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L^p-FOURIER TRANSFORMS ON NILPOTENT LIE GROUPS AND SOLVABLE LIE GROUPS ACTING ON SIEGEL DOMAINS

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We study Fourier transforms of L^p-functions $(1 < p \leq 2)$ on nilpotent Lie groups and affine automorphism groups of Siegel domains. We get an estimate for the norm of the L^p-Fourier transform for certain classes of nilpotent Lie groups. For affine automorphism groups, which are nonunimodular, we give an explicit definition of L^p-Fourier transform, and obtain an estimate for the norm.

Introduction. First of all, let us recall some known results of the L^p-Fourier transform on unimodular groups. For such groups, the classical Hausdorff-Young theorem was generalized by Kunze [**13**]. Following a description of Lipsman [**14**], we briefly mention the generalization. Let G be a separable locally compact unimodular group of type I, and $\widehat{G}$ be the unitary dual endowed with the Mackey Borel structure. Denote by dg a Haar measure on G, and by μ the Plancherel measure on $\widehat{G}$ associated with dg. That is, μ is uniquely determined by the abstract Plancherel formula; for $\varphi \in L^1(G) \cap L^2(G)$,

$$\int_G |\varphi(g)|^2 \, dg = \int_{\widehat{G}} \operatorname{tr}(\pi(\varphi)^* \pi(\varphi)) \, d\mu(\pi), \tag{0.1}$$

where $\pi(\varphi) = \int_G \varphi(g)\pi(g)\,dg$. We consider the Fourier transform $\mathscr{P}$ to be a mapping of $L^1(G)$ to a space of μ-measurable field of bounded operators on $\widehat{G}$; $(\mathscr{P}\varphi)(\pi) = \pi(\varphi)$, for $\varphi \in L^1(G)$, $\pi \in \widehat{G}$. Let $1 < p < 2$ and $q = p/(p-1)$, and for a μ-measurable field of bounded operators F on $\widehat{G}$, let

$$\|F\|_q = \left(\int_{\widehat{G}} \|F(\pi)\|_{C_q}^q \, d\mu(\pi) \right)^{1/q},$$

where $\|F(\pi)\|_{C_q} = (\operatorname{tr}(F(\pi)^* F(\pi))^{q/2})^{1/q}$. Denote by $L^q(\widehat{G})$ the Banach space defined by the space of measurable fields F such that $\|F\|_q < \infty$ in the usual way (with norm $\|\cdot\|_q$). Then the Hausdorff-Young type inequality

$$\|\mathscr{P}\varphi\|_q \leq \|\varphi\|_p \tag{0.2}$$

holds for $\varphi \in L^1(G) \cap L^p(G)$. Thus the Hausdorff-Young theorem asserts that the map $\varphi \to \mathscr{P}\varphi$ from $L^1(G) \cap L^p(G)$ to $L^q(\widehat{G})$ extends to a continuous operator $\mathscr{P}^p \colon L^p(G) \to L^q(\widehat{G})$ and its norm

$$\|\mathscr{P}^p(G)\| = \sup_{\|\varphi\|_p \leq 1} \|\mathscr{P}^p(\varphi)\|_q \leq 1. \tag{0.3}$$

Next, let us consider the norm $\|\mathscr{P}^p(G)\|$. For the case of $G = \mathbf{R}^n$ (the classical Fourier transform), Babenko [1] and Beckner [2] obtained the norm

$$\|\mathscr{P}^p(\mathbf{R}^n)\| = A_p^n, \quad \text{where } A_p = \left(\frac{p^{1/p}}{q^{1/q}}\right)^{1/2}. \tag{0.4}$$

On the other hand, by a result of Fournier [8], the following statements (1) and (2) are equivalent for a locally compact unimodular group G:

(1) $\|\mathscr{P}^p(G)\| = 1$.

(2) G has a compact open subgroup.

For various examples which do not have compact open subgroups, Russo obtained estimates for the norm in [18], [19] and [20].

In §1, we deal with connected and simply connected nilpotent Lie groups G with Lie algebras $\mathfrak{g}$. We first treat irreducible representations of G, and give an estimate for $\|\pi(\varphi)\|_{C_q}$ $(\varphi \in L^1(G) \cap L^p(G))$ for irreducible representations π satisfying the condition (C1) (Proposition 1.2). Then we give an estimate for $\|\mathscr{P}^p(G)\|$ for groups G satisfying the condition (C2) (Theorem 1.3) as follows:

$$\|\mathscr{P}^p(G)\| \leq A_p^{(2\dim G - m)/2}, \tag{0.5}$$

where m is the dimension of generic coadjoint orbits of G in $\mathfrak{g}^*$ (the dual space of $\mathfrak{g}$). Here let us note that the Plancherel measure is supported on the set of representations corresponding to generic orbits in $\mathfrak{g}^*$ by the Kirillov mapping. Applying Theorem 1.3 to the Heisenberg groups and the nilpotent groups of real upper triangular matrices, for example, we get the same estimates as those obtained by Russo in [19].

Section 2 is devoted to a nonunimodular case. We will treat connected and simply connected Lie groups whose Lie algebras are normal j-algebras (see 2.1 for definition). In the sequel, let $G = \exp \mathfrak{g}$ be such a group.

An extension of the Hausdorff-Young theorem to general (i.e., not necessarily unimodular) locally compact groups was given by Terp [21] in terms of the spatial theory of von Neumann algebras. But we will give an explicit realization of the L^p-Fourier transform based on the Plancherel theorem of Duflo and Moore **[5]**. For each irreducible representation π corresponding to one of the generic coadjoint orbits, which are open, we modify the map $\varphi \to \pi(\varphi)$ using the operator called the formal degree of π **[5]**, and define L^p-Fourier transform $\mathscr{F}^p$. Then the following estimate for the norm is obtained:

$$\|\mathscr{F}^p(G)\| \leq A_p^{\dim G/2} \tag{0.6}$$

(Theorem 2.2.1). This result (0.6) is compatible with (0.5) for $m = \dim G$.

Let us remark that Eymard and Terp **[7]** and Russo **[20]** developed their L^p-Fourier analysis for the $ax+b$ group (the group of all affine transformations of the real line), and we are generalizing their results to our G.

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Notations. Let G be a Lie group and dg a left Haar measure on G. We denote by $\Delta = \Delta_G$ the modular function of G, i.e., $d(gx) = \Delta(x)\,dg$. If φ is a function on G and $1 \leq p < \infty$, we put $\varphi^{*(p)}(g) = \Delta(g)^{-1/p}\overline{\varphi(g^{-1})}$ for $g \in G$. (We often use φ^* for $\varphi^{*(1)}$.) We regard $L^p(G)$ as equipped with the involution $\varphi \to \varphi^{*(p)}$.

Let $\mathscr{H}$ be a Hilbert space. Then we denote by $\mathscr{B}(\mathscr{H})$ the space of bounded operators with the operator norm $\|\cdot\|_\infty$, and by $\mathscr{L}\mathscr{C}(\mathscr{H})$ the space of compact operators. For $1 \leq p < \infty$, $C_p(\mathscr{H})$ is the space of $T \in \mathscr{B}(\mathscr{H})$ satisfying $\|T\|_{C_p} = (\mathrm{tr}((T^*T)^{p/2}))^{1/p} < \infty$, where $\mathrm{tr}(\cdot)$ denotes the trace. It is a Banach space with the C_p-norm $\|\cdot\|_{C_p}$.

1. The norm of the L^p-Fourier transform for nilpotent Lie groups. Here we treat connected and simply connected nilpotent Lie groups. First of all, let us summarize the Plancherel theorem for such groups in terms of the orbit method. (For details, we refer to Chapter 4 of **[4]**.)

Let $\mathfrak{g}$ be a nilpotent Lie algebra, $G = \exp \mathfrak{g}$, $\Theta\colon \mathfrak{g}^*/G \to \widehat{G}$ be the Kirillov mapping which assigns the coadjoint orbit $G \cdot f$ $(f \in \mathfrak{g}^*)$ to the class of $\pi_f = \mathrm{ind}_{B_f}^G \chi_f$: the representation of G induced by a character χ_f of $B_f = \exp \mathfrak{b}_f$, where $\mathfrak{b}_f$ is a real polarization at f and $\chi_f(\exp X) = e^{\sqrt{-1}f(X)}$ $(X \in \mathfrak{b}_f)$.

Let $\{X_1, \dots, X_n\}$ be a strong Malcev basis for $\mathfrak{g}$ (i.e., $\mathfrak{g}_i = \mathbf{R}\text{-span}\{X_1, \dots, X_i\}$ is an ideal of $\mathfrak{g}$ for each i), and let $\{l_1, \dots, l_n\}$

be the dual basis for $\mathfrak{g}^*$. For each $l \in \mathfrak{g}^*$, define $S(l) = \{2 \le j \le n;\ \mathfrak{g}_{j-1} + \mathfrak{g}(l) \ne \mathfrak{g}_j + \mathfrak{g}(l)\}$, where $\mathfrak{g}(l) = \{X \in \mathfrak{g};\ l([X, \mathfrak{g}]) = \{0\}\}$, the radical of $l([\cdot, \cdot])$.

Then there are disjoint sets of indices S, T with $S \cup T = \{1, \dots, n\}$, and a G-invariant Zariski-open set $\mathscr{U}$ such that $S(l) = S$ for all $l \in \mathscr{U}$. Define the Pfaffian $\mathrm{Pf}(l)$ for $l \in \mathscr{U}$ by

$$|\mathrm{Pf}(l)|^2 = \det(l([X_i, X_j]))_{i,j \in S}.$$

Let $V_T = \mathbf{R}\text{-span}\{l_i;\ i \in T\}$ and dl be the Lebesgue measure on V_T such that the unit cube spanned by $\{l_i;\ i \in T\}$ has volume 1. Then for a function $\varphi \in L^1(G) \cap L^2(G)$, we have the Plancherel formula

$$\|\varphi\|_2^2 = \int_{\mathscr{U} \cap V_T} \|\pi_{2\pi l}(\varphi)\|_{C_2}^2 |\mathrm{Pf}(l)|\, dl. \tag{1.1}$$

Thus we get the following description:

$$\|\mathscr{P}^p(\varphi)\|_q = \left(\int_{\mathscr{U} \cap V_T} \|\pi_{2\pi l}(\varphi)\|_{C_q}^q |\mathrm{Pf}(l)|\, dl \right)^{1/q}. \tag{1.2}$$

Before computing (1.2), we treat the C_q-norm of $\pi(\varphi)$ for an irreducible representation π.

Definition 1.1. Let $\mathfrak{h}$ be an ideal of $\mathfrak{g}$ and dX be a Lebesgue measure on $\mathfrak{h}$ and $l \in \mathfrak{h}^*$ such that $l([\mathfrak{h}, \mathfrak{h}]) = \{0\}$. For $\varphi \in L^1(G)$, define a function $\mathscr{F}_{\mathfrak{h}}\varphi(l)(\cdot)$ on G associated to $l \in \mathfrak{h}^*$ by

$$\mathscr{F}_{\mathfrak{h}}\varphi(l)(g) = \int_{\mathfrak{h}} e^{\sqrt{-1}l(X)} \varphi((\exp X)g)\, dX \tag{1.3}$$

for almost all $g \in G$.

Since $\mathscr{F}_{\mathfrak{h}}\varphi(l)(hg) = e^{-\sqrt{-1}l(Y)}\mathscr{F}_{\mathfrak{h}}\varphi(l)(g)$ for $h = \exp Y \in H = \exp \mathfrak{h}$, we regard $|\mathscr{F}_{\mathfrak{h}}\varphi(l)(\cdot)|$ as a function on $H\backslash G$.

Proposition 1.2. *Let $f \in \mathfrak{g}^*$ and π_f be the corresponding irreducible representation of G. Suppose the following condition:*

(C1) *there exists an ideal $\mathfrak{h}$ satisfying $\mathfrak{g}(f) \subset \mathfrak{h}$ and $f([\mathfrak{h}, \mathfrak{h}]) = \{0\}$.*

Let $\mathfrak{h}^f = \{X \in \mathfrak{g};\ f([X, \mathfrak{h}]) = \{0\}\}$, which is a subalgebra, and $H^f = \exp \mathfrak{h}^f$. Taking Lebesgue measures on $\mathfrak{h}$ and $\mathfrak{h}^f$, let $|\mathrm{Pf}(\mathfrak{h}^f/\mathfrak{h}, f)| = (\det(f[Y_i, Y_j]))^{1/2}$, where $\{Y_i\}$ is a unit basis for $\mathfrak{h}^f/\mathfrak{h}$ of volume 1. Giving a Haar measure on G, we take the invariant measures on H, H^f, $H\backslash G$ and $H^f\backslash G$ normalized by the Lebesgue measures on $\mathfrak{h}$ and $\mathfrak{h}^f$ through the exponential map and the *transitivity of invariant measures. Then the following inequality holds for* $1 < p \leq 2$, $\frac{1}{p} + \frac{1}{q} = 1$ *and* $\varphi \in L^1(G) \cap L^p(G)$:

(1.4) $$\|\pi_f(\varphi)\|_q \leq C(\mathfrak{g}, \mathfrak{h}, f, p) \left\| \left(\int_{H^f \backslash G} |\mathscr{F}_{\mathfrak{h}}\varphi(g^{-1} \cdot (f|_{\mathfrak{h}}))(\cdot)|^q \, d\dot{g} \right)^{1/q} \right\|_{L^p(H\backslash G)},$$

where

$$C(\mathfrak{g}, \mathfrak{h}, f, p) = ((2\pi)^{1/q} A_p)^{(\dim \mathfrak{g} + \dim \mathfrak{g}(f) - 2\dim \mathfrak{h})/2} |\operatorname{Pf}(\mathfrak{h}^f/\mathfrak{h}, f)|^{-1/q},$$

and $d\dot{g}$ *is the invariant measure on* $H^f\backslash G$. (*We regard* $g \to |\mathscr{F}_{\mathfrak{h}}\varphi(g^{-1} \cdot (f|_{\mathfrak{h}}))(\cdot)|$ *as a function on* $H^f\backslash G$.)

If $p = 2$, *equality holds in* (1.4).

Proof. The proof is by induction on the dimension of $\mathfrak{g}$. The proposition is trivial for $\dim \mathfrak{g} = 1$ (regarding $c(\mathfrak{g}, \mathfrak{h}, f, p) = 1$ in this case). Assume that the proposition is valid for all dimensions of $\mathfrak{g}$ less than or equal to $n-1$, and that $\dim \mathfrak{g} = n$. Let $\mathfrak{z}$ be the center of $\mathfrak{g}$ and $Z = \exp \mathfrak{z}$.

Case 1. Suppose that $\mathfrak{z} \cap \ker(f) \neq \{0\}$. Taking $0 \neq Z \in (\mathfrak{z} \cap \ker f)$, let $\dot{\mathfrak{g}} = \mathfrak{g}/\mathbf{R}Z$ with the quotient map $\operatorname{pr}: \mathfrak{g} \to \dot{\mathfrak{g}}$, and $\dot{G} = \exp \dot{\mathfrak{g}}$ with $P: G \to \dot{G}$. We factor down f and π into $\dot{f} \in \dot{\mathfrak{g}}^*$ and $\dot{\pi} \in \widehat{\dot{G}}$ respectively. Then the radical $\dot{\mathfrak{g}}(\dot{f}) = \operatorname{pr}(\mathfrak{g}(f))$, $(\operatorname{pr}(\mathfrak{h}))^{\dot{f}} = \operatorname{pr}(\mathfrak{h}^f)$, and the coadjoint orbit $\dot{G} \cdot \dot{f}$ corresponds to $\dot{\pi}$.

For $\varphi \in C_c(G)$ (compactly supported continuous functions on G), define the function $\dot{\varphi} \in C_c(\dot{G})$ by

$$\dot{\varphi}(\dot{g}) = \int_{\mathbf{R}} \varphi((\exp tZ)g) \, dt, \qquad g \in G.$$

Then, writing $\dot{\mathfrak{h}} = \operatorname{pr}(\mathfrak{h})$ and $\dot{H} = \exp \dot{\mathfrak{h}}$, and taking the invariant measures on $\dot{G}$, $\dot{H}\backslash\dot{G}$ and $\dot{H}^{\dot{f}}\backslash\dot{G}$ associated to those on G, $H\backslash G$ and $H^f\backslash G$ through the projection P respectively, we have $\pi(\varphi) = \dot{\pi}(\dot{\varphi})$ and by the induction hypothesis,

$$\begin{aligned}
\|\pi(\varphi)\|_{C_q} &= \|\dot{\pi}(\dot{\varphi})\|_{C_q} \\
&\leq C(\dot{\mathfrak{g}}, \dot{\mathfrak{h}}, \dot{f}, p) \left\| \left(\int_{\dot{H}^{\dot{f}} \backslash \dot{G}} |\mathscr{F}_{\dot{\mathfrak{h}}}\dot{\varphi}(\dot{g}^{-1} \cdot (\dot{f}|_{\dot{\mathfrak{h}}}), \cdot)|^q \, d\dot{g} \right)^{1/q} \right\|_{L^p(\dot{H}\backslash\dot{G})} \\
&= C(\mathfrak{g}, \mathfrak{h}, f, p) \left\| \left(\int_{H^f \backslash G} |\mathscr{F}_{\mathfrak{h}}\varphi(g^{-1} \cdot (f|_{\mathfrak{h}}), \cdot)|^q \, dg \right)^{1/q} \right\|_{L^p(H\backslash G)},
\end{aligned}$$

where $d\dot{g}$ and dg are the invariant measures on $\dot{H}^{\dot{f}}\backslash\dot{G}$ and $H^f\backslash G$. (The last equality is verified by the property of the quotient spaces.)

Case 2. Suppose that $\ker(f)\cap\mathfrak{z} = \{0\}$. Since $\mathfrak{z}$ is 1-dimensional, we can take X_0, Y, $Z \in \mathfrak{g}$ such that $\mathfrak{z} = \mathbf{R}Z$, $[\mathfrak{g}, Y] = \mathfrak{z}$, $[X_0, Y] = Z$, and $f(Y) = 0$. Regarding $\mathfrak{g}$ as acting on $\mathfrak{h}$, we may assume

$$(1.5) \qquad Y \in \mathfrak{h} \quad \text{if } \mathfrak{h} \neq \mathfrak{z}.$$

Let $\mathfrak{g}_1 = \ker(\operatorname{ad} Y)$ and $G_1 = \exp\mathfrak{g}_1$. Then $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathbf{R}X_0$, the radical of $f' = f|_{\mathfrak{g}_1}$: $\mathfrak{g}_1(f') = \mathfrak{g}(f) + \mathbf{R}Y$, and $\mathfrak{h} \subset \mathfrak{g}_1$. Let π_1 denote the irreducible representation of G_1 corresponding to $G_1 \cdot f'$.

Using the supplementary basis X_0 to $\mathfrak{g}_1$, we realize π as induced from π_1, whose space is denoted by $\mathscr{H}_1$. That is, for $\xi = \xi(t) \in L^2(\mathbf{R}, \mathscr{H}_1)$, the space of $\mathscr{H}_1$-valued L^2-functions on $\mathbf{R}$ with a Lebesgue measure dt, define the action of $G = G_1 \exp\mathbf{R}X_0$ by

$$\pi(g_1 \exp sX_0)\xi(t) = \pi_1(g_1^t)\xi(t+s),$$

where $g_1 \in G_1$, $g_1^t = (\exp tX_0)g_1(\exp -tX_0)$.

Then we have $\pi(\varphi)$ for $\varphi \in C_c(G)$ as the integral operator

$$\pi(\varphi)\xi(t) = \int_{\mathbf{R}} k_\varphi(t, s)\xi(s)\, ds,$$

where $k_\varphi(t, s) = \int_{G_1} \pi_1(g_1)\varphi(g_1^{-t}\exp(s-t)X_0)\, dg_1$, dg_1 is the Haar measure on G_1 such that $dg = dg_1\, dt$ for $g = g_1 \exp tX_0$. For each fixed $t, s \in \mathbf{R}$, putting $\varphi^{t,s}(g_1) = \varphi(g_1^{-t}\exp(s-t)X_0) \in C_c(G_1)$, we regard the integral kernel as

$$k_\varphi(t, s) = \pi_1(\varphi^{t,s}).$$

Here let us recall an inequality of Hausdorff-Young type for integral operators due to Fournier and Russo **[9]**. Let $\mathscr{H}$ be a complex Hilbert space, $\mathscr{B}(\mathscr{H})$ be the space of bounded operators on $\mathscr{H}$, and M be a σ-finite measure space. Denote by $L^2(M, \mathscr{H})$ the Hilbert space of square integrable $\mathscr{H}$-valued functions on M. We consider an integral operator K on $L^2(M, \mathscr{H})$ with operator-valued kernel k, a $\mathscr{B}(\mathscr{H})$-valued function on $M \times M$, by letting

$$K\xi(x) = \int_M k(x, y)\xi(y)\, dy$$

for all $\xi \in L^2(M, \mathscr{H})$, and almost all $x \in M$.

If $1 \le p,\ r,\ s < \infty$, define the norm $\|\cdot\|_{C_p, r, s}$ by

$$\|k\|_{C_p, r, s} = \left\{ \int_M \left(\int_M (\|k(x, y)\|_{C_p})^r \, dx \right)^{s/r} dy \right\}^{1/s}.$$

We get from **[9]** the following estimate for the norm of K. Let $1 < p \le 2$, $\frac{1}{p} + \frac{1}{q} = 1$. Suppose $\|k\|_{C_q, p, q} < \infty$ and $\|k^*\|_{C_q, p, q} < \infty$, where $k^*(x, y) = k(y, x)$. Then the integral operator K with kernel k belongs to $C_q(L^2(M, \mathscr{H}))$, and

$$\|K\|_{C_q} \le \|k\|_{C_q, p, q}^{1/2} \|k^*\|_{C_q, p, q}^{1/2}. \tag{1.6}$$

If $p = 2$, equality holds in (1.6).

Now we return to the proof. Giving $\mathbf{R}X_0$ the Lebesgue measure such that X_0 has volume 1, let dX_1 be a Lebesgue measure on $\mathfrak{g}_1$ adapted to the direct sum decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathbf{R}X_0$.

Subcase 2.1. Here we suppose that $\mathfrak{h} = \mathfrak{z} = \mathfrak{g}(f)$. Let $\mathfrak{z}_1 = \mathfrak{g}_1(f') = \mathbf{R}Z + \mathbf{R}Y$, which coincides with the center of $\mathfrak{g}_1$. We apply the induction hypothesis to G_1 with the Haar measure $dg_1 = d(\exp X_1) = dX_1$, π_1, $\mathfrak{z}_1$ with the Lebesgue measure normalized as $\mathfrak{z}_1 = \mathbf{R}Y \oplus \mathbf{R}Z$ and $\varphi^{t,s}$. Putting a basis $\{Y_i\}_{1 \le i \le n-3}$ of $\mathfrak{g}_1/\mathfrak{z}_1$ whose unit cube has volume 1, and writing $f_1 = f|_{\mathfrak{z}_1}$, we get

$$\|\pi_1(\varphi^{t,s})\|_{C_q(\mathscr{H}_1)} \le c_1 \|\mathscr{F}_{\mathfrak{z}_1}\varphi^{t,s}(f_1)(\cdot)\|_{L^p(Z_1 \backslash G_1)}, \tag{1.7}$$

where

$$\begin{aligned} c_1 &= c(\mathfrak{g}_1, \mathfrak{z}_1, f', p) \\ &= (((2\pi)^{1/q} A_p)^{n-3} |\det(f_1([Y_i, Y_k]))_{1 \le i, k \le n-3}|^{-1/q})^{1/2}, \end{aligned}$$

and get equality in (1.7) for $p = 2$. For $g_1 \in G_1$,

$$\begin{aligned} &\mathscr{F}_{\mathfrak{z}_1}\varphi^{t,s}(f_1)(g_1) \\ &= \int_{\mathbf{R}^2} \varphi((\exp(zZ + yY)g_1)^{-t} \exp(s - t)X_0) e^{\sqrt{-1} f(zZ + yY)} \, dz \, dy \\ &= \mathscr{F}_{\mathfrak{z}_1}\varphi(\lambda Z^* + t\lambda Y^*)(g_1^{-t} \exp(s - t)X_0), \end{aligned}$$

where $\{Z^*, Y^*\} \subset \mathfrak{z}_1^*$ is the dual basis of $\{Z, Y\}$, and $\lambda = f(Z)$.

We first calculate the norm $\|k_\varphi^*\|_{C_q,p,q}$:

$$\left(\int_{\mathbf{R}}\left(\int_{\mathbf{R}}\|k_\varphi^*(t,s)\|^p_{C_q(\mathscr{H}_1)}\,dt\right)^{q/p}ds\right)^{1/q}$$
$$=\left(\int\left(\int\|\pi_1(\varphi^{s,t})\|^p_{C_q(\mathscr{H}_1)}\,dt\right)^{q/p}ds\right)^{1/q}$$
$$\leq\left(\int_{\mathbf{R}}\left(\int_{\mathbf{R}}(c_1\|\mathscr{F}_{\mathfrak{z}_1}\varphi^{s,t}(f_1)(\cdot)\|_{L^p(Z_1\backslash G_1)})^p\,dt\right)^{q/p}ds\right)^{1/q}\quad\text{(by (1.7))}$$
$$=c_1\left(\int_{\mathbf{R}}\left(\int_{\mathbf{R}}\int_{Z_1\backslash G_1}|\mathscr{F}_{\mathfrak{z}_1}\varphi(\lambda Z^*+s\lambda Y^*)(g_1^{-s}\exp(t-s)X_0)|^p\,d\dot{g}_1\,dt\right)^{q/p}ds\right)^{1/q}$$
$$=c_1\left(\int_{\mathbf{R}}\left(\int_{\mathbf{R}}\int_{Z_1\backslash G_1}|\mathscr{F}_{\mathfrak{z}_1}\varphi(\lambda Z^*+s\lambda Y^*)(g_1\exp tX_0)|^p\,d\dot{g}_1\,dt\right)^{q/p}ds\right)^{1/q}$$
$$\leq c_1\left(\int_{\mathbf{R}}\int_{Z_1\backslash G_1}\left(\int_{\mathbf{R}}|\mathscr{F}_{\mathfrak{z}_1}\varphi(\lambda Z^*+s\lambda Y^*)(g_1\exp tX_0)|^q\,ds\right)^{p/q}d\dot{g}_1\,dt\right)^{1/p}$$

(by the generalized Minkowski's inequality)

$$\leq(2\pi)^{1/q}A_p\lambda^{-1/q}c_1\left(\int_{\mathbf{R}}\int_{Z_1\backslash G_1}\int_{\mathbf{R}}|\mathscr{F}_{\mathfrak{z}}\varphi(\lambda Z^*)(\exp yYg_1\exp tX_0)|^p\,dy\,d\dot{g}_1\,dt\right)^{1/p}$$

(by (0.4) with our normalization of the Lebesgue measures)

$$=(2\pi)^{1/q}A_p\lambda^{-1/q}c_1\|\mathscr{F}_{\mathfrak{z}}\varphi(f|_{\mathfrak{z}})(\cdot)\|_{L^p(Z\backslash G)}.$$

(If $p=2$, equality holds in the above estimate.) Noticing that the unit cube of the basis $\{Y_0=Y, Y_1,\dots,Y_{n-3}, Y_{n-2}=X_0\}$ of $\mathfrak{g}/\mathfrak{z}$ has volume 1, and that $[Y_i,Y_0]=0$ for $i\leq n-3$, $[Y_{n-2},Y_0]=Z$, we get

$$\det(f([Y_i,Y_k]))_{0\leq i,k\leq n-2}=\lambda^2\det(f([Y_i,Y_k]))_{1\leq i,k\leq n-3},$$

and

$$(2\pi)^{1/q}A_p\lambda^{-1/q}c_1 = (((2\pi)^{1/q}A_p)^{n-1}|\det(f([Y_i,Y_k]))_{0\leq i,k\leq n-2}|^{-1/q})^{1/2}.$$

Thus we have

$$\|k_\varphi^*\|_{C_q,p,q}\leq C(\mathfrak{g},\mathfrak{z},f,p)\|\mathscr{F}_{\mathfrak{z}}\varphi(f_1)(\cdot)\|_{L^p(Z\backslash G)}.$$

On the other hand, remarking that $k_\varphi^*=\overline{k_{\varphi^*}}$, we also have $\|k_\varphi\|_{C_q,p,q}\leq C(\mathfrak{g},\mathfrak{z},f,p)\|\mathscr{F}_{\mathfrak{z}}\varphi^*(f_1)(\cdot)\|_{L^p(Z\backslash G)}$. Since $\|\mathscr{F}_{\mathfrak{z}}\varphi(f_1)(\cdot)\|_{L^p(Z\backslash G)}=\|\mathscr{F}_{\mathfrak{z}}\varphi^*(f_1)(\cdot)\|_{L^p(Z\backslash G)}$, we conclude that

$$\|\pi(\varphi)\|_{C_q}\leq C(\mathfrak{g},\mathfrak{z},f,p)\|\mathscr{F}_{\mathfrak{z}}\varphi(f)(\cdot)\|_{L^p(Z\backslash G)},$$

and that equality holds for $p=2$.

Subcase 2.2. We next suppose that $\mathfrak{h} \neq \mathfrak{z}$. Recalling that $Y \in \mathfrak{h}$ (1.5), we note that $\mathfrak{h}^f \subset \mathfrak{g}_1$. As in Subcase 2.1, we will estimate the norm $\|k_\varphi^*\|_{C_q, p, q}$. Apply the induction hypothesis to G_1, π_1, $\mathfrak{h}$ and $\varphi^{s,t}$ using the Haar measure on G_1 adapted to the decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathbf{R}X_0$ and the invariant measures on $H^f \backslash G_1$ (denoted by $d\dot{g}_1$) and on $H \backslash G_1$ suitably normalized:

$$\|\pi_1(\varphi^{s,t})\|_q \\ \leq C(\mathfrak{g}_1, \mathfrak{h}, f', p) \left\| \left(\int_{H^f \backslash G_1} |\mathscr{F}_\mathfrak{h} \varphi^{s,t}(g_1^{-1} \cdot f_1)(\cdot)|^q \, d\dot{g}_1 \right)^{1/q} \right\|_{L^p(H \backslash G_1)},$$

where $f_1 = f|_\mathfrak{h}$. Since

$$\int_{H^f \backslash G_1} |\mathscr{F}_\mathfrak{h} \varphi^{s,t}(g_1^{-1} \cdot f_1)(x)|^q \, d\dot{g}_1 \\ = \int_{H^f \backslash G_1} |\mathscr{F}_\mathfrak{h} \varphi((g_1 \exp sX_0)^{-1} \cdot f_1)(x^{-s} \exp(t-s)X_0)|^q \, d\dot{g}_1,$$

for $x \in G_1$,

$$\left(\int_\mathbf{R} \left\| \left(\int_{H^f \backslash G_1} |\mathscr{F}_\mathfrak{h} \varphi^{s,t}(g_1^{-1} \cdot f_1)(\cdot)|^q \, d\dot{g}_1 \right)^{1/q} \right\|_{L^p(H \backslash G_1)}^p dt \right)^{1/p} \\ = \left\| \left(\int_{H^f \backslash G_1} |\mathscr{F}_\mathfrak{h} \varphi((g_1 \exp sX_0)^{-1} \cdot f_1)(\cdot)|^q \, d\dot{g}_1 \right)^{1/q} \right\|_{L^p(H \backslash G)}.$$

Thus,

$$C(\mathfrak{g}_1, \mathfrak{h}, f', p)^{-1} \|k_\varphi^*\|_{C_q p, q} \\ \leq \left(\int_\mathbf{R} \left\| \left(\int_{H^f \backslash G_1} |\mathscr{F}_\mathfrak{h} \varphi((g_1 \exp sX_0)^{-1} \cdot f_1)(\cdot)|^q \, d\dot{g}_1 \right)^{1/q} \right\|_{L^p(H \backslash G)}^q ds \right)^{1/q}$$

(by the induction hypothesis)

$$\leq \left\| \left(\int_\mathbf{R} \int_{H^f \backslash G_1} |\mathscr{F}_\mathfrak{h} \varphi((g_1 \exp sX_0)^{-1} \cdot f_1)(\cdot)|^q \, d\dot{g}_1 \, ds \right)^{1/q} \right\|_{L^p(H \backslash G)}$$

(by the generalized Minkowski's inequality)

$$= \left\| \left(\int_{H^f \backslash G} |\mathscr{F}_\mathfrak{h} \varphi(g^{-1} \cdot f_1)(\cdot)|^q \, d\dot{g} \right)^{1/q} \right\|_{L^p(H \backslash G)}.$$

Since $\dim \mathfrak{g}_1 + \dim \mathfrak{g}_1(f') = \dim \mathfrak{g} + \dim \mathfrak{g}(f)$, we get $C(\mathfrak{g}, \mathfrak{h}, f, p) = C(\mathfrak{g}_1, \mathfrak{h}, f', p)$. As the proof of Subcase 2.1, the inequality (1.4) is verified. □

Now we get an estimate for $\|\mathscr{P}^p(\varphi)\|_q$ when $\mathfrak{g}$ admits an ideal $\mathfrak{h}$ such that the condition (C1) is satisfied for almost all $f \in \mathfrak{g}^*$ with $\mathfrak{h}$. Remark that if a subspace $\mathfrak{l}$ of $\mathfrak{g}$ satisfies that $f([\mathfrak{l}, \mathfrak{l}]) = \{0\}$ for all $f \in \mathscr{U} \subset \mathfrak{g}^*$, where $\mathscr{U}$ is a dense subset of $\mathfrak{g}^*$, then $\mathfrak{l}$ is an abelian subalgebra.

THEOREM 1.3. *Let $\mathfrak{g}$ be a nilpotent Lie algebra of dimension n, $G = \exp \mathfrak{g}$ and m be the dimension of the generic orbits. Suppose that $\mathfrak{g}$ satisfies the following condition.* (C2) *There exists an open dense subset $\mathscr{U}$ of $\mathfrak{g}^*$ such that the ideal generated by $\bigcup_{f \in \mathscr{U}} \mathfrak{g}(f)$ is abelian.*
Then the inequality

$$\|\mathscr{P}^p(G)\| \leq A_p^{(2n-m)/2}$$

holds for $1 < p < 2$.

COROLLARY 1.4. *Let $G = \exp \mathfrak{g}$ be a connected and simply connected nilpotent Lie group with the center $Z = \exp \mathfrak{z}$. Suppose that G has irreducible square integrable* (*mod the center*) *representations. Then*

$$\|\mathscr{P}^p(G)\| \leq A_p^{(\dim G + \dim Z)/2}.$$

Proof. An irreducible representation π is square integrable mod Z (i.e., π occurs discretely in the induced representation $\mathrm{ind}_Z^G \pi_Z$ by the central character π_Z of π) if and only if the dimension of the corresponding orbit Ω is $\dim \mathfrak{g}/\mathfrak{z}$, that is, $\mathfrak{g}(f) = \mathfrak{z}$ for $f \in \Omega$ (e.g., **[4]**(4.5)). And then square integrable representations correspond to the generic orbits. Thus the condition (C2) is satisfied in this case.

REMARK 1.5. There are nilpotent Lie groups which satisfy (C2) but do not have square integrable (mod the center) representations. For example, the nilpotent Lie group N_n of n real upper triangular matrices with ones on the main diagonal is such a group for $n \geq 4$. In this case,

$$\|\mathscr{P}^p(N_n)\| \leq A_p^{[n/2]\cdot[(n+1)/2]}.$$

In **[19]**, Russo obtained similar estimates for $\|\mathscr{P}^p(G)\|$ for the Heisenberg groups, the group N_n and some low dimensional nilpotent Lie groups. The results are based on estimates for $\|\pi(\varphi)\|_{C_q}$ for each irreducible representation π using the inequality (1.6) of integral operators under explicit realization of π. Our method, where we also use (1.6), is a generalization of the computation in **[19]**.

Proof of Theorem 1.3. Let $\mathfrak{h}$ be an abelian ideal satisfying $\mathfrak{h} \supset \mathfrak{g}(f)$ for all $f \in \mathscr{U}$, and $H = \exp\mathfrak{h}$. We may choose a Malcev basis $\{X_1, \dots, X_n\}$ for $\mathfrak{g}$ such that $\mathfrak{h} = \mathbf{R}\text{-span}\{X_1, \dots, X_k\}$ for some k. We use the notations in the Plancherel theorem (1.1). Noting $T \subset \{1, \dots, k\}$, let $S' = \{1, \dots, k\}\backslash T$, $V_{S'} = \mathbf{R}\text{-span}\{l_j; j \in S'\}$ and $p_{S'}$ be the projection of $\mathfrak{h}^* = V_T \oplus V_{S'}$ to $V_{S'}$. If $f \in \mathscr{U}$, then $\mathfrak{h}^\perp + G\cdot f = G\cdot f$, where $\mathfrak{h}^\perp = \{l \in \mathfrak{g}^*; l|_\mathfrak{h} = 0\}$, since $\mathfrak{h} \supset \mathfrak{g}(f)$. Thus, considering the coadjoint action of G on $\mathfrak{h}^*$, we get a parametrization of generic orbits in $\mathscr{U}$ from Chapter 3 of **[4]**. (We may assume that $\mathscr{U}$ is included in the set of generic orbits treated in the reference.): The set $\mathscr{U}' = \{l|_\mathfrak{h}; l \in \mathscr{U}\}$ is dense in $\mathfrak{h}^*$ and every G-orbit in $\mathscr{U}'$ meets V_T in a unique point. Furthermore, there is a diffeomorphism $\Psi: (\mathscr{U}' \cap V_T) \times V_{S'} \to \mathscr{U}'$ such that $(p_{S'} \circ \Psi)(f, \lambda) = \lambda$, and the Jacobian determinant of Ψ is identically 1. Let $S_f^0 = \{1 \le i \le n; \mathfrak{g}_{i-1} + \mathfrak{h}^f \neq \mathfrak{g}_i + \mathfrak{h}^f\}$ and $T_f^0 = \{1, \dots, n\}\backslash S_f^0$. We take the invariant measures on G, H and $H^f\backslash G$ defined by the Lebesgue measures on $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{h}^f\backslash\mathfrak{g}$ such that $\{X_1, \dots, X_n\}$, $\{X_1, \dots, X_k\}$ and $\{X_j; j \in S_f^0\}$ span unit cubes of volume 1 respectively. Identifying G with $H \times (H\backslash G)$, let us treat $\varphi = \varphi_0 \otimes \varphi_1 \in C_c(H) \otimes C_c(H\backslash G)$. Writing $\hat{\varphi}_0(l) = \int_\mathfrak{h} e^{\sqrt{-1}l(X)}\varphi_0(\exp X)\, dX$, we have

$$\begin{aligned}\int_{V_{S'}} &|\hat{\varphi}_0(\Psi(f, \lambda))|^q \, d\lambda \\ &= |\operatorname{Pf}(\mathfrak{h}^f/\mathfrak{h}, f)|^{-1} \int_{H^f\backslash G} |\hat{\varphi}_0(g^{-1}\cdot(f|_\mathfrak{h}))|^q \, d\dot{g} |\operatorname{Pf}(f)|,\end{aligned}$$

where $d\lambda$ is the Lebesgue measure such that $\{l_i; i \in S'\}$ spans a unit cube of volume 1, and $d\dot{g}$ is the G-invariant measure on $H^f\backslash G$. In fact, the Jacobian determinant of the map $H^f\backslash G \to \{f\} \times V_{S'}: \dot{g} \to \Psi^{-1}(g^{-1}\cdot(f|_\mathfrak{h}))$ is

$$|\det(g^{-1}\cdot f([X_i, X_j]))_{i\in S_f^0, j\in S'}| = |\det(f([X_i, X_j]))_{i\in S_f^0, j\in S'}|,$$

and

$$\begin{aligned}|\operatorname{Pf}(f)|^2 &= |\det(f([X_i, X_j]))_{i,j\in S}| \\ &= |\det(f([X_i, X_j]))_{i\in S_f^0, j\in S'}|^2 |\operatorname{Pf}(\mathfrak{h}^f/\mathfrak{h}, f)|^2.\end{aligned}$$

Thus from Proposition 1.2 and (0.4),

$$\begin{aligned}
\int_{\mathscr{U}\cap V_T} & \|\pi_{2\pi f}(\varphi)\|_{C_q}^q |\operatorname{Pf}(f)|\, df \\
&\le (2\pi)^{-\dim \mathfrak{h}} A_p^{q(2n-m-2\dim\mathfrak{h})/2} \\
&\quad \cdot \int_{\mathscr{U}\cap V_T} |\operatorname{Pf}(\mathfrak{h}^f/\mathfrak{h}, f)|^{-1} \\
&\quad \cdot \int_{H^f\backslash G} |\hat{\varphi}_0(g^{-1}\cdot(f|_{\mathfrak{h}}))|^q\, d\dot{g} \|\varphi_1\|_{L^p(H\backslash G)}^q |\operatorname{Pf}(f)|\, df \\
&= A_p^{q(2n-m)/2}(2\pi A_p^q)^{-\dim\mathfrak{h}} \\
&\quad \cdot \int_{\mathscr{U}\cap V_T}\int_{V_{S'}} |\hat{\varphi}_0(\Psi(f, \lambda))|^q\, d\lambda\, df \|\varphi_1\|_{L^p(H\backslash G)}^q \\
&= A_p^{q(2n-m)/2}(2\pi A_p^q)^{-\dim\mathfrak{h}} \int_{\mathfrak{h}^*} |\hat{\varphi}_0(l)|^q\, dl \|\varphi_1\|_{L^p(H\backslash G)}^q \\
&\le A_p^{q(2n-m)/2} \|\varphi\|_{L^p(G)}^q .
\end{aligned}$$

This implies the theorem. □

2. The L^p-Fourier transform on affine automorphism groups of Siegel domains.

2.1. *Preliminaries.* Concerning affine homogeneous Siegel domains, let us recall the notion of normal j-algebras introduced by Pyatetskii-Shapiro:

DEFINITION 2.1.1. A triple $(\mathfrak{g}, j, \omega)$ is a normal j-algebra if

(1) $\mathfrak{g}$ is a real completely solvable Lie algebra (i.e., $\mathfrak{g}$ admits a decreasing series of ideals $\mathfrak{g}_i$ such that $\dim \mathfrak{g}_i/\mathfrak{g}_{i+1} = 1$),
(2) $j: \mathfrak{g} \to \mathfrak{g}$ is a complex structure,
(3) $[jX, jY] = [X, Y] + j[jX, Y] + j[X, jY]$ for all $X, Y \in \mathfrak{g}$,
(4) $\omega \in \mathfrak{g}^*$ has the properties
 (a) $\omega([Y, jY]) > 0$ for all $Y \in \mathfrak{g} - \{0\}$,
 (b) $\omega([jX, jY]) = \omega([X, Y])$ for all $X, Y \in \mathfrak{g}$.

It is known that the connected and simply connected Lie group $G = \exp \mathfrak{g}$ with a normal j-algebra $(\mathfrak{g}, j, \omega)$ can be realized as an affine automorphism group acting simply and transitively on a Siegel domain of type II, and vice versa. (For details, see e.g. **[11]**, **[16]**.) Thus, starting from a normal j-algebra $(\mathfrak{g}, j, \omega)$, which we often denote by $\mathfrak{g}$ only, we study the corresponding group $G = \exp \mathfrak{g}$.

Here we summarize fundamental facts of the structures of normal j-algebras and unitary representations of corresponding groups.

For a normal j-algebra $(\mathfrak{g}, j, \omega)$, let Λ be the symmetric positive definite bilinear form $\Lambda(X, Y) = \omega([X, jY])$ on $\mathfrak{g}$, and let $\mathfrak{a}$ be the orthogonal complement of $[\mathfrak{g}, \mathfrak{g}]$ with respect to Λ. Then $\mathfrak{a}$ is an abelian subalgebra of $\mathfrak{g}$, and the adjoint representation of $\mathfrak{a}$ on $[\mathfrak{g}, \mathfrak{g}]$ is real diagonalizable. There exists a unique element $S \in \mathfrak{a}$ such that $\operatorname{ad} S|_{j\mathfrak{a}} = \mathbf{1}_{j\mathfrak{a}}$. The eigenvalues of $\operatorname{ad} S$ are at most 1, $\frac{1}{2}$ and 0. Denoting each eigenspace by $\mathfrak{g}_k$, $k = 1, \frac{1}{2}, 0$, we have

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_{1/2} \oplus \mathfrak{g}_0, \quad \text{and}$$
$$j\mathfrak{g}_1 = \mathfrak{g}_0, \quad j\mathfrak{g}_{1/2} = \mathfrak{g}_{1/2}, \quad [\mathfrak{g}_i, \mathfrak{g}_k] \subset \mathfrak{g}_{i+k},$$

with the convention that if $i + k \neq 1, \frac{1}{2}$ nor 0, $\mathfrak{g}_{i+k} = \{0\}$, **[11]**, **[16]**.

We next consider unitary representations of $G = \exp \mathfrak{g}$. Since G is an exponential group (i.e., the exponential mapping is a diffeomorphism of $\mathfrak{g}$ onto G), its unitary dual $\widehat{G}$ is parametrized by the coadjoint orbits of G on $\mathfrak{g}^*$ through the Kirillov-Bernat mapping. In the case of a normal j-algebra, G has open orbits, whose union is dense in $\mathfrak{g}^*$. They correspond to the classes of square integrable representations of G. (The criterion of square integrability used in the proof of Corollary 1.4 holds for exponential groups **[6]**.)

Let us give a more detailed description of open orbits. Notice that the subgroup $G_0 = \exp \mathfrak{g}_0$ acts on $\mathfrak{g}_1^*$ by the coadjoint action since $\mathfrak{g}_1$ is an ideal of $\mathfrak{g}$. Let $l \in \mathfrak{g}^*$, and $l_1 = l|_{\mathfrak{g}_1}$. Then the orbit $G \cdot l$ is open in $\mathfrak{g}^*$ if and only if $G_0 \cdot l_1$ is open in $\mathfrak{g}_1^*$. Thus, regarding $\mathfrak{g}^* = \mathfrak{g}_1^* \oplus \mathfrak{g}_{1/2}^* \oplus \mathfrak{g}_0^*$ according to the direct sum decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_{1/2} \oplus \mathfrak{g}_0$, we have

$$G \cdot l = G_0 \cdot l_1 + \mathfrak{g}_{1/2}^* + \mathfrak{g}_0^*,$$

for an open orbit $G \cdot l$ **[15]** (1.3), **[17]** (Proposition 3.3.1).

Throughout §2, $\mathfrak{g}$ is a normal j-algebra, $G = \exp \mathfrak{g}$ and dg is a left Haar measure on G.

2.2. *A Hausdorff-Young theorem for* G. Let π be an irreducible square integrable representation of G in a Hilbert space $\mathscr{H}$. Then from **[5]**, there exists a unique operator K_π in $\mathscr{H}$, self-adjoint positive, semi-invariant with weight Δ^{-1}, i.e.,

$$\pi(g) K_\pi \pi(g)^{-1} = \Delta(g)^{-1} K_\pi \quad \text{for all } g \in G, \tag{2.1}$$

and satisfying that

$$\int_G |\langle \xi, \pi(g)\eta\rangle|^2 \, dg = \|\xi\|^2 \|K_\pi^{-1/2}\eta\|^2 \tag{2.2}$$

for all $\xi \in \mathscr{H}$ and $\eta \in \operatorname{dom} K_\pi^{-1/2}$, the domain of $K_\pi^{-1/2}$. The operator K_π is called the formal degree of π.

Using the formal degree, we state our Hausdorff-Young theorem as follows.

THEOREM 2.2.1. *Let $\mathfrak{g}$ be a normal j-algebra, $G = \exp \mathfrak{g}$, and dg be a left Haar measure on G. Taking a set of representatives of classes of irreducible square integrable representations of G, $\{(\pi_i, \mathscr{H}_{\pi_i}); i \in I\}$, let K_{π_i} be the formal degree of π_i in the sense of* [**5**]. *Let p, q be exponents such that $1 < p \leq 2$, $\frac{1}{p} + \frac{1}{q} = 1$.*

(1) *Let $\varphi \in L^1(G) \cap L^p(G)$. Then the operator $\pi_i(\varphi)K_{\pi_i}^{1/q}$ can be extended to a C_q-class operator, denoted by $[\pi_i(\varphi)K_{\pi_i}^{1/q}]$, and satisfies the following inequality;*

$$\left(\sum_{i\in I} \|[\pi_i(\varphi)K_{\pi_i}^{1/q}]\|_{C_q}^q\right)^{1/q} \leq A_p^{\dim G/2} \|\varphi\|_p. \tag{2.3}$$

If $p = 2$, equality holds in (2.3).

(2) *The mapping $\varphi \to \pi_i^p(\varphi) = [\pi_i(\varphi)K_{\pi_i}^{1/q}]$ extends uniquely to a continuous mapping $\pi_i^p\colon L^p(G) \to C_q(\mathscr{H}_{\pi_i})$, $i \in I$.*

Let $\mathscr{P}^p\colon L^p(G) \to \bigoplus_{i\in I} C_q(\mathscr{H}_{\pi_i})$ be the mapping defined by $\varphi \to \mathscr{P}^p(\varphi) = \bigoplus_{i\in I} \pi_i^p(\varphi)$. Then $\mathscr{P}^p$ is continuous and injective, and the image $\mathscr{P}^p(L^p(G))$ is dense in $\bigoplus_{i\in I} C_q(\mathscr{H}_{\pi_i})$.

The involutions of $L^p(G)$ and $\bigoplus_{i\in I} C_q(\mathscr{H}_{\pi_i})$ are preserved, i.e.,

$$\mathscr{P}^p(\varphi^{*(p)}) = \bigoplus_{i\in I} \pi_i^p(\varphi)^* = \mathscr{P}^p(\varphi)^* \tag{2.4}$$

In the case of $p = 2$, $\mathscr{P}^2$ is a surjective isometry.

REMARK 2.2.2. It is obtained from the Plancherel theorem of Duflo and Moore [**5**] that $\mathscr{P}^2$ is a surjective isometry. But we will prove it simultaneously in the course of establishing the inequality (2.3).

On the $ax+b$ group, Eymard and Terp [**7**] and Russo [**20**] obtained similar results. The former is based on the Plancherel theorem of Duflo and Moore, but the latter is based on that of Kleppner and Lipsman [**12**]. In order to obtain L^p-estimates, they used the integral operator inequality (1.6), which we will also use, and got the same estimate with that of our $n = 2$ case.

We give here a representative of each class of irreducible square integrable representations, and an explicit description of the formal degree, to be used later.

Recalling that the classes of irreducible square integrable representations correspond to open coadjoint orbits, let Ω be an open coadjoint orbit. Put an element $f \in \Omega$ and take a real polarization $\mathfrak{b}_f$ at f satisfying the Pukanszky condition [3]. Defining a character χ_f of $B = \exp \mathfrak{b}_f$ by $\chi_f(\exp X) = e^{\sqrt{-1}f(X)}$ for $X \in \mathfrak{b}_f$, construct the induced representation $\pi = \mathrm{ind}_B^G \chi_f$ of G from χ_f. The representation π is irreducible and its class is the corresponding one under the Kirillov-Bernat mapping. Remark that we can always take such a polarization $\mathfrak{b}_f$ such that $\mathfrak{g}_1 \subset \mathfrak{b}_f \subset \mathfrak{g}_{1/2}$ (see [10], Remark 2.5). Thus putting $\mathfrak{n} = \mathfrak{g}_1 + \mathfrak{g}_{1/2}$, which is a nilpotent ideal, and $N = \exp \mathfrak{n}$, we regard π as induced from the irreducible representation $\sigma = \mathrm{ind}_B^N \chi_f$ of N, $\pi = \mathrm{ind}_N^G \sigma$.

Regarding G as the semidirect product $G = NG_0$, we take a right Haar measure dg_0 on G_0 and dn on N such that $\Delta^{-1}(g)\,dg = dn\,dg_0$, for $g = ng_0$, $n \in N$, $g_0 \in G_0$. Letting $\mathscr{H}_\sigma$ be a space of σ, we realize π in the space $L^2(G_0, \mathscr{H}_\sigma, dg_0)$ of $\mathscr{H}_\sigma$-valued L^2-functions on G_0, acting on the right:

$$(\pi(ng_0)\xi)(x_0) = \sigma(x_0 n x_0^{-1})\xi(x_0 g_0)$$

for $\xi \in L^2(G_0, \mathscr{H}_\sigma, dg_0)$, $ng_0 \in G = NG_0$, $x_0 \in G_0$.

We next choose the Lebesgue measure dX on $\mathfrak{g}$ such that $dg = \mu_G(X)\,dX$, where $\mu_G(X) = |\det((1 - e^{-\mathrm{ad}\,X})/\mathrm{ad}\,X)|$, $g = \exp X$ [3]. Letting $\{X_i\}_{1\le i\le n}$ be a basis of $\mathfrak{g}$ such that the unit cube has volume 1, define a function $l \to D_l$ on $\mathfrak{g}^*$ by

$$D_l = |\det(l([X_i, X_k]))_{1\le i,k\le n}| \qquad (l \in \mathfrak{g}^*). \tag{2.5}$$

Putting a unit basis $\{X_1, \dots, X_{n_1}, V_1, \dots, V_{n_2}, Y_1, \dots, Y_{n_1}\}$, where $\mathfrak{n} = \mathrm{span}\{V_i, Y_k;\ 1 \le i \le n_2,\ 1 \le k \le n_1\}$ and $\mathfrak{g}_1 = \mathrm{span}\{Y_k;\ 1 \le k \le n_1\}$, we get

$$D_l = |\det(l([V_i, V_k]))_{1\le i,k\le n_2}|\,|\det(l[X_i, Y_k])_{1\le i,k\le n_1}|^2 \tag{2.6}$$

since $[\mathfrak{g}_1, \mathfrak{n}] = \{0\}$.

DEFINITION 2.2.3. Let K_f be the operator in $L^2(G_0, \mathscr{H}_\sigma, dg_0)$ defined by multiplication by the function $c_f\Delta^{-1}$, where $c_f = ((2\pi)^{-\dim\mathfrak{g}}D_f)^{1/2}$.

Then K_f is the formal degree of π (see [6]).

2.3. *Proof of Theorem* 2.1. Let Ω be an open coadjoint orbit. Taking an element $f \in \Omega$ and a real polarization $\mathfrak{b}_f$ at f, we realize the corresponding irreducible representation π in $\mathscr{H}$ and define the formal degree K_f as we mentioned in 2.2.

DEFINITION 2.3.1. Let $\psi \in L^1(G)$ and dX be a Lebesgue measure on $\mathfrak{g}_1$. We define the partial Euclidean Fourier transform $\mathscr{F}_1\psi$ on $G_1 = \exp\mathfrak{g}_1$ with dX by

$$\mathscr{F}_1\psi(l)(g) = \int_{\mathfrak{g}_1} e^{\sqrt{-1}l(X)}\psi((\exp X)g)\,dX,$$

for $l \in \mathfrak{g}_1^*$ and almost all $g \in G$.

Let $\varphi \in C_c(G)$, and $\xi \in \mathscr{H} = L^2(G_0, \mathscr{H}_\sigma, dg_0)$ such that $K_f^{1/q}\xi \in \mathscr{H}$. From the semi-invariance (2.1),

$$\begin{aligned}(\pi(\varphi)K_f^{1/q}\xi)(x_0) &= \int_G (\pi(g)K_f^{1/q}\xi)(x_0)\varphi(g)\,dg \\ &= \int_G \Delta^{-1/q}(g)(K_f^{1/q}\pi(g)\xi)(x_0)\varphi(g)\,dg \\ &= (K_f^{1/q}\pi(\Delta^{-1/q}\varphi)\xi)(x_0)\end{aligned} \tag{2.7}$$

Let us identify G with $\mathfrak{g}_1 \times (G_1\backslash G)$ by taking a global section $\mathfrak{s}$ of $G_1\backslash G$, and choose a right Haar measure $d\dot{g}$ on $G_1\backslash G$ such that $\Delta^{-1}(g)\,dg = dX\,d\dot{g}$ for $g = (\exp X)\mathfrak{s}(\dot{g})$ with $X \in \mathfrak{g}_1$, $\dot{g} \in (G_1\backslash G)$.

We next suppose that $\varphi = \varphi_1 \otimes \dot{\varphi} \in C_c(\mathfrak{g}_1) \otimes C_c(G_1\backslash G)$ and that the Euclidean Fourier transform of φ_1, denoted by $\hat{\varphi}_1$, is of support compact. Letting $\lambda\colon G_0 \to \mathfrak{g}_1^*$ be the mapping defined by $\lambda(x_0) = x_0^{-1}\cdot f_1$, where $x_0 \in G_0$ and $f_1 = f|_{\mathfrak{g}_1}$, and noting that $\pi(\exp X)\eta(x_0) = e^{\sqrt{-1}f(x_0\cdot X)}\eta(x_0)$ for $X \in \mathfrak{g}_1$ and $\eta \in \mathscr{H}$, we get

$$\begin{aligned}
\|\pi(\varphi)K^{1/q}\xi\|^2 &= \int_{G_0} \left\| \int_G \Delta^{-1/q}(x_0)(\pi(g)\xi)(x_0)\Delta^{-1/q}(g)\varphi(g)\,dg \right\|^2 dx_0 \\
&= \int_{G_0} \left\| \int_{\mathfrak{g}_1} \int_{G_1\backslash G} \Delta^{-1/q}(x_0) e^{\sqrt{-1}f(x_0\cdot X)}(\pi(s(\dot g))\xi)(x_0)\varphi((\exp X)s(\dot g)) \right. \\
&\qquad \left. \cdot\, \Delta^{1/p}(s(g))\,dX\,d\dot g \right\|^2 dx_0 \\
&= \int_{G_0} \left\| \int_{G_1\backslash G} \Delta^{-1/q}(x_0)(\pi(s(\dot g))\xi)(x_0)\Delta^{1/p}(s(\dot g)) \right. \\
&\qquad \left. \cdot\, \mathscr{F}_1\varphi(x_0\cdot f_1)(s(\dot g))\,d\dot g \right\|^2 dx_0 \\
&\le \sup_{x_0\in G_0} |\Delta^{-1/q}(x_0)\hat\varphi_1(x_0^{-1}\cdot f_1)|^2 \int_{G_0}\int_{G_1\backslash G} \|(\pi(s(\dot g))\xi)(x_0)\|^2 \\
&\qquad \cdot\, |\dot\varphi(\dot g)\Delta^{1/p}(s(\dot g))|^2\,d\dot g\,dx_0 \\
&= \sup_{l\in G_0\cdot f_1} |\Delta^{-1/q}(\lambda^{-1}(l))\hat\varphi_1(l)|^2 \|\dot\varphi\Delta^{1/p}\|^2_{L^2(G_1\backslash G)}\|\xi\|^2.
\end{aligned}$$

(Note that λ is a diffeomorphism of G_0 onto $G_0\cdot f_1$.) Here we regard the function $l \to D_l$ on $\mathfrak{g}^*$ as defined on $\mathfrak{g}_1^*$ remarking that $D_{l+m} = D_l$ for any $m\in\mathfrak{g}_1^\perp$. Then

$$D_l = D_{\lambda^{-1}(l)\cdot f_1} = \Delta^{-1}(\lambda^{-1}(l))D_f, \qquad l\in G_0\cdot f_1.$$

Thus

$$\begin{aligned}
&\sup_{l\in G_0\cdot f_1} |\Delta^{-1/q}(\lambda^{-1}(l))\hat\varphi_1(l)|^2 \\
&\qquad \le \sup_{l\in(G_0\cdot f_1\cap\operatorname{supp}\hat\varphi_1)} (D_l D_f^{-1})^{2/q} \sup|\hat\varphi_1|^2 < \infty,
\end{aligned}$$

which implies that the operator $\pi(\varphi)K^{1/q}$ extends to a bounded operator, denoted by $\pi^p(\varphi)$.

REMARK 2.3.2. For such a φ, it holds that

$$\pi^p(\varphi)^* = K^{1/q}\pi(\varphi^*) = \pi^p(\varphi^{*(p)}), \tag{2.8}$$

$$\pi^p(\varphi) = K^{1/q}\pi(\Delta^{-1/q}\varphi). \tag{2.9}$$

Proof. Let ξ, $\eta \in \mathscr{H}$ and suppose that $\xi \in \operatorname{dom} K^{1/q}$. Then

$$\langle \pi(\varphi)K^{1/q}\xi, \eta\rangle = \langle K^{1/q}\xi, \pi(\varphi^*)\eta\rangle.$$

Noting that $K^{1/q}$ is self-adjoint, we conclude that $\pi(\varphi^*)\eta \in \operatorname{dom} K^{1/q}$ and that $\pi^p(\varphi)^* = K^{1/q}\pi(\varphi^*)$. Using (2.7), the second equality of (2.8) and (2.9) are obtained. □

Now, we will estimate the C_q-norm of $\pi^p(\varphi)$. Identifying G with NG_0 according to the realization of π, we get for $\xi \in \operatorname{dom} K^{1/q}$

$$\begin{aligned}
&\pi(\varphi)K_f^{1/q}\xi(x_0)\\
&\quad = c_f^{1/q}\int_N\int_{G_0}\sigma(x_0nx_0^{-1})\Delta^{-1/q}(x_0g_0)\xi(x_0g_0)\varphi(ng_0)\Delta(g_0)\,dn\,dg_0\\
&\quad = c_f^{1/q}\int_N\int_{G_0}\sigma(x_0nx_0^{-1})\xi(g_0)\varphi(nx_0^{-1}g_0)\\
&\qquad\qquad\qquad\qquad\cdot\Delta^{1/p}(g_0)\Delta_0(x_0)\Delta^{-1}(x_0)\,dn\,dg_0\\
&\quad = c_f^{1/q}\int_N\int_{G_0}\sigma(n)\xi(g_0)\varphi(x_0^{-1}ng_0)\Delta^{1/p}(g_0)\,dn\,dg_0,
\end{aligned}$$

where $\Delta_0(g_0) = |\det(\mathrm{ad}_{\mathfrak{g}_0} x_0)|^{-1}$. Letting $\varphi^{x_0}(ng_0) = \varphi(x_0^{-1}ng_0)$ and for each fixed x_0, $g_0 \in G_0$, regarding $\varphi^{x_0}(ng_0)$ as a function on N, define a Fourier transform for σ;

$$\sigma(\varphi^{x_0}(ng_0)) = \int_N \sigma(n)\varphi^{x_0}(ng_0)\,dn,$$

which is a bounded operator of $\mathscr{H}_\sigma$. Then

$$\pi^p(\varphi)\xi(x_0) = c_f^{1/q}\int_{G_0}\sigma(\varphi^{x_0}(ng_0))\xi(g_0)\Delta^{1/p}(g_0)\,dg_0,$$

and we regard $\pi^p(\varphi)$ as the integral operator with integral kernel

$$k_\varphi(x_0, g_0) = c_f^{1/q}\sigma(\varphi^{x_0}(ng_0))\Delta^{1/p}(g_0).$$

Let us remark that the representation $\sigma = \operatorname{ind}_B^N \chi_f$ is square integrable. In fact, from $D_f \neq 0$ and (2.6), the singular space of the bilinear form $f([\cdot, \cdot])$ on $\mathfrak{n}$ is $\mathfrak{g}_1$, which is the center of $\mathfrak{n}$. Choose a Haar measure $d\dot{n}$ on $G_1\backslash N$ such that the transitivity holds with measures dn and dX. Applying Proposition 1.2 to σ, for each fixed x_0, $g_0 \in G_0$, we get

$$\|\sigma(\varphi^{x_0}(ng_0))\|_{C_q(\mathscr{H}_\sigma)} \leq A_p^{\dim(\mathfrak{n}/\mathfrak{g}_1)/2} c(\sigma)^{1/q} \|\mathscr{F}_1 \varphi^{x_0}(f_1)(ng_0)\|_{L^p(G_1\backslash N)},$$

where $c(\sigma) = (2\pi)^{(\dim \mathfrak{g} - \dim \mathfrak{z})/2} |\operatorname{Pf}(\mathfrak{g}/\mathfrak{z}, f)|^{-1}$. Using the notations $\Delta_1(x_0) = |\det \operatorname{ad}_{\mathfrak{g}_1} x_0|^{-1}$, $\Delta_N(x_0) = |\det \operatorname{ad}_{\mathfrak{n}} x_0|^{-1}$ for $x_0 \in G_0$, we get

$$\begin{aligned}
\mathscr{F}_1 \varphi^{x_0}(f_1)(ng_0) &= \int_{\mathfrak{g}_1} e^{\sqrt{-1} f(X)} \varphi^{x_0}((\exp X) n g_0)\, dX \\
&= \int_{\mathfrak{g}_1} e^{\sqrt{-1} f(x_0 \cdot X)} \varphi((\exp X) x_0^{-1} n g_0) \Delta_1^{-1}(x_0)\, dX \\
&= \mathscr{F}_1 \varphi(x_0^{-1} \cdot f_1)(x_0^{-1} n g_0) \Delta_1^{-1}(x_0)
\end{aligned}$$

and

$$\begin{aligned}
&\|\mathscr{F}_1 \varphi^{x_0}(f_1)(ng_0)\|_{L^p(G_1\backslash N)} \\
&\quad = \left(\int_{G_1\backslash N} |\mathscr{F}_1 \varphi(x_0^{-1} \cdot f_1)(x_0^{-1} n g_0)|^p \, d\dot{n} \right)^{1/p} \Delta_1^{-1}(x_0) \\
&\quad = \left(\int_{G_1\backslash N} |\mathscr{F}_1 \varphi(x_0^{-1} \cdot f_1)(n x_0^{-1} g_0)|^p \, d\dot{n} \right)^{1/p} (\Delta_1/\Delta_N)^{1/p}(x_0) \Delta_1^{-1}(x_0).
\end{aligned}$$

Thus we have the inequality

$$\begin{aligned}
(2.10)\quad \|k_\varphi(x_0, g_0)\|_{C_q(\mathscr{H}_\sigma)} \leq \alpha &\left(\int_{G_1\backslash N} |\mathscr{F}_1 \varphi(x_0^{-1} \cdot f_1)(n x_0^{-1} g_0)|^p \, d\dot{n} \right)^{1/p} \\
&\cdot \Delta_N^{-1/p}(x_0) \Delta_1^{-1/q}(x_0) \Delta^{1/p}(g_0),
\end{aligned}$$

where

$$\alpha = A_p^{(\dim \mathfrak{n}/\mathfrak{g}_1)/2} c(\sigma)^{1/q} c_f^{1/q},$$

for x_0, $g_0 \in G_0$.

As the proof of Proposition 1.2, we first estimate the norm $\|k_\varphi^*\|_{C_q,p,q}$.

$$
\begin{aligned}
&\|k_\varphi^*\|_{C_q,p,q} \\
&= \left(\int_{G_0}\left(\int_{G_0}\|k_\varphi^*(g_0\,,\,x_0)\|_{C_q}^p\,dg_0\right)^{q/p} dx_0\right)^{1/q} \\
&\le \alpha\left(\int_{G_0}\left(\int_{G_0}\int_{G_1\backslash N}|\mathscr{F}_1\varphi(x_0^{-1}\cdot f_1)(nx_0^{-1}g_0)|^p\,d\dot{n}\Delta(g_0)\,dg_0\right)^{q/p}\right. \\
&\qquad\qquad \left.\cdot\,\Delta_N^{-q/p}(x_0)\Delta_1^{-1}(x_0)\,dx_0\right)^{1/q} \qquad \text{(by (2.10))} \\
&= \alpha\left(\int_{G_0}\left(\int_{G_0}\int_{G_1\backslash N}|\mathscr{F}_1\varphi(x_0^{-1}\cdot f_1)(ng_0)|^p\,d\dot{n}\Delta(g_0)\,dg_0\right)^{q/p}\Delta_1^{-1}(x_0)\,dx_0\right)^{1/q} \\
&\le \alpha\left(\int_{G_0}\int_{G_1\backslash N}\left(\int_{G_0}|\mathscr{F}_1\varphi(x_0^{-1}\cdot f_1)(ng_0)|^q\Delta_1^{-1}(x_0)\,dx_0\right)^{p/q} d\dot{n}\Delta(g_0)\,dg_0\right)^{1/p}
\end{aligned}
$$

(by the generalized Minkowski's inequality for measures $\Delta_1^{-1}(x_0)\,dx_0$ and $d\dot{n}\Delta(g_0)dg_0$).

Choose a Lebesgue measure d_0X on $\mathfrak{g}_0$ such that

$$dg_0 = d(\exp X) = \mu_{G_0}(X)d_0X\,,$$

for $g_0 = \exp X \in G_0$, where $\mu_{G_0}(X) = |\det((1-e^{-\operatorname{ad}X})/\operatorname{ad}X)|$. Let $\{X_i\}$ (resp. $\{Y_i\}$) be a basis of $\mathfrak{g}_0$ (resp. $\mathfrak{g}_1$) whose unit cube has volume 1. Then under the mapping $x_0 \to \lambda(x_0) = x_0^{-1}\cdot f_1$ from G_0 (with the Haar measure dg_0) to $\mathfrak{g}_1^*$ (with the Lebesgue measure $d\lambda$), which is the dual space of $\mathfrak{g}_1$ with the Lebesgue measure dX, we have $d\lambda(x_0) = \Delta_1^{-1}(x_0)|\det(f([X_i\,,\,Y_k]))_{i,k}|\,dx_0$. Thus we have

$$
\begin{aligned}
&\|k_\varphi^*\|_{C_q,p,q} \\
&\quad\le \alpha|\det(f([X_i\,,\,Y_k]))_{i,k}|^{-1/q} \\
&\quad\cdot\left(\int_{G_0}\int_{G_1\backslash N}\left(\int_{\Omega'}|\mathscr{F}_1\varphi(\lambda)(ng_0)|^q\,d\lambda\right)^{p/q} d\dot{n}\Delta(g_0)\,dg_0\right)^{1/p},
\end{aligned}
$$

where $\Omega' = G_0\cdot f_1 \subset \mathfrak{g}_1^*$. Again noticing that $k_\varphi^* = \overline{k_{\varphi^{*(p)}}}$ from (2.8), we get

$$\begin{aligned}&\|k_\varphi\|_{C_q,p,q}\\ &\quad\le \alpha|\det(f([X_i, Y_k]))_{i,k}|^{-1/q}\\ &\qquad\cdot\left(\int_{G_0}\int_{G_1\backslash N}\left(\int_{\Omega'}|\mathscr{F}_1\varphi^{*(p)}(\lambda)(ng_0)|^q\,d\lambda\right)^{p/q}d\dot{n}\Delta(g_0)\,dg_0\right)^{1/p}.\end{aligned}$$

By some simple calculation, it can be shown that the right-hand sides of the above two inequalities coincide.

Recall that

$$\alpha = A_p^{(\dim \mathfrak{n}/\mathfrak{g}_1)/2}c(\sigma)^{1/q}c_f^{1/q},$$

and that

$$c(\sigma) = ((2\pi)^{\dim \mathfrak{n}/\mathfrak{g}_1}|\det(f([V_i, V_k]))|^{-1})^{1/2},$$

where $\{V_i\}$ is a basis of $\mathfrak{n}/\mathfrak{g}_1$ whose unit cube has volume 1. Using (2.6), we get

$$\begin{aligned}c_f &= ((2\pi)^{-\dim \mathfrak{g}}D_f)^{1/2}\\ &= (2\pi)^{-\dim \mathfrak{g}/2}|\det(f([X_i, Y_k]))|\,|\det(f([V_i, V_k]))|^{1/2},\end{aligned}$$

and

$$\alpha|\det(f([X_i, Y_k]))|^{-1/q} = (2\pi)^{-\dim \mathfrak{g}_1/q}A_p^{(\dim \mathfrak{n}/\mathfrak{g}_1)/2}.$$

Thus identifying with $\mathfrak{g}_1 \times (G_1\backslash G)$,

$$\begin{aligned}(2.11)\quad &\|\pi(\varphi)K_f^{1/q}\|_{C_q}\\ &\quad\le (2\pi)^{-\dim \mathfrak{g}_1/q}A_p^{(\dim \mathfrak{n}/\mathfrak{g}_1)/2}\\ &\qquad\cdot\left(\int_{G_1\backslash G}\left(\int_{\Omega'}|\mathscr{F}_1\varphi(\lambda)(ng_0)|^q\,d\lambda\right)^{p/q}\Delta(g)\,d\dot{g}\right)^{1/p}.\end{aligned}$$

Taking a system of representatives $\{f_i\,;\, i \in I\}$ of open orbits Ω_i $(i \in I)$, let $(\pi_i, \mathscr{H}_i, K_{f_i})$ be the associated representation and the operator of 2.2. Then, recalling that $\varphi = \varphi_1 \otimes \dot{\varphi}$, $\varphi \in C_c(\mathfrak{g}_1)$ and $\dot{\varphi} \in C_c(G_1\backslash G)$, we obtain the following:

$$\begin{aligned}
&\sum_{i\in I}\|\pi_i^p(\varphi)\|_{C_q}^q \\
&\quad\le (2\pi)^{-\dim\mathfrak{g}_1}A_p^{q(\dim\mathfrak{n}/\mathfrak{g}_1)/2} \\
&\qquad\cdot\sum_{i\in I}\left(\int_{G_1\backslash G}\left(\int_{\Omega'}|\mathcal{F}_1\varphi_1(\lambda)\dot\varphi(\dot g)|^q\,d\lambda\right)^{p/q}\Delta(\dot g)\,d\dot g\right)^{q/p} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(by (2.11))} \\
&\quad= (2\pi)^{-\dim\mathfrak{g}_1}A_p^{q(\dim\mathfrak{n}/\mathfrak{g}_1)/2} \\
&\qquad\cdot\int_{\mathfrak{g}_1^*}|\mathcal{F}_1\varphi_1(\lambda)|^q\,d\lambda\left(\int_{G_1\backslash G}|\dot\varphi(\dot g)|^p\Delta(\dot g)\,d\dot g\right)^{q/p} \\
&\quad\le A_p^{q(\dim\mathfrak{g})/2}\left(\int_{\mathfrak{g}_1}|\varphi_1(X)|^p\,dX\int_{G_1\backslash G}|\varphi_0(\dot g)|^p\Delta(\dot g)\,d\dot g\right)^{q/p} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(by (0.4))} \\
&\quad= A_p^{q(\dim\mathfrak{g})/2}\|\varphi\|_p^q.
\end{aligned}$$

This proves (2.3), and the mapping $\varphi\to[\pi(\varphi)K_f^{1/q}]$ extends uniquely to the continuous mapping $\pi^p\colon L^p(G)\to C_q(\mathcal{H}_{\pi_i})$. If $p=2$, equality holds in (0.4), (2.10) and the Minkowski's inequality. Thus we get equality in (2.3) for $p=2$.

Using Remark 2.3.2, we can obtain the following

LEMMA 2.3.3. *$\pi^p(\varphi)^*=\pi^p(\varphi^{*(p)})$ for $\varphi\in L^p(G)$.*

We next prove that $\pi^p(L^p)$ is dense in $C_q(\mathcal{H}_\pi)$. Noting that $L^1(G)*L^p(G)\subset L^p(G)$, we get

$$\pi(\psi)\pi^p(\varphi)=\pi^p(\psi*\varphi)\quad\text{for } \psi\in L^1(G),\ \varphi\in L^p(G).$$

Let $T\in C_p(\mathcal{H})$ such that $\operatorname{tr}(\pi^p(\varphi)T)=0$ for all $\varphi\in L^p(G)$. Then

$$\operatorname{tr}(\pi(\psi)\pi^p(\varphi)T)=0\quad\text{for all } \psi\in L^1(G).$$

Since $\pi(L^1(G))\cap l\mathcal{C}(\mathcal{H})$ is dense in $l\mathcal{C}(\mathcal{H})$, whose dual is $C_1(\mathcal{H})$, it follows that $\pi^p(\varphi)T=0$ for all $\varphi\in L^p(G)$.

From Remark 2.3.2, it holds for $\varphi\in C_c(G)$ that

$$\pi^p(\varphi)=K_f^{1/q}\pi(\Delta^{-1/q}\varphi).$$

Thus $\pi(\varphi)T=0$ for all $\varphi\in C_c(G)$, that is, letting $\xi\in\mathcal{H}$, we have $0=\langle\pi(\varphi)T\xi,\eta\rangle=\int_G\langle\pi(g)T\xi,\eta\rangle\varphi(g)\,dg$, for all $\eta\in\mathcal{H}$ and $\varphi\in C_c(G)$. This implies that $\langle\pi(g)T\xi,\eta\rangle=0$ for all $g\in G$ and $\eta\in\mathcal{H}$, i.e., $T=0$. This proves that $\pi^p(L^p)$ is dense.

REMARK 2.3.4. Concerning the case $p = 2$, we have proved that $\mathscr{P}^2\colon L^2(G) \to \bigoplus_{i\in I} C_2(\mathscr{H}_{\pi_i})$ is a norm-preserving mapping of image dense. This concludes that $\mathscr{P}^2$ is a surjective isometry.

LEMMA 2.3.5. *Let $\kappa \in C_c(G)$ and $\varphi \in L^p(G)$. Then $\varphi * \kappa \in L^2(G)$ and*

$$\mathscr{P}^2(\varphi * \kappa) = \mathscr{P}^p(\varphi)\mathscr{P}^{1/(1/q+1/2)}(\Delta^{-1/q}\kappa). \tag{2.12}$$

Proof. Assume that $\varphi \in C_c(G)$. Then $\varphi * \kappa \in C_c(G)$, and

$$\begin{aligned}\pi_i^2(\varphi * \kappa) &= [\pi_i(\varphi * \kappa)K_i^{1/2}] = [\pi_i(\varphi)\pi_i(\kappa)K_i^{1/2}] \\ &= [\pi_i(\varphi)K_i^{1/q}K_i^{1/2-1/q}\pi_i(\Delta^{-1/2}\kappa)] \\ &= \pi_i^p(\varphi)\pi_i^{1/(1/2+1/q)}(\Delta^{-1/q}\kappa)\end{aligned}$$

for $i \in I$. Thus

$$\mathscr{P}^2(\varphi * \kappa) = \mathscr{P}^p(\varphi)\mathscr{P}^{1/(1/q+1/2)}(\Delta^{-1/q}\kappa),$$

and using our Hausdorff-Young theorem,

$$\begin{aligned}\|\mathscr{P}^2(\varphi * \kappa)\|_{C_2} &= \|\mathscr{P}^p(\varphi)\mathscr{P}^{1/(1/2+1/q)}(\Delta^{-1/q}\varphi)\|_{C_2} \\ &\le \|\mathscr{P}^p(\varphi)\|_{C_q}\|\mathscr{P}^{1/(1/2+1/q)}(\Delta^{-1/q}\kappa)\|_{C_{1/(1/2-1/q)}} \\ &\le C(\kappa, p)\|\varphi\|_p,\end{aligned}$$

where $C(\kappa, p)$ is a constant depending only on κ and p.

Noting that the equality $\|\mathscr{P}^2(\varphi * \kappa)\|_{C_2} = \|\varphi * \kappa\|_2$ holds, we get

$$\|\varphi * k\|_2 \le C(\kappa, p)\|\varphi\|_p,$$

which implies that the mapping $\varphi \to \varphi * \kappa \in L^2(G)$ can be extended to a continuous mapping of $L^p(G)$ into $L^2(G)$. This verifies the assertion of the lemma for all $\varphi \in L^p(G)$. □

Now we prove that $\mathscr{P}^p$ is injective. Suppose that $\mathscr{P}^p(\varphi) = 0$ ($\varphi \in L^p(G)$). Then using (2.12), it holds for every $\kappa \in C_c(G)$ that $\mathscr{P}^2(\varphi * \kappa) = 0$, which implies that $\varphi * \kappa = 0$ since $\mathscr{P}^2$ is injective. Thus $\varphi = 0$. □

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