DISTRIBUTIVE LATTICES WITH A THIRD OPERATION DEFINED

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1. Introduction. If L is the direct union of two distributive lattices, one may define a new operation $*$ between any two elements (a,b) and (c,d) of L by

$$(a,b) * (c,d) = (a \cap c,\ b \cup d). \tag{1}$$

This operation $*$ is:

$P1$. Idempotent
$P2$. Commutative
$P3$. Associative
$P4$. Distributive with $*$, $\cup$, $\cap$ in all possible ways.

The main results of this paper are the following.

First (Theorem 16), this is essentially the only way in which an operation with properties $P1$-$P4$ can arise in a distributive lattice. That is, if L is a distributive lattice with a binary operation $*$ having properties $P1$-$P4$, then L is a sublattice of the direct union of two distributive lattices, and the operation $*$ is given by equation (1).

Second (Theorem 9), if

$P5$. L contains an identity element e for the operation $*$,

then L is the entire direct union. Here $P5$ is sufficient but not necessary; a necessary and sufficient condition is given in Theorem 17. In case $*$ is identical with $\cup$ or $\cap$, Theorems 9, 16, and 17 still hold, but give trivial decompositions.

Finally, Section 5 shows that the presence of an operation $*$ is equivalent to the existence of a partial ordering with certain properties, so that our theorems may be restated so as to apply to distributive lattices with an auxiliary partial ordering.

2. Preliminary considerations. Throughout the paper, L is a distributive lattice with an operation $*$ having at least properties $P1$-$P4$. By an isomorphism

Received February 10, 1950. Published with the approval of the Oregon State College Monographs Committee, *Research Paper No. 152*, Department of Mathematics, School of Science.

Pacific J. Math. 1 (1951), 33-41.

between two such algebraic structures, we shall mean a one-to-one correspondence which preserves the operations $\cup$, $\cap$, $*$; as is customary, in the direct union $A \times B$ of two such algebraic structures, all operations act coordinatewise; for example, $(a,b) * (c,d) = (a * c,\ b * d)$.

For later reference, we collect here several simple consequences of $P1$-$P4$. The proofs consist of repeated applications of the idempotent and other laws, and will be presented briefly and without annotation of the separate steps. These results will be used frequently in later proofs without any explicit reference being made. In these theorems, small Latin letters represent arbitrary elements of L.

THEOREM 1. $x \cap y \leq x * y \leq x \cup y$.

Proof. We have

$$(x \cap y) \cup (x * y) = [(x \cap y) \cup x] * [(x \cap y) \cup y] = x * y;$$

thus $x \cap y \leq x * y$. Similarly for the other inequality.

THEOREM 2. *If* $x_1 \leq x_2$ *and* $y_1 \leq y_2$, *then* $x_1 * y_1 \leq x_2 * y_2$.

Proof. We have

$$(x_1 * y_1) \cap (x_1 * y_2) = x_1 * (y_1 \cap y_2) = x_1 * y_1;$$

thus $x_1 * y_1 \leq x_1 * y_2$. Similarly, $x_1 * y_2 \leq x_2 * y_2$, and the theorem follows.

THEOREM 3. $x * (x \cup y) = x \cup (x * y)$ *and* $x * (x \cap y) = x \cap (x * y)$.

Proof. Clearly,

$$x * (x \cup y) = (x * x) \cup (x * y) = x \cup (x * y).$$

Similarly for the other equation.

THEOREM 4. $x * y = (x \cap y) * (x \cup y)$.

Proof. The result follows from the continued equation,

$$\begin{aligned}(x \cap y) * (x \cup y) &= [(x \cap y) * x] \cup [(x \cap y) * y] \\ &= [(x * y) \cap x] \cup [(x * y) \cap y] \\ &= (x * y) \cap (x \cup y) = x * y.\end{aligned}$$

THEOREM 5. $x * (x * y) = x * y$.

Proof. Clearly,

$$x * (x * y) = (x * x) * y = x * y.$$

THEOREM 6. *If* $x \leq u \leq x * y \leq v \leq y$, *then* $u * v = x * y$.

Proof. Since $x \leq u$ and $x * y \leq v$, Theorem 2 shows that

$$x * y = x * (x * y) \leq u * v.$$

Similarly,

$$u * v \leq (x * y) * y = x * y.$$

3. **The operation $*$ has properties $P1$-$P5$.** In this section, we prove one of the main results of the paper (Theorem 9), using the assumption that L contains an element e which is an identity element for the operation $*$.

THEOREM 7. *If* $a \leq e$ *and* $c \leq e$, *then* $a * c = a \cap c$.

Proof. Since $a * c = (a \cap c) * (a \cup c)$, it is sufficient to consider the case $a \leq c \leq e$ and prove $a * c = a$. But then $a \leq a \leq a * e \leq c \leq e$, and Theorem 6 shows that $a * c = a * e = a$.

THEOREM 8. *If* $b \geq e$ *and* $d \geq e$, *then* $b * d = b \cup d$.

The proof is similar to that of Theorem 7.

THEOREM 9. *If L is a distributive lattice with a binary operation $*$ having properties $P1$-$P5$, then L is isomorphic to the direct union of two distributive lattices A, B each with an operation $*$ having properties $P1$-$P5$; and if (a,b), (c,d) are any two elements of $A \times B$, then $(a,b) * (c,d) = (a \cap c,\ b \cup d)$.*

Proof. Set

$$A = \{a \mid a \leq e\}, \quad B = \{b \mid b \geq e\};$$

then, with the same operations as in L, A and B are distributive lattices each with an operation $*$ having properties $P1$-$P5$.

We prove that the correspondence $(a,b) \to a * b$ is the required isomorphism from $A \times B$ onto L. It is clearly a single valued correspondence from $A \times B$ into L. It covers L because, for any element x of L, we have $x \cap e \in A$, $x \cup e \in B$ and, by Theorem 4, $(x \cap e) * (x \cup e) = x * e = x$. It is one-to-one because, for any $a \in A$, $b \in B$, we have $e \cap (a * b) = (e \cap a) * (e \cap b) = a * e = a$. Thus if $a * b = c * d$, $c \in A$, $d \in B$, then $a = c$. Similarly, $b = d$.

This correspondence preserves the three operations $\cup$, $\cap$, $*$. For instance,

$$(a,b) \cup (c,d) = (a \cup c,\ b \cup d) \to (a \cup c) * (b \cup d) = (a \cup c) * [(b \cup d) * (b \cup d)].$$

By Theorem 8, $b \cup d = b * d$; making this replacement in one parenthesis only, and rearranging the factors connected by $*$, we have $[(a \cup c) * b] * [(b \cup d) * d]$. But $b = b \cup c$, $d = a \cup d$, so that we have

$$[(a \cup c) * (b \cup c)] * [(b \cup d) * (a \cup d)]$$
$$= [(a * b) \cup c] * [(b * a) \cup d] = (a * b) \cup (c * d),$$

whence the operation $\cup$ is preserved by our correspondence. Similarly for $\cap$.

For the operation $*$,

$$(a,b) * (c,d) = (a * c,\ b * d) \to (a * c) * (b * d) = (a * b) * (c * d).$$

Thus our correspondence is an isomorphism.

By Theorems 7 and 8, $(a,b) * (c,d) = (a \cap c,\ b \cup d)$. This completes the proof.

REMARK. The element e will be the I in A and the O in B. The lattice A will have an O if and only if L has one; B will have an I if and only if L has one.

4. The operation $*$ has properties $P1$-$P4$. In this section, we prove one of the main results of the paper (Theorem 16). The method employed is to complete L in such a way that $*$ has properties $P1$-$P5$ and then to apply Theorem 9. Several preliminary definitions and theorems will be of use.

DEFINITION 1. We extend the operations $\cup$, $\cap$, $*$ to act on any subsets H, K of L by defining $H \cup K = \{x \cup y \mid x \in H,\ y \in K\}$, and similarly for the other operations.

Notice that $H \cup K$, for example, is a subset of L, and is usually neither the supremum of the elements in the subsets H and K nor the point set union of H and K.

DEFINITION 2. A subset P of L is a *$*$-ideal* if $P * L \subset P$.

For any fixed $a \in L$, the set $a * L$ is a $*$-ideal; it is called the *principal $*$-ideal generated by a.*

THEOREM 10. *An element x of L is in the principal $*$-ideal A generated by a if and only if $a * x = x$.*

The sufficiency is evident. To prove necessity, we note that if $x \in A$, then $x = a * y$; by Theorem 5 it follows that $a * x = a * (a * y) = x$.

DEFINITION 3. A subset H of L is *intervally closed* if $x \in H$, $y \in H$, and $x \leq z \leq y$ imply $z \in H$. The *interval closure* of a set G is the smallest intervally closed subset containing G.

It is easily seen that the interval closure of any set is the collection of elements which lie between two elements of the set.

DEFINITION 4. A subset R of L is *special* if it is

(a) a $*$-ideal,
(b) a sublattice, and
(c) intervally closed.

THEOREM 11. *Each principal $*$-ideal is also a special subset.*

Proof. Let A be the principal $*$-ideal generated by a, and let x, y be any two elements of A. Then $a * (x \cup y) = (a * x) \cup (a * y) = x \cup y$, and $x \cup y \in A$, by Theorem 10. Similarly $x \cap y \in A$, and A is a sublattice of L.

If x, y are any two elements of A, and $x \leq z \leq y$, we must show that z is in A. Since A contains $a \cap x$ and $a \cup y$, there is no loss in generality in supposing that $x \leq a \leq y$. Set $a * (a \cup z) = u$. We prove first that $u = a \cup z$. Since $a * y = y$ and $a \leq u \leq y$, Theorem 6 shows that $u * y = y$, so that

$$(a \cup z) \cap (u * y) = (a \cup z) \cap y = a \cup z. \tag{2}$$

By Theorem 5, $u * (a \cup z) = u$. From the definition of u and Theorem 1, we have $u \leq a \cup z$, so that

$$[(a \cup z) \cap u] * [(a \cup z) \cap y] = u * (a \cup z) = u. \tag{3}$$

But, from the distributive law, the left-hand members of equations (2) and (3) are equal, and $u = a \cup z$.

We now proceed with the proof that $z \in A$. Set

$$v = z \cap (a * z) = z * (a \cap z).$$

Then, by Theorem 1, $a \cap z \leq v \leq z$, and

$$a \cup v = a \cup [z * (a \cap z)] = (a \cup z) * [a \cup (a \cap z)] = (a \cup z) * a = u = a \cup z.$$

But now

$$v = (a \cap z) \cup v = (a \cup v) \cap (z \cup v) = (a \cup z) \cap (z \cup v) = (a \cup z) \cap z = z.$$

That is, $z \cap (a * z) = z$, whence $z \leq a * z$. Similarly, $z \geq a * z$. Thus $z = a * z$ and, by Theorem 10, $z \in A$.

THEOREM 12. *If P is any $*$-ideal, the interval closure of the sublattice generated by P is a special subset of L.*

Proof. Let Q be the sublattice generated by P, and let $\bar{Q}$ be the interval closure of Q. Then evidently $\bar{Q}$ is intervally closed.

$\bar{Q}$ is a sublattice because if $x, y \in \bar{Q}$, there exist elements u_1, u_2, v_1, v_2 of Q such that $u_1 \leq x \leq v_1$, $u_2 \leq y \leq v_2$. Then $u_1 \cup u_2 \leq x \cup y \leq v_1 \cup v_2$ and, since Q

is a lattice, $x \cup y \in \bar{Q}$. Similarly $x \cap y \in \bar{Q}$, so that $\bar{Q}$ is a sublattice of L.

$\bar{Q}$ is a $*$-ideal. Since $*$ distributes over $\cup$, $\cap$, Q is a $*$-ideal, and Theorem 2 then shows that $\bar{Q}$ is a $*$-ideal. This completes the proof.

REMARK. It is evident that any special subset containing P must contain $\bar{Q}$. Thus Theorem 12 gives a construction for the smallest special subset containing (generated by) a given $*$-ideal.

THEOREM 13. *If R, S are special subsets of L, then $R * S$, $R \cup S$, $R \cap S$ are special subsets of L.*

Proof. That each of the sets $R * S$, $R \cup S$, $R \cap S$ is a $*$-ideal is a simple consequence of the distributive laws and Definition 1.

To see that $R * S$ is a special subset, note that $R * S$ is contained in both R and S, since both are $*$-ideals; but clearly $R * S$ contains the point-set intersection of R and S since $*$ is idempotent. Thus $R * S$ is this intersection, which is easily seen to be a special subset of L.

$R \cup S$ is intervally closed because, if $r_1 \cup s_1 \leq x \leq r_2 \cup s_2$, then

$$r_1 \cap r_2 \leq x \cap r_2 \leq r_2$$

and, since R is a special subset, $x \cap r_2 \in R$. Similarly, $x \cap s_2 \in S$. But then

$$(x \cap r_2) \cup (x \cap s_2) = x \cap (r_2 \cup s_2) = x$$

lies in $R \cup S$, and $R \cup S$ is intervally closed.

$R \cup S$ is a sublattice of L because, if $r_1 \cup s_1$, $r_2 \cup s_2$ are any two elements of $R \cup S$, clearly $(r_1 \cup s_1) \cup (r_2 \cup s_2) = (r_1 \cup r_2) \cup (s_1 \cup s_2)$ lies in $R \cup S$. Also, since $r_1 \cap r_2 \leq (r_1 \cup s_1) \cap (r_2 \cup s_2)$ and $s_1 \cap s_2 \leq (r_1 \cup s_1) \cap (r_2 \cup s_2)$, we have

$$(r_1 \cap r_2) \cup (s_1 \cap s_2) \leq (r_1 \cup s_1) \cap (r_2 \cup s_2) \leq (r_1 \cup r_2) \cup (s_1 \cup s_2).$$

But the two extreme elements of this sequence of inequalities lie in $R \cup S$; thus, since $R \cup S$ is intervally closed, the center element also lies in $R \cup S$. This completes the proof that $R \cup S$ is a special subset; dually, $R \cap S$ is a special subset.

DEFINITION 5. $\mathfrak{L} = \{R, S, T, \cdots\}$ is the collection of all special subsets of L with the three operations $\cup$, $\cap$, $*$.

THEOREM 14. *The set $\mathfrak{L}$ with the operations $\cup$, $\cap$ is a distributive lattice and $*$ has properties $P1$-$P5$.*

Proof. Theorem 13 shows that $\mathfrak{L}$ is closed under the operations $\cup$, $\cap$, $*$. To show that $\mathfrak{L}$ is a distributive lattice, we prove

1. $R \cup R = R,\ R \cap R = R,$
2. $R \cup S = S \cup R,\ R \cap S = S \cap R,$
3. $(R \cup S) \cup T = R \cup (S \cup T),\ (R \cap S) \cap T = R \cap (S \cap T),$
4. $R \cup (R \cap S) = R,$
5. $R \cup (S \cap T) = (R \cup S) \cap (R \cup T).$

Numbers 1, 2, and 3 are evident. To prove 4, we note that clearly

$$R \cup (R \cap S) \supset R;$$

we show that $R \cup (R \cap S) \subset R$. If $x = r_1 \cup (r_2 \cap s)$ is any element of $R \cup (R \cap S)$, then $r_1 \leq x \leq r_1 \cup r_2$, and $x \in R$.

To prove 5, we note that clearly $(R \cup S) \cap (R \cup T) \supset R \cup (S \cap T)$; we show that $(R \cup S) \cap (R \cup T) \subset R \cup (S \cap T)$. If $x = (r_1 \cup s) \cap (r_2 \cup t)$ is any element of $(R \cup S) \cap (R \cup T)$, then $(r_1 \cap r_2) \cup (s \cap t) \leq x \leq (r_1 \cup r_2) \cup (s \cap t)$, and

$$x \in R \cup (S \cap T).$$

The proofs that the operation $*$ has properties $P1$-$P4$ are similar to those just given and will be omitted. For $P5$, the lattice L itself is a special subset of L and acts as the identity element for the operation $*$ in $\mathfrak{L}$.

THEOREM 15. *The correspondence $x \to$ the principal $*$-ideal generated by x is an isomorphism of L onto a sublattice of $\mathfrak{L}$ which identifies the operations $*$ in L and the sublattice of $\mathfrak{L}$.*

Proof. By Theorem 10, if x, y generate the same principal $*$-ideal, then $y = x * y = y * x = x$, so that the above correspondence is one-to-one.

To prove that this correspondence is an isomorphism, let

$$x \to X = x * L,\quad y \to Y = y * L;\quad \text{then } x \cup y \to (x \cup y) * L = Z.$$

Clearly $Z \subset X \cup Y$. Conversely, if $w = (x * u) \cup (y * v)$ is any element of $X \cup Y$, then $(x \cup y) * (u \cap v) \leq w \leq (x \cup y) * (u \cup v)$, and $w \in Z$. The proofs for $\cap$, $*$ are similar, and will be omitted.

Theorems 9, 14, and 15 give immediately our main result:

THEOREM 16. *If L is any distributive lattice with an operation $*$ having properties $P1$-$P4$, then L is isomorphic to a sublattice of the direct union of two distributive lattices A, B, each with an operation $*$ having properties $P1$-$P5$; and if (a,b), (c,d) are any two elements of $A \times B$, then*

$$(a,b) * (c,d) = (a \cap c,\ b \cup d).$$

THEOREM 17. *If L is any distributive lattice with an operation $*$ having properties $P1$-$P4$, then L is isomorphic to a direct union in which the operation $*$*

is given by equation (1) *if and only if each pair of elements of L is contained in some principal $*$-ideal.*

Necessity. If L is a direct union with $*$ given by equation (1), the two arbitrary elements (a,b), (c,d) are contained in the principal $*$-ideal generated by $(a \cup c, b \cap d)$.

Sufficiency. By Theorem 16, L may be considered as a sublattice of a direct union in which $*$ is given by equation (1). Let (x_1, y_1) be any fixed element of L, and define $A = \{x \mid (x,y_1) \in L\}$, $B = \{y \mid (x_1,y) \in L\}$; then $A \times B \subset L$. In fact, if (x,y_1) and (x_1,y) are in the principal $*$-ideal generated by (a,b), then

$$a \geq x \cup x_1, \quad b \leq y \cap y_1,$$

and L contains

$$[(x,y_1) \cap (a,b)] * [(x_1,y) \cup (a,b)] = (x,b) * (a,y) = (x,y).$$

Conversely, $L \subset A \times B$, for if (x,y) is any element of L, and (a,b) generates a principal $*$-ideal containing (x,y) and (x_1,y_1), then L contains

$$\begin{aligned}&[(x,y) \cap (a,b)] \cup \{(x_1,y_1) \cap [(x,y) * (x_1,y_1)]\} \\ &\qquad = (x,b) \cup \{(x_1,y_1) \cap (x \cap x_1, y \cup y_1)\} = (x,y_1).\end{aligned}$$

Similarly, (x_1,y) is in L, and $(x,y) \in A \times B$.

CAUTION. The decomposition of L will be trivial (one of A, B consisting of a single element) if and only if $*$ is identical with $\cup$ or $\cap$.

5. The ordering equivalent to $*$. In any distributive lattice L with an operation $*$ having properties $P1$-$P4$, we may define an auxiliary order relation by making $x \succ y$ mean $x * y = y$. It is easily seen that this order relation has the following properties:

$O1$. $x \succ x$;
$O2$. $x \succ y$, $y \succ x$ imply $x = y$;
$O3$. $x \succ y$, $y \succ z$ imply $x \succ z$;
$O4$. Any two elements x, y of L have a greatest lower bound (namely $x * y$);
$O5$. The operation of taking the greatest lower bound is distributive with itself and with the two lattice operations in all possible ways.

Conversely, if L is any distributive lattice (with no additional operation $*$ defined) with an auxiliary order relation having properties $O1$-$O5$, then the operation $*$ defined in L by setting $x * y$ equal to the greatest lower bound of x and y has properties $P1$-$P4$. Moreover, the operation $*$ will have property $P5$ if and only if the order relation satisfies:

$O6$. There is a greatest element e in L.

Our results may thus be restated as theorems concerning distributive lattices with an auxiliary order relation.

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