefficients in \mathbb{Z}_2) we have to establish the existence of a nontrivial A-system for a product of two maps with A-systems with coefficients in a field \mathbb{F} (cf. Definition I.9).

This will be done first. Let $(K_i, \hat{\tau}^i)$ and $(L_i, \hat{\mu}^i)$, $i = 1, 2$, be finite block complexes with fixed f.s.b.d. $\hat{\tau}^i, \hat{\mu}^i$, respectively (cf. Definition I.7). We consider the product f.s.b.d.

$$\theta = \hat{\tau}^1 \times \hat{\tau}^2 = \{\theta_k = \tau_k^1 \times \tau_k^2 | k = 0, 1, \dots\}$$

on $K_1 \times K_2$, and

$$\bar{\theta} = \hat{\mu}^1 \times \hat{\mu}^2 = \{\bar{\theta}_k = \mu_k^1 \times \mu_k^2 | k = 0, 1, \dots\}$$

on $L_1 \times L_2$, respectively (cf. Remark I.6.3).

The tensor product $C_* = C_*(K_1, \tau_k^1) \otimes C_*(K_2, \tau_k^2)$ (over the field \mathbb{F}) is a chain complex with differential $\partial(e_1 \otimes e_2) = \partial e_1 \otimes e_2 + (-1)^{\dim e_1} e_1 \otimes \partial e_2$, $e_j \in \tau_k^j$, cf. [8, 9.1, p. 161]. Using [8, 9.21(4), p. 166] we may

identify the chain complexes $C_*(K_1 \times K_2, \theta_k)$ and C_* via the chain isomorphism defined on the generators by $e_1 \times e_2 \mapsto e_1 \otimes e_2$.

Following [26], we consider the chain subdivision maps

$$b_i(k, l): C_*(K_i, \tau_k^k) \rightarrow C_*(K_i, \tau_l^i)$$

and a chain approximation of the identity (see I.2) for $l > k$

$$\chi_i(l, k): C_*(K_i, \tau_l^i) \rightarrow C_*(K_i, \tau_k^i).$$

For the chain complex $C_*(K_1 \times K_2, \theta)$ we consider analogously the chain maps $b(k, l)$ and $\chi(l, k)$ defined by

$$\begin{aligned} b(k, l) &= b_1(k, l) \otimes b_2(k, l), \\ \chi(l, k) &= \chi_1(l, k) \otimes \chi_2(l, k) \end{aligned}$$

which gives the chain subdivision and a chain approximation of the identity map for a f.s.b.d. θ on $K_1 \times K_2$. We call it the tensor product subdivision and a tensor product approximation of the identity.

For the block complexes $(L_i, \hat{\mu}^i = \{\mu_k^i\})$, $i = 1, 2$, we consider $C_*(L_1 \times L_2, \bar{\theta})$ with the tensor product subdivision $\bar{b}(k, l)$ and a tensor product approximation of the identity $\bar{\chi}(l, k)$.

LEMMA 2. *Let $(K_i, \hat{\tau}^k)$, $(L_i, \hat{\mu}^i)$ be block complexes with f.s.b.d. and let $F_i: K_i \rightarrow L_i$ be u.s.c. maps with A-systems $A(F_i)$ with coefficients in \mathbb{F} , $i = 1, 2$. Then there exists an A-system with coefficients in \mathbb{F} for the u.s.c. map $F_1 \times F_2: K_1 \times K_2 \rightarrow L_1 \times L_2$ with respect to the product f.s.b.d. $\theta = \hat{\tau}^1 \times \hat{\tau}^2$, $\bar{\theta} = \hat{\mu}^1 \times \hat{\mu}^2$ and with Kronecker index $(KIA(F_1))(KIA(F_2))$.*

Proof. The map $F_1 \times F_2$ is defined by $(F_1 \times F_2)(x, y) = F_1(x) \times F_2(y)$ for $(x, y) \in K_1 \times K_2$. Define

$$A(F_1)_k \otimes A(F_2)_k = \{\varphi_1 \otimes \varphi_2 \mid \varphi_1 \in A(F_1)_k, \varphi_2 \in A(F_2)_k\}.$$

The tensor product of φ_1 and φ_2

$$\varphi_1 \otimes \varphi_2: C_*(\tau_k^1) \otimes C_*(\tau_k^2) \rightarrow C_*(\mu_k^1) \otimes C_*(\mu_k^2)$$

is a chain map, cf. [8, p. 162]. We shall show that $A(F_1) \otimes A(F_2) = \{A(F_1)_k \otimes A(F_2)_k \mid k \in \mathbb{N}\}$ is an A-system for the u.s.c. map $F_1 \times F_2$ with respect to the f.s.b.d. θ and $\bar{\theta}$.

Let $k \in \mathbb{N}$ and $\varphi_1 \otimes \varphi_2 \in A(F_1)_k \otimes A(F_2)_k$. Since $\varphi_i \in A(F_i)_k$, $i = 1, 2$, there exists an integer $l > k$ (it is no restriction to assume that l is the same for φ_1 and φ_2) such that $\varphi_i = \bar{\varphi}_i b_i(k, l)$ and the chain maps $\bar{\varphi}_i$ are (τ_k^i, μ_k^i) -chain approximations of F_i , respectively,

cf. Definition I.9. Writing $\varphi_1 \otimes \varphi_2 = (\overline{\varphi}_1 \otimes \overline{\varphi}_2)b(k, l)$ we obtain a $(\theta_k, \overline{\theta}_k)$ -chain approximation of the map $F_1 \times F_2$. This proves part (a) of Definition I.9.

Let $n \in \mathbb{N}$; there is an integer $k \geq n$ (again we assume k to be independent of i) such that for integers $m_1 \geq l_1 \geq m \geq l \geq k$, and $\varphi_i = \overline{\varphi}_i b_k(l, l_1) \in A(F_i)_l$, $\psi_i = \overline{\psi}_i b_i(m, m_1) \in A(F_i)_m$ the chain maps $\overline{\varphi}_i \chi_i(m_1, l_1)$ and $\overline{\chi}_i(m, l) \overline{\psi}_i$ are chain homotopic with homotopies D_i , $i = 1, 2$, satisfying the condition (b) of Definition I.9 for the A-system $A(F_i)$, respectively. According to [8, p. 163] the chain maps

$$(\overline{\varphi}_1 \otimes \overline{\varphi}_2) \chi(m_1, l_1) \quad \text{and} \quad \overline{\chi}(m, l) (\overline{\psi}_1 \otimes \overline{\psi}_2)$$

are homotopic via the chain homotopy

$$D = D_1 \otimes (\overline{\varphi}_2 \chi_2(m_1, l_1)) + (\chi_1(m, l) \overline{\psi}_1) \otimes D_2.$$

This homotopy satisfies the condition (b) of Definition I.9. Thus $A(F_1) \otimes A(F_2)$ is an A-system for the map $F_1 \times F_2$, with coefficients in the field \mathbb{F} , and with respect to the f.s.b.d. θ and $\overline{\theta}$. We call it the tensor product of the A-systems $A(F_1)$ and $A(F_2)$. Finally, let $\varphi_1 \otimes \varphi_2 \in A(F_1)_k \otimes A(F_2)_k$; then $KI\varphi_1 \otimes \varphi_2 = (KI\varphi_1)(KI\varphi_2) = (KIA(F_1))(KIA(F_2))$. \square

COROLLARY 3. *Let $F_i: K \rightarrow L_i$ be u.s.c. maps with A-systems $A(F_i)$ (coefficients in the field \mathbb{F}) with respect to f.s.b.d. $\hat{\tau}$ on K and $\hat{\mu}^i$ on L_i , $i = 1, 2$. Then the map*

$$\Phi = (F_1 \times F_2)\Delta: K \rightarrow L_1 \times L_2$$

has an A-system (with coefficients in \mathbb{F}) with respect to f.s.b.d. $\hat{\tau}$ and $\hat{\mu}^1 \times \hat{\mu}^2$ with Kronecker index $(KIA(F_1))(KIA(F_2))$. Here $\Delta: K \rightarrow K \times K$ denotes the diagonal map $\Delta(x) = (x, x)$.

Proof. Lemma 2 shows that the tensor product $A(F_1) \otimes A(F_2)$ is an A-system for the map $F_1 \times F_2$ with respect to f.s.b.d. $\hat{\tau} \times \hat{\tau}$, $\hat{\mu}^1 \times \hat{\mu}^2$.

Since the diagonal map Δ is single valued there exists an A-system $A(\Delta)$ with coefficients in \mathbb{F} for Δ with respect to $\hat{\tau}$ and $\hat{\tau} \times \hat{\tau}$, and Kronecker index $KIA(\Delta) = 1$, see [25, 26]. The composition $(A(F_1) \otimes A(F_2))A(\Delta)$ of A-systems provides an A-system for the map Φ with Kronecker index $(KIA(F_1))(KIA(F_2))$, cf. [25, Lemma 2.16]. The chain maps in the A-system $(A(F_1) \otimes A(F_2))A(\Delta)$ are compositions of chain maps of $A(\Delta)_k$ and $(A(F_1) \otimes A(F_2))_k$, cf. [25, I.2] for the definition of the composition of A-systems. \square

Let K be a block complex with f.s.b.d. $\hat{\tau} = \{\tau_k\}$. Then the product f.s.b.d. $\hat{\tau} \times \hat{\tau}$ on $K \times K$ is T -equivariant (Remark I.6.4). Therefore T induces a chain map

$$T: C_*(\tau_k) \otimes C_*(\tau_k) \rightarrow C_*(\tau_k) \otimes C_*(\tau_k)$$

defined by $T(e_1 \otimes e_2) = (-1)^{\dim e_1 \dim e_2} e_2 \otimes e_1$.

COROLLARY 4. *Let $F: K \rightarrow L$ be an u.s.c. map and $A(F)$ an A -system (with coefficients in \mathbb{F}) for F with respect to given f.s.b.d. $\hat{\tau}$, $\hat{\mu}$ on K and L , respectively. Then there exists an A -system with coefficients in \mathbb{F} for $F \times F$ w.r.t. $\hat{\tau} \times \hat{\tau}$, $\hat{\mu} \times \hat{\mu}$ which is T -equivariant, with Kronecker index $(KIA(F))(KIA(F))$.*

Proof. Let $A_T(F \times F)_k = \{\varphi \otimes \varphi | \varphi \in A(F)_k\}$ and $A_T(F \times F) = \{A_T(F \times F)_k | k \in \mathbb{N}\}$. Obviously $A_T(F \times F) \subset A(F) \otimes A(F)$. Since $A(F) \otimes A(F)$ is an A -system for $F \times F$ then $A_T(F \times F)$ is an A -system and it is T -equivariant; and $KIA_T(F \times F) = (KIA(F))(KIA(F))$. \square

3. Borsuk-Ulam theorem. In this subsection we prove the Borsuk-Ulam theorem for u.s.c. maps having A -systems with coefficients in \mathbb{Z}_2 .

PROPOSITION 5. *Let $F: S^n \rightarrow D_a^n$ be an u.s.c. map with A -system (coefficients in \mathbb{Z}_2), then the set*

$$B(F) = \{x \in S^n | F(x) \cap F(-x) \neq \emptyset\}$$

is not empty.

Proof. Let $\hat{\tau}$ be a f.s.b.d. induced by an α -equivariant block structure on S^n and $\hat{\mu}$ be the f.s.b.d. induced by the cubic subdivision of D_a^n (see Remark I.6.2). Without loss of generality we consider the A -system for F with respect to the f.s.b.d. $\hat{\tau}$ and $\hat{\mu}$.

On the space $D_a^n \times D_a^n$ we consider the map

$$\bar{d}: D_a^n \times D_a^n \rightarrow D_{2a}^n, \quad \text{with } \bar{d}(x, y) = x - y,$$

and on S^n the map

$$\bar{\Delta}: S^n \rightarrow S^n \times S^n, \quad \text{with } \bar{\Delta}(x) = (x, -x).$$

On the spaces $S^n \times S^n$ and $D_a^n \times D_a^n$ acts the involution T defined by $T(x, y) = (y, x)$. The map $\bar{\Delta}$ is (α, T) -equivariant, the map \bar{d} is (T, α) -equivariant. Since the map $\bar{\Delta}$ is single valued there exists an (α, T) -equivariant A -system (coefficients in \mathbb{Z}_2) $A(\bar{\Delta})$ with respect to

f.s.b.d. $\hat{\tau}$ and $\hat{\tau} \times \hat{\tau}$, $KIA(\bar{\Delta}) = 1$, this follows from Lemma IV.3 or from appropriate simplicial approximations of $\bar{\Delta}$.

For the map \bar{d} there exists a (T, α) -equivariant A-system $A(\bar{d})$ (coefficients in \mathbb{Z}_2) with respect to f.s.b.d. $\hat{\mu} \times \hat{\mu}$ and $\hat{\mu}$ and $KIA(\bar{d}) = 1$; again, this follows from appropriate simplicial approximations of \bar{d} . Corollary 4 guarantees the existence of a T -equivariant A-system (\mathbb{Z}_2 coefficients) $A_T(F \times F)$ with respect to f.s.b.d. $\hat{\tau} \times \hat{\tau}$ and $\hat{\mu} \times \hat{\mu}$ with Kronecker index $(KIA(F))(KIA(F))$.

Assume that the set $B(F)$ is empty and consider the map

$$\Psi = \bar{d}(F \times F)\bar{\Delta}: S^n \rightarrow D_{2a}^n.$$

The assumption $B(F) = \emptyset$ leads to

$$(F \times F)\bar{\Delta}(S^n) \cap \{(x, y) \in D_a^n \times D_a^n \times D_a^n | x = y\} = \emptyset$$

and the compactness of $(F \times F)\bar{\Delta}(S^n)$ ensures the existence of $k_0 \in \mathbb{N}$ such that the sets

$$N = St^2((F \times F)\bar{\Delta}(S^n), \mu_{k_0} \times \mu_{k_0})$$

and

$$D = St^2(\{(x, x) \in D_a^n \times D_a^n\}, \mu_{k_0} \times \mu_{k_0})$$

are disjoint. Since D is a T -equivariant block complex with block structure $\mu_{k_0} \times \mu_{k_0}$ then $N_0 = \overline{D_a^n \times D_a^n \setminus D}$ is a T -equivariant block complex with block structure $\mu_{k_0} \times \mu_{k_0}$ and $N \subset N_0$.

Due to the assumption $B(F) = \emptyset$ it follows that $0 \notin \bar{d}(N_0)$ and there exists $l_0 \in \mathbb{N}$ such that $St^2(0, \mu_{l_0}) \cap \bar{d}(N_0) = \emptyset$, here 0 denotes the point $(0, \dots, 0) \in D_{2a}^n$. Let $D_0 = \overline{D_{2a}^n \setminus St^2(0, \mu_{l_0})}$. Then D_0 is an α -equivariant block complex with block structure μ_{l_0} . The map $\rho: D_0 \rightarrow S^{n-1}$ defined by $\rho(x) = x/\|x\|$ is α -equivariant. Since it is single valued there exists an α -equivariant A-system (coefficients in \mathbb{Z}_2) $A(\rho)$ for ρ with respect to $\hat{\mu}' = \{\mu_l | l \geq l_0\}$ and $\hat{\tau}$.

It follows from [25, Lemma 2.16] that the composition

$$A(\rho)A(\bar{d})A_T(F \times F)A(\bar{\Delta})$$

is an A-system for the map $\rho\Psi: S^n \rightarrow S^{n-1}$ with respect to $\hat{\tau}' = \{\tau_i | i \geq m_0\}$, $m_0 = \max(l_0, k_0)$. This A-system with coefficients in \mathbb{Z}_2 is α -equivariant. This contradicts Proposition 1. \square

4. *Bourgin-Yang theorem.* Let $F: S^{n+k} \rightarrow \mathbb{R}^n$, $k \geq 0$, be an u.s.c. map. The set $B(F) = \{x \in S^{n+k} | F(x) \cap F(-x) \neq \emptyset\}$ is compact and α -equivariant. If $B(F) \neq \emptyset$ we consider the genus, $g(B(F), \alpha)$, of the space $(B(F), \alpha)$, where α is the antipodal map, see Remark II.2.

PROPOSITION 6. *Let $F: S^{n+k} \rightarrow \mathbb{R}^n$, $k \geq 0$, be an u.s.c. map with an A -system (coefficients in \mathbb{Z}_2). Then $g(B(F), \alpha) \geq k$.*

Proof. From Corollary 4 we have that $B(F) \neq \emptyset$. Following [12, Theorem 8.7], we assume $g(B(F), \alpha) = m < k$. Then there exists an α -equivariant single valued map $f: B(F) \rightarrow S^m$. Let $\tilde{f}: S^{n+k} \rightarrow \mathbb{R}^{m+1}$ be a continuous single valued extension of f , and consider the multivalued map

$$\Psi = (F \times \tilde{f})\Delta: S^{n+k} \rightarrow \mathbb{R}^n \times \mathbb{R}^{m+1} = \mathbb{R}^{m+n+1}$$

defined by $\Psi(x) = F(x) \times \tilde{f}(x)$, where $\Delta(x) = (x, x)$ is the diagonal map. The map Ψ is u.s.c. and Corollary 3 implies the existence of an A -system with coefficients in \mathbb{Z}_2 for Ψ . Since $m < k$ then $n+m+1 \leq n+k$. From Proposition 5 one concludes the existence of a point $x_0 \in S^{n+k}$ such that $\Psi(x_0) \cap \Psi(-x_0) \neq \emptyset$, which is equivalent to $F(x_0) \cap F(-x_0) \neq \emptyset$ and $\tilde{f}(x_0) = \tilde{f}(-x_0)$. But, then $x_0 \in B(F)$ and $\tilde{f}(x_0) = f(x_0) = \tilde{f}(-x_0) = f(-x_0)$. Since $f: B(F) \rightarrow S^m$ is α -equivariant we derive a contradiction. \square

COROLLARY 7. *Let $F: S^{n+k} \rightarrow \mathbb{R}^n$, $k \geq 0$, be an u.s.c. map with an A -system with coefficients in \mathbb{Z}_2 . Then $\dim B(F) \geq k$.*

Proof. This is a consequence of $\dim X \geq g(X, \gamma)$ [30], for any space with a free involution γ and of Proposition 6. \square

IV. Existence of A -systems, fixed point index.

1. *A-systems.* In this section we prove the existence of A -systems with coefficients in \mathbb{F} for (γ_1, γ_2) -equivariant m -acyclic maps (w.r.t. a field \mathbb{F}), and γ_1 a free involution, see I.3 for the notation.

LEMMA 1. *Let $(K, \hat{\tau})$ and $(L, \hat{\mu})$ be compact polyhedra with a given f.s.b.d., furthermore, let $F: K \rightarrow L$ be a (γ_1, γ_2) -equivariant m -acyclic map (w.r.t. a field \mathbb{F}), γ_1 a free involution, and multiplicity $m(F) = a \in \mathbb{F}$. Let $n, k_0, l_0 \in \mathbb{N}$. There are numbers $0 < k_0 < \dots < k_{n+1}$; $0 < l_0 < \dots < l_{n+1} \in \mathbb{N}$ such that for any block $e \in \tau_k$, there exists an integer $l_{i-1}(e)$ and a point $a(e) \in K$ such that*

- (a) $l_{i-1} < l_{i-1}(e) < l_i$ for $i = 1, \dots, n + 1$.
- (b) The covering $\{St^2(y, l_i(e)) | y \in L\}$ is finer than the covering $\{St(y, l_i) | y \in L\}$ and $e \subset St(a(e), k_{i-1})$ (cf. Remark I.7.2).
- (c) $\gamma_1(a(e)) = a(\gamma_1(e))$ and $l_{i-1}(e) = l_{i-1}(\gamma_1(e))$.

(d) $F(St^2(e, k_i)) \subset St(F(a(e)), l_{i-1}(e))$, and if $F(a(e)) = C_1(a(e)) \cup \dots \cup C_s(a(e))$, $s = s(a(e))$, then for all $u \neq v$, $u, v = 1, \dots, s$,

$$St^2(C_u(a(e)), l_{i-1}(e)) \cap St^2(C_v(a(e)), l_{i-1}(e)) = \emptyset,$$

and

$$m(a(e), C_u(a(e))) = \sum_{C(x) \subset St(C_u(a(e)), l_{i-1}(e))} m(x, C(x)).$$

(e) The inclusion $St^2(C_u(a(e)), l_{i-1}(e)) \subset St(C_u(a(e)), l_{i-1}(e))$ induces the zero homomorphism in the reduced homology with coefficients in \mathbb{F} .

The proof is immediate from [25, Lemma B], [26, Lemma 2] and [9, Lemma 4.1].

LEMMA 2. Let the assumptions of Lemma 1 be satisfied. For any $\tau_k \in \hat{\tau}$, $\mu_l \in \hat{\mu}$ there exists a $\bar{k} > k$, and a chain map

$$\varphi: C_*(K, \bar{k}) \rightarrow C_*(L, l)$$

such that

- (a) φ is (γ_1, γ_2) -equivariant,
- (b) $KI\varphi = m(F)$,
- (c) for any block $e \in \tau_{\bar{k}}$ there exists a point $t(e) \in K$ such that

$$\begin{aligned} e &\in St(t(e), k), \\ |\varphi(e)| &\subset St(F(t(e)), l), \\ t(\gamma_1(e)) &= \gamma_1(t(e)). \end{aligned}$$

Proof. Applying Lemma 1 for $k_0 = k$, $l_0 = l$, $n = \dim K$ we set $\bar{k} = k_{n+1}$. We shall construct successively chain maps

$$\varphi^j: (C_*((K, \bar{k}))^{(j)}) \rightarrow C_*(L, l_{n-j}),$$

where $(K, \bar{k})^{(j)}$ denotes the j -skeleton of (K, \bar{k}) , such that φ^j is (γ_1, γ_2) -equivariant, and for any block $e \in \tau_{\bar{k}}$ with $\dim e \leq j$ there exists a point $t(e) \in K$ such that $e \in St(t(e), k_{n-j})$, $t(\gamma_1(e)) = \gamma_1(t(e))$. If

$$F(t(e)) = C_1(t(e)) \cup \dots \cup C_s(t(e)), \quad s = s(t(e)),$$

then $\varphi^j(e) = \varphi^j(e)_1 + \dots + \varphi^j(e)_s$ with $|\varphi^j(e)_u| \subset St(C_u(t(e)), l_{n-j})$ and $KI\varphi^j(e)_u = m(t(e), C_u(t(e)))$ for all $u = 1, \dots, s$.

Step 1. We start with $j = 0$. Let $e \in \tau_{\bar{k}}$ and $\dim e = 0$; Lemma 1 for $i = n + 1$ guarantees the existence of a point $a(e) \in K$ such that (a)–(e) of Lemma 1 hold. We set $t(e) = a(e)$ and $t(\gamma_1(e)) = \gamma_1(t(e))$. Let a_n be a zero-dimensional block in $St(C_u(t(e)), l_n(e))$, where $C_u(t(e))$, $u = 1, \dots, s = s(t(e))$, denotes a connected component of $F(t(e))$. Therefore

$$\varphi^0(e) = \sum_{u=1}^s \varphi^0(e)_u$$

with

$$\varphi^0(e)_u = \chi(l_n(e), l_n)m(t(e), C_u(t(e)))a_u$$

is well defined and satisfies the above conditions.

Step 2. Let $j = 1$ and $e \in \tau_{\bar{k}}$ with $\dim e = 1$ and $\bar{e} = |e, k_n|$ (support of the block e in τ_{k_n}). Due to Lemma 1 there exists an $a(\bar{e}) \in K$ and $l_{n-1}(\bar{e})$ such that (a)–(e) of Lemma 1 hold; define $t(e) = a(\bar{e})$ and $t(\gamma_1(e)) = \gamma_1(t(e))$.

Let $\partial e = e_1 - e_0$ and $\varphi^0(e_\nu) = \sum_{u=1}^s \varphi^0(e_\nu)_u$, $s = s(t(e_\nu))$, for $\nu = 0, 1$. $\varphi^0(e_\nu)_u$ is a chain in $St(C_u(t(e_\nu)), l_n)$. Furthermore,

$$St(F(t(e_\nu)), l_n) \subset St^2(F(a(\bar{e})), l_{n-1}(\bar{e}))$$

and

$$|\varphi^0(e_\nu)| \subset St^2(F(a(\bar{e})), l_{n-1}(\bar{e})).$$

Now

$$St^2(F(a(\bar{e})), l_{n-1}(\bar{e})) = \bigcup_{r=1}^{s^*} St^2(C_r(a(\bar{e})), l_{n-1}(\bar{e})), \quad s^* = s(a(\bar{e})),$$

and we consider $\varphi^0(e_\nu) = \sum_{r=1}^{s^*} \varphi^0(e_\nu)^r$ with

$$\varphi^0(e_\nu)^r = \sum_{u \in I_r} \varphi^0(e_\nu)_u,$$

where $I_r = \{u | C_u(t(e_\nu)) \subset St^2(C_r(a(\bar{e})), l_{n-1}(\bar{e}))\}$, which gives

$$KI\varphi^0(e_\nu)^r = \sum_{u \in I_r} m(t(e_\nu, C_u(t(e_\nu)))) = m(a(\bar{e}), C_r(a(\bar{e}))).$$

Then $\chi(l_n, l_{n-1})(\varphi^0(e_0)^r - \varphi^0(e_1)^r)$ is a chain in the block complex $St(F(a(\bar{e})), l_{n-1})$ with Kronecker index zero. Therefore there exists a chain $c_r = \varphi^1(e)_r \in St(F(a(\bar{e})), l_{n-1})$ such that

$$\partial c_r = \chi(l_n, l_{n-1})(\varphi^0(e_0)^r - \varphi^0(e_1)^r),$$

and define $\varphi^1(e) = \sum_{r=1}^{s^*} \varphi^1(e)_r$ and $\varphi^1(\gamma_1(e)) = \gamma_2(\varphi^1(e))$. For a zero-dimensional block e we define $\varphi^1(e) = \chi(l_n, l_{n-1})\varphi^0(e)$. The homomorphism φ^1 is a chain map which satisfies the above conditions.

Now it is clear how to proceed for $j > 1$, see [25, Lemma 3.5], [26, Lemma 3], [9, Theorem 4.3]. □

REMARK 1. If F is an m -point map (or m -acyclic map, such that all connected components of $F(x)$ are R_δ -sets, cf. [13]), then the above construction holds even for rings \mathbb{A} , i.e. in Lemmata 1 and 2 the field \mathbb{F} may be replaced by the ring \mathbb{A} .

LEMMA 3. *Let the map F be as in Lemma 1, with f.s.b.d. $\hat{\tau}$ and $\hat{\mu}$. Then there exists a (γ_1, γ_2) -equivariant A -system with coefficients in \mathbb{F} for F such that $KIA(F) = m(F)$.*

The proof follows directly from the above lemmata and the proofs in [25, Lemma 3.8], [9, Proposition 5.3].

REMARK 2. 1. For m -point maps F , Lemma 3 implies the existence of A -systems with coefficients in a ring \mathbb{A} .

2. Let $\Phi: K \rightarrow L$ such that there exists an m -acyclic map (w.r.t. \mathbb{F}) $F: K \rightarrow L$ with $F(x) \subset \Phi(x)$, for all $x \in K$, i.e., F is a selector of the map Φ . Then Φ possesses an A -system with coefficients in \mathbb{F} , and Lemma 3 holds for Φ .

3. Let $F: K \rightarrow L$ be an m -acyclic map (w.r.t. \mathbb{Z}) and multiplicity function $m \geq 0$ and multiplicity $n > 0$. Then F may be decomposed into an u.s.c. acyclic map $\overline{F}: K \rightarrow SP^n L$ (w.r.t. \mathbb{Z}) and an m -point map $\pi: SP^n L \rightarrow L$ with multiplicity n , where (cf. Definition I.2)

$$\overline{F}(x) = \{x_1^{k_1} \cdots x_s^{k_s} | x_i \in C_i(x), k_i = m(x, C_i(x))\}, \quad s = s(x),$$

and

$$\pi(x_1^{l_1} \cdots x_s^{l_s}) = \{x_1, \dots, x_s\}.$$

Then $F = \pi\overline{F}$ and F has an A -system with coefficients in \mathbb{F} , because \overline{F} has one [25] and π has one induced by a simplicial approximation of π in [20].

4. For the maps in [5], [7], [16], the A -system can be constructed using simplicial approximation developed in [5], [7], [16], resp.

5. Let $p: K \rightarrow L$ be a continuous single valued map, K, L finite polyhedra. We say that p is a map with multiplicity (w.r.t. \mathbb{A}) if the map $p^{-1}: L \rightarrow K$ is an m -acyclic map (w.r.t. \mathbb{A}). If p^{-1} is

an m -point map it is sufficient to work with coefficients in a ring \mathbb{A} , otherwise the coefficients should be in a field \mathbb{F} . Let $A(p^{-1})$ be the \mathbb{A} -system for p^{-1} constructed in Lemma 3. Every chain map $\varphi \in A(p^{-1})$ induces a homomorphism $\varphi_*: H_*(L, \mathbb{A}) \rightarrow H_*(K, \mathbb{A})$ which is a transfer homomorphism for the map p , i.e. $p_*\varphi_* = m(p^{-1})id$, cf. [28]. Therefore, the homomorphism $p_*: H_*(K, \overline{\mathbb{A}}) \rightarrow H_*(L, \overline{\mathbb{A}})$ is an epimorphism ($\overline{\mathbb{A}}$ is the quotient field of the ring \mathbb{A} .) We will not go into more details.

PROPOSITION 4. *Let $F: (S^n, \alpha) \rightarrow (S^m\alpha)$ be an α -equivariant m -acyclic map w.r.t. \mathbb{Z}_2 ; then $m \geq n$.*

Proof. Lemma 3 implies the existence of an \mathbb{A} -system for F with coefficients in \mathbb{Z}_2 and Proposition III.1 yields $m \geq n$. □

PROPOSITION 5. *Let $F: S^{n+k} \rightarrow \mathbb{R}^n$, $k \geq 0$, be an m -acyclic map w.r.t. \mathbb{Z}_2 . then $\dim\{x \in S^{n+k} | F(x) \cap F(-x) \neq \emptyset\} \geq k$.*

Proof. The existence of an \mathbb{A} -system for F with \mathbb{Z}_2 -coefficients and Corollary III.7 prove the assertion. □

2. *Fixed point index.* For every map $\Phi: K \rightarrow K$ which has an \mathbb{A} -system $A(\Phi)$ with coefficients in the field \mathbb{F} there exists a fixed point index defined in [25, Definition II.2.6]. Let U be an open subset in K such that \overline{U} is a subcomplex in some $\tau_k \in \hat{\tau}$. Suppose $x \notin \Phi(x)$ for all $x \in \partial U$, i.e., the triple (K, Φ, U) is admissible.

DEFINITION 6. The fixed point index $K_A(K, \Phi, U)$ is defined as

$$I_A(K, \Phi, U) = \Lambda(\pi(\varphi|_{\overline{U}}))$$

where $\varphi \in A(\Phi)_l$, $\pi: C_*(K, \tau_l) \rightarrow C_*(\overline{U}, \tau_l)$ is a homomorphism (projections) such that $\pi(e) = 0$ for $e \notin \overline{U}$ and $\pi(e) = e$ for $e \in \overline{U}$, e is a block in τ_l ($l \geq k$), and $\Lambda(\pi(\varphi|_{\overline{U}}))$ is the Lefschetz number of the homomorphism $\pi(\varphi|_{\overline{U}})$.

This definition does not depend on l , π and τ for sufficiently large $l \in \mathbb{N}$ [25, Lemmata I.1.5 and II.2.5].

The fixed point index $I_A(K, \Phi, U)$ has all properties, i.e., additivity, homotopy invariance, normalization, commutativity, mod p -property, see [25]. Let $\mathcal{M}(\mathbb{F})$ be the set of m -acyclic maps of polyhedra w.r.t. the field \mathbb{F} . Let $F \in \mathcal{M}(\mathbb{F})$, $F: K \rightarrow K$, and U an open subset in k such that the triple (K, F, U) is admissible. Applying

Lemma 3 to the map F , γ_1 and γ_2 being the identity, we obtain an A -system $A(F)$ with $KIA(F) = m(F) = 1$.

DEFINITION 7. The fixed point index $i(K, F, U)$ is defined as

$$i(K, F, U) = I_A(K, F, U)$$

w.r.t. A -system $A(F)$ for every $F \in \mathcal{M}(\mathbb{F})$.

For m -acyclic maps the index $i(K, F, U)$ is independent from the choice of A -system $A(F)$, since $KIA(F) = 1$, see [25, 26].

PROPOSITION 8. Let $F \in \mathcal{M}(\mathbb{F})$, i.e., F is an m -acyclic map w.r.t. the field \mathbb{F} . Then the fixed point index $i(K, F, U)$ has the following properties:

(a) Homotopy invariance. Let $H \in \mathcal{M}(\mathbb{F})$, $H: \bar{U} \times I \rightarrow K$, be a homotopy joining the maps $\Phi_1|_{\bar{U}}, \Phi_2|_{\bar{U}} \in \mathcal{M}(\mathbb{F})$ such that for all $t \in I$ and $H_t: U \rightarrow K$, $H_t(x) = H(x, t)$, the triple (K, H_t, U) is admissible. Then

$$i(K, \Phi_1, U) = i(K, \Phi_2, U).$$

(b) Additivity. Let $\Phi \in \mathcal{M}(\mathbb{F})$, $\Phi: K \rightarrow K$, and U_1, U_2 be open disjoint subsets of U . If $\text{Fix}(\Phi|_U) \subset U_1 \cup U_2$ then

$$i(K, \Phi, U) = i(K, \Phi, U_1) + i(K, \Phi, U_2).$$

(c) Normalization.

$$i(K, \varphi, K) = \Lambda(A(\Phi));$$

here $\Lambda(A(\Phi))$ denotes the Lefschetz number $\Lambda(\varphi)$ of a chain map $\varphi \in A(\Phi)_k$.

(d) Commutativity. Let $\Phi_1: K \rightarrow L, \Phi_2: L \rightarrow K$ be in $\mathcal{M}(\mathbb{F})$, and let $(K, \Phi_2\Phi_1, U)$ and $(L, \Phi_1\Phi_2, \Phi_2^{-1}(U))$ be admissible. Assume that for all $y \in \text{Fix}(\Phi_1\Phi_2) \setminus \Phi_2^{-1}(U)$:

$$\Phi_2(y) \cap \text{Fix}(\Phi_2\Phi_1|_{\bar{U}}) = \emptyset.$$

Then

$$I_A(K, \Phi_2\Phi_1, U) = I_A(L, \Phi_1\Phi_2, \Phi_2^{-1}(U)),$$

where the fixed point index is defined with the A -systems $A(\Phi_2)A(\Phi_1), A(\Phi_1)A(\Phi_2)$, respectively.

(e) Mod p -property. Let $p \in \mathbb{N}$ be prime and let $\Phi \in \mathcal{M}(\mathbb{Z}_p)$, $(K, \Phi, U), (K, \Phi^p, U)$ be admissible. Assume that for all $y \in \text{Fix}(\Phi^p) \setminus U$:

$$\Phi^k(y) \cap \text{Fix}(\Phi^p(\bar{U})) = \emptyset, \quad 1 \leq k < p.$$

Then

$$i(K, \Phi, U) = I_A(K, \Phi^p, U) \quad \text{in } \mathbb{Z}_p.$$

Here the fixed point index for Φ^p is defined with the A -system $A(\Phi) \cdots A(\Phi)$ (p -times).

(f) *Multiplicity.* Let $\Phi_i: K_i \rightarrow L_i$, $\Phi_i \in \mathcal{M}(\mathbb{F})$, and (K_i, Φ_i, U_i) be admissible. Then

$$i(K_1 \times K_2, \Phi_1 \times \Phi_2, U_1 \times U_2) = i(K_1, \Phi_1, U_1)i(K_2, \Phi_2, U_2).$$

Here the index for $\Phi_1 \times \Phi_2$ is defined with the A -system $A(\Phi_1) \otimes A(\Phi_2)$, see III.2, and is independent from the choice of A -systems.

Proof. The properties (a)–(e) follow from [25]: Homotopy invariance from II.15, additivity from II.2.8, normalization from Definition 5, commutativity from II.2.17, mod- p -property from II.2.19.(f), and multiplicity from [26, III.9]. \square

REMARK 3. 1. For maps $f: K \rightarrow SP^n K$ the fixed point index is defined in [18].

2. In [9] the fixed point index for $(1, n)$ -valued maps F , i.e., $F(x)$ consists of 1 or n acyclic components, is defined; for n -valued maps see [24].

3. For the composition of m -maps see [5], [16].

4. The m -point maps F (w.r.t. \mathbb{Z}), or more generally m -acyclic maps for which the connected components of $F(x)$ are R_δ -sets, [13], have A -systems w.r.t. \mathbb{Z} (Remark 2.1). Therefore the fixed point index $i(K, F, U)$ is integer valued.

The fixed point index for this kind of map with nonnegative multiplicity is constructed in [13].

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