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ESSENTIAL NONLINEARITY IN FIELD THEORY AND CONTINUUM MECHANICS. SECOND- AND FIRST-ORDER GENERALLY-COVARIANT MODELS

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Abstract. Discussed is the problem of the mutual relationship of differentially first-order and second-order field theories and quantum-mechanical concepts. We show that unlike the real history of physics, the theories with algebraically second-order Lagrangians are primary, and in any case more adequate. It is shown that in principle, the primary Schrödinger idea about Lagrangians which are quadratic in derivatives, and leading to second-order differential equations, is not only acceptable, but just it opens some new perspective in field theory. This has to do with using the Lorentz-conformal or rather its universal covering $\mathrm{SU}(2,2)$ as a gauge group. This has also some influence on the theory of defects in continua.

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1. Summary

It is well-known that fundamental laws of motion of discrete or continuous systems of material points are given by the second-order differential equations, in any case second-order in time. The equations of electrodynamics, when formulated in a proper four-potential way are also second-order. Therefore, the whole tradition of classical physics created the paradigm of second-order differential equations as a fundamental tool of theoretical physics. When formulating the ideas of wave mechanics in the sense of de Broglie and others, Schrödinger followed this paradigm and formulated what is today known as Klein-Gordon equation, with the potential term of course. The results, although in a sense qualitatively reasonable, were not encouraging, in any case they were worse than the traditional Bohr-Sommerfeld spectrum. Then Schrödinger resigned and, basing on the analogy suggested by the Hamilton-Jacobi equation, formulated what is today known as the Schrödinger equation. The results were beautiful. But the spin phenomena were discovered and to describe them in a relativistic way, Dirac suggested the wave equation which is known today under his name. But this is a first-order differential equation in space-time variables, however imposed onto the four-component wave function. In any case, this was a revolutionary step of introducing first-order differential equation as a fundamental law of physics. This was reinforced by the whole school of more or less sophisticated use of Clifford-algebraic paradigm. We are going to show that this triumph might have been relatively premature.

2. Heuristics of the $\mathrm{U}(1)$ Gauge Group

Let us begin with the academic, or rather scholar discussion. We deal with the complex scalar field $\Psi\colon M \to \mathbb{C}$ over the space-time manifold M, subjected to the natural multiplicative action of the group $\mathrm{U}(1)$. If M is endowed with the metric tensor g, then the globally $\mathrm{U}(1)$-invariant Lagrangian of Ψ without external interactions or self-interaction is given by

$$L_{\mathrm{m}} = \left(\frac{1}{2}\, g^{\mu\nu}\, \overline{\partial_\mu \Psi}\, \partial_\nu \Psi - \frac{c}{2}\, \overline{\Psi}\Psi\right) \sqrt{|g|}. \tag{1}$$

Now we assume the theory to be locally $\mathrm{U}(1)$-invariant, i.e., we admit the multiplication factor to be x^μ (space-time point)-dependent. To achieve this, we must introduce the gauge covector E_μ and replace the derivative ∂_μ by the covariant one D_μ

$$D_\mu \Psi = \partial_\mu \Psi - \mathrm{i} q E_\mu \Psi. \tag{2}$$

Here q denotes the coupling constant, the "charge" so to the speak. The Lagrangian (1) is replaced by its local counterpart, namely

$$L_{\mathrm{m}}\left[\Psi, E\right] = \frac{1}{2}\, g^{\mu\nu}\, \overline{D_\mu \Psi}\, D_\nu \Psi \sqrt{|g|} - \frac{c}{2}\, \overline{\Psi}\Psi \sqrt{|g|}. \tag{3}$$

And, of course, to speak we must admit the Lagrangian for the field E

$$L_{\mathrm{E}}\left[E\right] = -\frac{1}{4}\, g^{\mu\varkappa} g^{\nu\lambda} F_{\mu\nu} F_{\varkappa\lambda} \sqrt{|g|} \tag{4}$$

where $F = \mathrm{d}E$, i.e., coordinates-wise

$$F_{\mu\nu} = \partial_\mu E_\nu - \partial_\nu E_\mu. \tag{5}$$

After performing the usual variational procedure on the action built in terms of the total Lagrangian L

$$L = L_{\mathrm{m}}\left[\Psi, E\right] + L_{\mathrm{E}}\left[E\right] \tag{6}$$

then we obtain the following field equations

$$q\mathrm{i} E^\mu \partial_\mu \Psi - \left(\frac{c}{2} - \frac{q^2}{2} E^\mu E_\mu - \frac{\mathrm{i}q}{2} E^\mu{}_{;\mu} \right) \Psi - \frac{1}{2} g^{\mu\nu} \Psi_{;\mu\nu} = 0 \tag{7}$$

$$\partial_\nu F^{\mu\nu} = \frac{q\mathrm{i}}{2} \left(\overline{\Psi} \partial^\mu \Psi - \left(\partial_\mu \overline{\Psi} \right) \Psi \right) + q^2 E^\mu \overline{\Psi} \Psi. \tag{8}$$

Obviously, the tensor indices in (7) (8) are moved in the sense of the metric tensor $g_{\mu\nu}$ and the covariant derivatives in (7) are meant in the sense of the Levi-Civita connection built of g. This is the system of second-order partial differential equations imposed onto (Ψ, E_μ).

Let us observe that in the asymptotic situation of slowly-varying fields, when derivative terms are small in comparison with algebraic ones, equations (7) and (8) approximately reduce to

$$\mathrm{i} E^\mu \partial_\mu \Psi - \left(\frac{c}{2q} - \frac{q}{2} E^\mu E_\mu - \frac{\mathrm{i}}{2} E^\mu{}_{;\mu} \right) \Psi = 0 \tag{9}$$

$$\partial_\nu F^{\mu\nu} = q^2 E^\mu \overline{\Psi} \Psi. \tag{10}$$

But those are first-order differential equations which may be derived from the auxiliary "Lagrangian"

$$\begin{aligned} L'_{\mathrm{m}} \;&=\; q\, g^{\mu\nu} E_\mu\, \frac{\mathrm{i}}{2} \left(\overline{\Psi} \partial_\nu \Psi - \left(\partial_\nu \overline{\Psi} \right) \Psi \right) \sqrt{|g|} \\ &\quad - \left(\frac{c}{2} - \frac{q^2}{2}\, g^{\mu\nu} E_\mu E_\nu \right) \overline{\Psi} \Psi \sqrt{|g|}. \end{aligned} \tag{11}$$

Obviously, it is easy to see that $L_{\mathrm{m}}[\Psi, E]$ is related to $L'_{\mathrm{m}}[\Psi, E]$ as follows

$$L_{\mathrm{m}}[\Psi, E] = L'_{\mathrm{m}}[\Psi, E] + \frac{1}{2} g^{\mu\nu} \overline{\partial_\mu \Psi}\, \partial_\nu \Psi. \tag{12}$$

It is easy to see that if one omit the Clifford-ideology, the simplified equations (9) and (10) following from the Lagrangian (11) for slowly-varying fields are structurally similar to the Dirac-Maxwell system of equations. The current in (10) is algebraically built of Ψ, and E^μ in (9) is an analogue of the tetrad field.

3. $\mathrm{SU}(2,2)$ as a Gauge Group

Of course, the system (9), (10) following from (11) is rather non-physical. But structurally, it resembles the system for the bispinor-tetrad object. This suggests us the idea that perhaps the first-order Dirac equation is also an approximation to some second-order system of equations. And this again raises the question: what is the proper order of the fundamental field equations: first-order or second-order? This is in a sense very important and natural question. Remind that in the first years of quantum mechanics the Klein-Gordon equation invented by Schrödinger was disqualified, but later on, within the framework of quantum field theory it was in a sense rehabilitated as one describing quantum phenomena. Let us begin with a short repetition of Dirac theory on a manifold. For simplicity we do not use the sophisticated language of fibre bundles, but rather, almost exclusively, the analytical representation.

Degrees of freedom are described by the triple of objects

1. The $\mathbb{C}^4$-valued bispinors field $\Psi\colon M \to \mathbb{C}^4$.
2. The tetrad, equivalently cotetrad field, $e^\mu{}_A$ or $e^A{}_\mu$, where $e^A{}_\mu e^\mu{}_B = \delta^A{}_B$, i.e., the objects are essentially identical via duality.
3. The $\mathrm{SL}(2,\mathbb{C})$-ruled spinor connection given by the system of differential one-forms $\omega^r{}_{s\mu}$.

Let us stress: the Greek indices μ run over the range $0, 1, 2, 3$, the capital Latin ones - as well, but they are referred to the $\mathbb{R}^4$ space, not to the tangent spaces $T_x M$, and the small Latin indices like r, s refer to the bispinor space, analytically $\mathbb{C}^4$.

The $\mathbb{R}^4$-space is endowed with the Lorentz metric η

$$[\eta_{AB}] = \mathrm{diag}(1, -1, -1, -1). \tag{13}$$

Similarly, in $\mathbb{C}^4$ we assume the geometry based on the sesquilinear Hermitian product G of signature $(++--)$, e.g., $\mathrm{diag}(1, 1, -1, -1)$. In $\mathbb{C}^4$, we introduce a family of Dirac matrices, to be more precise, of Dirac linear operators. It is spanned by a system of matrices γ^A satisfying the Clifford anticommutation relations

$$\left\{\gamma^A, \gamma^B\right\} = \gamma^A \gamma^B + \gamma^B \gamma^A = 2\eta^{AB} I_4. \tag{14}$$

They are assumed to be Hermitian with respect to the bispinor scalar product G in the sense that

$$\Gamma^{A}{}_{\bar{r}s} = \overline{\Gamma^{A}{}_{\bar{s}r}}, \qquad \Gamma^{A}{}_{\bar{r}s} = G_{\bar{r}w}\gamma^{Aw}{}_{s}. \tag{15}$$

And finally, on the basis of those objects we introduce some spatio-temporal objects, namely the metric tensor $g_{\mu\nu}$ and the Einstein-Cartan affine connection $\Gamma^{\alpha}{}_{\beta\mu}$, or, equivalently, the spinor connection $\omega^{r}{}_{s\mu}$. They are defined and interrelated as follows

$$g_{\mu\nu} = \eta_{AB}\, e^{A}{}_{\mu} e^{B}{}_{\nu} \tag{16}$$

$$\Gamma^{\alpha}{}_{\beta\mu} = e^{\alpha}{}_{A}\Gamma^{A}{}_{B\mu}e^{B}{}_{\beta} + e^{\alpha}{}_{A}e^{A}{}_{\beta,\mu} \tag{17}$$

$$\Gamma^{A}{}_{B\mu} = \frac{1}{2}\,\gamma^{Ar}{}_{s}\,\omega^{s}{}_{w\mu}\,\gamma_{B}{}^{w}{}_{r} = \frac{1}{2}\,Tr\left(\gamma^{A}\,\omega_{\mu}\,\gamma_{B}\right) \tag{18}$$

$$\omega_{\mu} = \frac{1}{2}\,\Gamma_{AB\mu}\,\Sigma^{AB} = \frac{1}{2}\,\eta_{AC}\,\Gamma^{C}{}_{B\mu}\,\Sigma^{AB}. \tag{19}$$

$$\Sigma^{AB} = \frac{1}{4}\,\left(\gamma^{A}\gamma^{B} - \gamma^{B}\gamma^{A}\right) = \frac{1}{4}\left[\gamma^{A},\gamma^{B}\right]. \tag{20}$$

The capital indices are moved with the help of Minkovski tensor η_{AB} in $\mathbb{R}^4$.

We have said above that Γ is an Einstein-Cartan connection. Therefore, the following holds

$$\nabla_{\mu}\, g_{\alpha\beta} = 0, \qquad \eta_{AC}\Gamma^{C}{}_{B\mu} + \eta_{BC}\Gamma^{C}{}_{B\mu} = 0. \tag{21}$$

If the Levi-Civita connection built of g is denoted by $\left\{ \begin{array}{c} \alpha \\ \beta\mu \end{array} \right\}$, then it is clear that

$$\Gamma^{\alpha}{}_{\beta\mu} = \left\{ \begin{array}{c} \alpha \\ \beta\mu \end{array} \right\} + S^{\alpha}{}_{\beta\mu} + S_{\beta\mu}{}^{\alpha} - S_{\mu}{}^{\alpha}{}_{\beta}. \tag{22}$$

Let us introduce the Dirac-conjugate bispinor field $\widetilde{\Psi}\colon M \to \mathbb{C}^{4\star}$ given by

$$\widetilde{\Psi}_{r} = \overline{\Psi}^{\bar{r}} G_{\bar{s}r}. \tag{23}$$

Therefore, at any $x \in M$, $\widetilde{\Psi}(x)$ is a covector in the sense of the internal space $\mathbb{C}^4$.

An important remark: in majority of textbooks one identifies G with γ^0. It is a mistake! The matrix γ^0 represents a linear operator in the internal bispinor space $\mathbb{C}^4$, whereas the matrix G represents a sesquilinear Hermitian form. The mapping $\Psi \to \widetilde{\Psi}$ is antilinear. The numerical identification of γ^0 and G is simply the peculiarity of the commonly used representations.

The fact that G and its signature are fixed, focuses our attention on the group of pseudounitary transformations $\mathrm{U}(4,G) \simeq \mathrm{U}(2,2)$. Its semisimple subgroup $\mathrm{SU}(4,G) \simeq \mathrm{SU}(2,2)$ is the covering group of $\mathrm{CO}(1,3)$, the Lorentz-conformal group in $\mathbb{R}^4$.

The Lagrangian of the bispinor field is given by

$$L_m = \frac{\mathrm{i}}{2}\, e^{\mu}{}_{A}\gamma^{Ar}{}_{s}\left(\widetilde{\Psi}_r D_\mu \Psi^s - \left(D_\mu \widetilde{\Psi}_r\right)\Psi^s\right)\sqrt{|g|} - m\widetilde{\Psi}_r \Psi^r \sqrt{|g|} \tag{24}$$

where, obviously, D_μ is the symbol of the spinorial covariant derivative

$$D_\mu \Psi^r = \partial_\mu \Psi^r + \omega^r{}_{s\mu}\Psi^s. \tag{25}$$

And similarly for all other tensor objects in $\mathbb{C}^4$. Let us stress that the standard Dirac theory for spinor fields in a Riemannian manifold is based on the invariance under the group $\mathrm{SL}(2,\mathbb{C})$, the universal covering group of $\mathrm{SO}^{\uparrow}(1,3)$. Let the covering projection of $\mathrm{SL}(2,\mathbb{C})$ onto $\mathrm{SO}^{\uparrow}(1,3)$ be denoted by $P\colon \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SO}^{\uparrow}(1,3)$. The group $\mathrm{SL}(2,\mathbb{C})$ acts in a natural way on the capital indices in (24). Bispinors are then affected by the group injection $U\colon \mathrm{SL}(2,\mathbb{C}) \to \mathrm{U}(4,G) \simeq \mathrm{U}(2,2)$. It is chosen in such a way that

$$U(A)\gamma_K U(A)^{-1} = \gamma_L\; P(A)^L{}_K. \tag{26}$$

The resulting homomorphisms of Lie algebras will be denoted by $\mathfrak{u}\colon \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{u}(4,G) \simeq \mathfrak{u}(2,2)$. Then (26) is represented by

$$[u(a),\gamma_K] = \gamma_L p(a)^L{}_K. \tag{27}$$

One can show that

$$P(A)^K{}_L = \frac{1}{4}\;\mathrm{Tr}\left(\gamma^K U(A)\gamma_L U(A)^{-1}\right) \tag{28}$$

$$p(a)^K{}_L = \frac{1}{2}\;\mathrm{Tr}\left(\gamma^K u(a)\gamma_L\right) \tag{29}$$

$$u(a) = \frac{1}{2}\, p(a)^A{}_B \Sigma_A{}^B. \tag{30}$$

One can show that in the sense of this action of $\mathrm{SL}(2,\mathbb{C})$, the Lagrangian (24) is invariant. It is also real-valued, therefore, it is really good Lagrange function.

Obviously, the connection forms transform under this action in an inhomogeneous way, namely, the tetrad transformation by $A \in \mathrm{SL}(2,\mathbb{C})$

$$[e^{\mu}{}_A] \mapsto \left[e^{\mu}{}_B A^B{}_A\right] \tag{31}$$

resulting in

$$[\omega^r{}_{s\mu}] \mapsto \left[U(A)^r{}_z\,\omega^z{}_{w\mu}\,U(A)^{-1w}{}_s - \frac{\partial U(A)^r{}_z}{\partial x^\mu}\,U(A)^{-1z}{}_s\right] \tag{32}$$

$$\left[\Gamma^A{}_{B\mu}\right] \mapsto \left[P(A)^A{}_C\,\Gamma^C{}_{D\mu}\,P(A)^{-1D}{}_B - \frac{\partial P(A)^A{}_C}{\partial x^\mu}\,P(A)^{-1C}{}_B\right]. \tag{33}$$

The particular structure of $U(A)$, $u(a)$ etc. depends on the used representation. For example, if we use the Weyl-Van der Warden representation

$$[G_{\bar{r}s}] = \begin{bmatrix} 0 & I_2 \\ I_2 & 0 \end{bmatrix} \tag{34}$$

then using the usual matrix symbols we have

$$U(A) = \begin{bmatrix} A & 0 \\ 0 & A^{-1+} \end{bmatrix}, \qquad u(a) = \begin{bmatrix} a & 0 \\ 0 & -a^{+} \end{bmatrix}. \tag{35}$$

And when the Dirac representation is used and

$$[G_{\bar{r}s}] = \begin{bmatrix} I_2 & 0 \\ 0 & -I_2 \end{bmatrix} \tag{36}$$

then in the analytic matrix form we have

$$U(A) = \frac{1}{2}\begin{bmatrix} A + A^{-1+} & A - A^{-1+} \\ A - A^{-1+} & A + A^{-1+} \end{bmatrix}, \qquad u(a) = \begin{bmatrix} a - a^{+} & a + a^{+} \\ a + a^{+} & a - a^{+} \end{bmatrix}. \tag{37}$$

Let us mention that (34) and (35) is more convincing if one begins the analysis from the two-component Weyl spinors and the Ur-philosophy of Weizsa̋cker. And (36), (37) is more natural from the point of view of the theory of Pauli description.

The Lagrangian (24) is quite similar to the scalar field Lagrangian L'_{m} (11). This just motivates the question whether L_{m} (24) is not just a slowly-valued approximation to some second-order Lagrangian structurally similar to (1) (3). The more so, just like L'_{m} it is a contraction of the quantity which looks as a bosonic current given by the tensorial expression

$$J_r{}^s{}_\mu := \left(\widetilde{\Psi}_r D_\mu \Psi^s - \left(D_\mu \widetilde{\Psi}\right)_r \Psi^s\right)\sqrt{|g|} \tag{38}$$

a contraction which seems to play a role similar to the gauge field E^μ in (3), (7) and (8). Everything, including the tensor structure of $J_r{}^s{}_\mu$ seems to suggest that (38) is the $\mathrm{SU}(2,2)$-ruled bosonic current.

The co-tetrad $e^A{}_\mu$ is interpreted as a kind of gauge field in our $\mathrm{SL}(2,\mathbb{C})$-based theory. But in all real gauge theories the field of frames does not occur explicitly in Lagrangian. Unlike this, in the tetrad model (24) the cotetrad/tetrad field $e^A{}_\mu/e^\mu{}_A$ occurs explicitly and plays on essential role. And all models of reinterpreting it as a kind of "external friction" are strange and get us far outside the usual framework of gauge theories. It seems natural to replace the Poincare gauge theory by the semisimple over-group $\mathrm{CO}(1,3)$ of conformal-Minkowskian mappings, or rather, for quantum purposes, by its universal covering $\mathrm{SU}(4,G) \simeq \mathrm{SU}(2,2)$. The point is only that just as $\mathrm{SL}(2,\mathbb{C})$ as the spinorial gauge group, this $\mathrm{SU}(2,2)$ is to be a group of purely internal symmetries, acting only on internal degrees of freedom, but not on the space-time variables x^μ. Though in a general manifold, there is no place for Lorentz- and Lorentz-conformal mappings as space-time transformations.

They act on parameters of internal degrees of freedom. On the wave fields they act simply as

$$(A\Psi)^r(x) = A^r{}_s\Psi^s(x) \tag{39}$$

but x itself being non-affected. It would be funny to expect the action like

$$\left(W(A)\Psi\right)^r(x) = A^r{}_s\Psi^s\left(p\left(A^{-1}(x)\right)\right) \tag{40}$$

with $p : \mathrm{SU}(4, G) \simeq \mathrm{SU}(2, 2) \to \mathrm{C}(1, 3)$ as a projection, because in a manifold there is nothing like $\mathrm{C}(1, 3)$. The only possibility is that the action (39) is accompanied with some action on the appropriate internal degrees of freedom. Just like in $\mathrm{SL}(2, \mathbb{C})$-ruled theory, it is accompanied with the action on tetrad variables which seem to be a counterpart of space-time points.

The above remarks concerned the material sector of the theory. But obviously, the dynamics of the gravitational sector is essential, because the total Lagrangian is given by

$$L\left[\Psi, e, \omega\right] = L_{\mathrm{m}}\left[\Psi, e, \omega\right] + L_{\mathrm{gr}}\left[\Psi, e, \omega\right] \tag{41}$$

and in the $\mathrm{SL}(2, \mathbb{C})$-ruled theory the choice of $L_{\mathrm{gr}}\left[e, \omega\right]$, the gravitational Lagrangian, is a challenge. Namely, there are a few a priori possible terms suggested by the $\mathrm{SL}(2, \mathbb{C})$-gauge philosophy

1. First of all, according to the general gauge paradigm, we expect the Yang-Mills Lagrangian quadratic in the curvature tensor built of Γ

$$\begin{aligned} L_{\mathrm{YM}}(e, \omega) &= \frac{1}{\ell} R(\Gamma)^{\varkappa}{}_{\mu\alpha\beta} R(\Gamma)^{\mu}{}_{\varkappa\gamma\delta}\, g^{\alpha\gamma} g^{\beta\delta} \sqrt{|g|} \\ &= \frac{1}{\ell} R^{A}{}_{B\alpha\beta} R^{B}{}_{A\gamma\delta}\, g^{\alpha\gamma} g^{\beta\delta} \sqrt{|g|} \end{aligned} \tag{42}$$

   This term is expected to describe the microscopic gravitation. Obviously, $R(\Gamma)$ is the curvature tensor of $g[e]$.

2. To give an account of the macroscopic and cosmic scales of gravitation, it is suggested to admit a kind of the Palatini-Einstein-Cartan expression

$$L_{\mathrm{PEC}}(e, \omega) = \frac{1}{k}\, g[e]^{\mu\nu} R(\Gamma)^{\varkappa}{}_{\mu\varkappa\nu} \sqrt{|g|}. \tag{43}$$

3. Three terms which are quadratic in torsion and are linear in Weitzenböck invariants quadratic in torsion

$$J_1(e, \omega) = a\, g_{\mu\alpha}\, g^{\nu\beta} g^{\varkappa\gamma} S^{\mu}{}_{\nu\varkappa}\, S^{\alpha}{}_{\beta\gamma} \sqrt{|g|} \tag{44}$$

$$J_2(e, \omega) = b\, g^{\mu\nu} S^{\varkappa}{}_{\alpha\mu}\, S^{\alpha}{}_{\varkappa\nu} \sqrt{|g|} \tag{45}$$

$$J_3(e, \omega) = c\, g^{\mu\nu} S^{\varkappa}{}_{\varkappa\mu}\, S^{\lambda}{}_{\lambda\nu} \sqrt{|g|}. \tag{46}$$

4. A possible cosmological term
$$L_{\mathrm{cosm}}(e,\omega) = \Lambda\sqrt{|g|} \tag{47}$$
where Λ is a constant.
5. Perhaps some more complicated terms, like, e.g., ones quadratic in the scalar curvature
$$L^{(2)}(e,\omega) = \frac{1}{k'}\, g^{\mu\nu} g^{\varkappa\lambda} R(\Gamma)^{\alpha}{}_{\mu\alpha\nu} R(\Gamma)^{\beta}{}_{\varkappa\beta\lambda}\sqrt{|g|}. \tag{48}$$

It is clear that all the above terms contribute the geometric Lagrangian L_{gr} with their own constant coefficients. This is a not quite advantageous consequence of the non semisimplicity of the Poincare group.

Let us now try to follow the above-suggested idea of the $\mathrm{SU}(4,G) \simeq \mathrm{SU}(2,2)$ gauge invariance, or more generally, of the $\mathrm{U}(4,G) \simeq \mathrm{U}(2,2)$ gauge invariance. By definition, elements of this group are linear mappings of $\mathbb{C}^4$ into itself, preserving the Hermitian sesquilinear form G
$$G_{\bar{r}s} = G_{\bar{z}t}\,\overline{U}^{\bar{z}}{}_{\bar{r}}\,U^{t}{}_{s} = G_{\bar{z}t}\,\overline{U^{z}{}_{r}}\,U^{t}{}_{s}. \tag{49}$$

The group $\mathrm{SU}(2,2)$ consists of such linear mapping U which in addition to (49) have also a determinant equal to 1. General elements of $\mathrm{U}(2,2)$ have determinant of the absolute value 1, so that
$$\det U = \exp(\mathrm{i}\varphi), \qquad \varphi \in \mathbb{R}. \tag{50}$$

It is clear that for any $A \in \mathrm{SL}(2,\mathbb{C})$, the mappings $U(A)$ in (26) are pseudounitary. However, they are special cases acting separately in two $\mathbb{C}^2$-subspaces of $\mathbb{C}^4$, as explicitly shown in (35). But even in the usual spinor theory this is insufficient when admitting the spatial mirror reflections. Then one must use two copies of $\mathbb{C}^2$ as linear subspaces of $\mathbb{C}^4$ and define mirror reflections as some transformations interchanging the two $\mathbb{C}^2$-copies among themselves. They belong to $\mathrm{SU}(2,2)$, but do not have the form $U(A)$ for some $A \in \mathrm{SL}(2,\mathbb{C})$. But if so, this is an additional argument for the $\mathrm{SU}(2,2)$ conformal symmetry.

The Lie algebra of $\mathrm{SU}(2,2)$ consists of linear mappings u of $\mathbb{C}^4$ into itself which are anti-Hermitian with respect to the G-scalar product, i.e., satisfy
$$G_{\bar{r}z}\,u^{z}{}_{s} + \overline{G_{\bar{s}z}\,u^{z}{}_{r}} = 0. \tag{51}$$

Therefore, there is a canonical isomorphism of $\mathfrak{su}(2,2)$ onto $\mathrm{iHerm}(4,G)$, where $\mathrm{Herm}(4,G)$ denotes the linear space of G-Hermitian linear mappings of $\mathbb{C}^4$ into itself.

It is clear what a nonsense is the idea of external space-time gauge transformations. Even in the case of the Lorentz group (or rather its covering $\mathrm{SL}(2,\mathbb{C})$) they are redefined as internal transformations of the matrix group (just numerical matrices, not space-time Lorentz mappings) acting on the tetrads, not separately on the tetrad legs. The formally mentioned problems with external translations were, rather unfortunately, tried to be solved by using affine tangent spaces instead of the linear spaces. But in the case of the conformal group the situation becomes completely hopeless because of the singularity of Minkowskian conformal boosts. Let us remind that the standard conformal boosts with respect to the null element of $\mathbb{R}^4$ are given by b_0

$$x'^{\mu} = x^{\mu} \|x\|^{-2}, \qquad \|x\|^2 = g_{\alpha\beta}\, x^{\alpha} x^{\beta}. \tag{52}$$

The total four-dimensional group of boosts is obtained from b_0 by moving the null element to all possible positions by the group of space-time translations

$$b_0 \mapsto t_v \circ b_0 \circ t_{-v} \tag{53}$$

v denoting the translation vector. This is quite meaningless, because the gauge transformations must be smooth in space-time manifold. But it is clear that (52) is badly singular on the light cone. The only possibility to avoid this and be compatible with the spinor concept is to use the covering group $\mathrm{SU}(2,2)$ from the very beginning as really internal gauge symmetry. It is to act on some tetrad-like internal object, separately at every space-time point and without any interference with the space-time transformations.

There are again two ways to achieve this, namely in terms of first-order or second-order matter equations. We feel emotionally attached to the second-order possibility, nevertheless, we quote here the both of them.

In both cases, the matter field is analytically represented by the wave field $\Psi : M \to \mathbb{C}^4$. The $\mathrm{U}(4,G) \simeq \mathrm{U}(2,2)$-gauge field is analytically given by the $\mathrm{SU}(4,G) \simeq \mathrm{SU}(2,2)$-valued differential one-form

$$M \ni x \mapsto \vartheta_x \in \mathrm{L}(T_x M, \mathfrak{u}(4,G)) \tag{54}$$

with values in the Lie algebra $\mathfrak{su}(4,G)$, or, for simplicity, as denoted above - with values in the total Lie algebra $\mathfrak{u}(4,G)$. Analytically, we use the symbols Ψ^r, $\vartheta^r{}_{s\mu}$ - components of those objects. Let us remind that the range of indices is $\mu = 0,1,2,3$, $r = 1,2,3,4$. We repeat once again that those objects are ruled by the pseudounitary group $\mathrm{U}(4,G) \simeq \mathrm{U}(2,2)$, no longer by $\mathrm{SL}(2,\mathbb{C})$. The group $\mathrm{SL}(2,\mathbb{C})$ injected into $\mathrm{U}(4,G) \simeq \mathrm{U}(2,2)$ reappears again as an injection (26) U of $\mathrm{SL}(2,\mathbb{C})$ into the pseudo-unitary group, when we consider the usual spinorial asymptotics. The local action of $\mathrm{U}(4,G) \simeq \mathrm{U}(2,2)$, local in the sense of x^{μ} - dependent action

$$U\colon M \to \mathrm{U}(4,G) \simeq \mathrm{U}(2,2)$$

on quantities Ψ and Θ is in both versions of the theory given by

$$(U\Psi)(x) = U(x)\Psi(x), \qquad (U\vartheta)_x \;\; = U(x)\vartheta_x U(x)^{-1} - \mathrm{d}U_x\, U(x)^{-1}. \tag{55}$$

Analytically it is represented by

$$
\begin{aligned}
(U\Psi)^r(x) &= U^r{}_s(x)\Psi^s(x) \\
(U\vartheta)^r{}_s(x) &= U^r{}_z(x)\,\vartheta^z{}_{t\mu}(x)\,U^{-1t}{}_s(x) - \frac{\partial U^r{}_z(x)}{\partial x^\mu}\,U^{-1z}{}_s(x).
\end{aligned}
\tag{56}
$$

As usual, the action on the connection form is inhomogeneous. Obviously, the metric field is invariant under this action

$$Ug = g. \tag{57}$$

The covariant differentiation of wave functions Ψ is defined by the following equation:

$$
\begin{aligned}
\nabla_\mu \Psi &= \partial_\mu \Psi + g\left(\vartheta_\mu - \frac{1}{4}\,\mathrm{Tr}\,\vartheta_\mu I\right)\Psi + \frac{q}{4}\,\mathrm{Tr}\,\vartheta_\mu \Psi \\
&= \partial_\mu \Psi + g\,\vartheta_\mu \Psi + \frac{q-g}{4}\,\mathrm{Tr}\,\vartheta_\mu \Psi.
\end{aligned}
\tag{58}
$$

In this case of the $\mathrm{SU}(2,2)$-subgroup obviously only the two first terms survive, and

$$\nabla_\mu \Psi = \partial_\mu \Psi + g\,\vartheta_\mu \Psi \tag{59}$$

where ϑ_μ is an $\mathfrak{su}(2,2)$ - valued differential form. But it is clear that the exterior covariant differential of ϑ, $F = D\vartheta$ "does not feel" the g-term,

$$F_{\mu\nu} = D\vartheta_{\mu\nu} = \mathrm{d}\vartheta_{\mu\nu} + g\,[\vartheta_\mu, \vartheta_\nu] = \partial_\mu \vartheta_\nu - \partial_\nu \vartheta_\mu + g\,[\vartheta_\mu, \vartheta_\nu]\,. \tag{60}$$

When dealing with the first-order version of the theory, we shall also use the $\mathrm{H}(4,G) \simeq \mathrm{i}\mathfrak{u}(2,2)$-valued differential form of generalized "conformal tetrad"

$$M \ni x \mapsto E_x \in \mathrm{L}(T_x M, \mathrm{H}(4,G)) \tag{61}$$

which transforms under U in a homogeneous way

$$E_x \mapsto U(x)\,E_x\,U(x)^{-1} \tag{62}$$

similarly to the $\mathrm{SL}(2,\mathbb{C})$-transformation of the usual tetrad. Because of this, the covariant exterior differential $G = DE$ is analytically given by

$$G_{\mu\nu} = \partial_\mu E_\nu - \partial_\nu E_\mu + g\,[\vartheta_\mu, E_\nu] - g\,[\vartheta_\nu, E_\mu]\,. \tag{63}$$

It is clear that it is just the object E which, just like Ψ, is a proper carrier of the internal gauge action of $\mathrm{SU}(2,2)$, the universal covering group of $\mathrm{CO}(1,3)$. In analogy to (23), it is convenient to introduce the ϑ- and E-objects with the G-lowered contravariant index

$$\widetilde{\vartheta}_{\bar{r}s\mu} = G_{\bar{r}z}\,\vartheta^z{}_{s\mu}, \qquad \widetilde{E}_{\bar{r}s\mu} = G_{\bar{r}z}\,E^z{}_{s\mu}. \tag{64}$$

Following (16), we could suggested to put the metric tensor in a first-order theory as

$$g_{\mu\nu} = \frac{1}{4}\,\mathrm{Tr}\,(E_\mu E_\nu) = \frac{1}{4}\,G^{s\bar{z}}G^{w\bar{r}}\widetilde{E}_{\bar{r}s\mu}\widetilde{E}_{\bar{z}w\nu}. \tag{65}$$

But this metric tensor is incompatible with the Clifford condition, namely

$$E_\mu E_\nu + E_\nu E_\mu \neq 2\,g_{\mu\nu} I_4. \tag{66}$$

And any additional extra condition which replaces this by equality, may be interpreted as an unjustified action at a distance introduction. Let me also mention about the project by Lämmerzahl [6] of experimental deviations from the Clifford paradigm.

4. Second-Order $\mathrm{SU}(2,2)$ Gauge Theory

Let us now discuss the second-order differential model suggested by our primary analogy, based on equations (1)÷(12).

The matter Lagrangian is assumed to be, accord with our mentioned above philosophy, given by

$$\begin{aligned} L_{\mathrm{m}}(\Psi;\Theta,g) &= \frac{b}{2}\,g^{\mu\nu}\nabla_\mu\widetilde{\Psi}\nabla_\nu\Psi\sqrt{|g|} - \frac{c}{2}\,\widetilde{\Psi}\Psi\sqrt{|g|} \\ &= \frac{b}{2}\,g^{\mu\nu}\nabla_\mu\overline{\Psi}^{\bar{r}}\nabla_\nu\Psi^s G_{\bar{r}s}\sqrt{|g|} - \frac{c}{2}\,G_{\bar{r}s}\overline{\Psi}^{\bar{r}}\Psi^s\sqrt{|g|}. \end{aligned} \tag{67}$$

The usual Yang-Mills Lagrangian for the gauge field Θ is as always

$$\begin{aligned} L_{\mathrm{YM}}(\Theta,g) &= \frac{A}{4}\,F^r{}_{s\mu\nu}F^s{}_{r\varkappa\lambda}\,g^{\mu\varkappa}g^{\nu\lambda}\sqrt{|g|} \\ &= \frac{A'}{2}\,F^r{}_{r\mu\nu}F^s{}_{s\varkappa\lambda}\,g^{\mu\varkappa}g^{\nu\lambda} + \sqrt{|g|}. \end{aligned} \tag{68}$$

Of course, in the case of the simple subgroup $\mathrm{SU}(2,2)$, the second term does not occur. The above Lagrangian (68) gives rise to the following Yang-Mills field momentum

$$\mathcal{X}^r{}_s{}^{\mu\nu} = \frac{\partial L_{\mathrm{YM}}}{\partial\vartheta^s{}_{r\mu,\nu}} = -A\,\Phi^r{}_{s\varkappa\lambda}\,g^{\varkappa\mu}g^{\lambda\nu} - A'\,\delta^r{}_s\Phi^z{}_{z\varkappa\lambda}\,g^{\varkappa\mu}g^{\lambda\nu}\sqrt{|g|}. \tag{69}$$

The $\mathrm{U}(2,2)$-gauge invariant matter current based on L_{m} is given by (38) or in the contravariant form by

$$J^r{}_s{}^\mu = \left(\left(D_\nu\widetilde{\Psi}_s\right)\Psi^r - \widetilde{\Psi}_s\,(D_\nu\Psi^r)\right)g^{\nu\mu}\sqrt{|g|}. \tag{70}$$

Performing the variation of action corresponding to (67), (68), one obtains the following field equations

$$g^{\mu\nu} \overset{g}{\nabla}_\mu \overset{g}{\nabla}_\nu + \frac{c}{b} \Psi = 0 \tag{71}$$

$$\overset{g}{\nabla}_\nu \mathcal{X}^{\mu\nu} = gJ^\mu + \frac{q-g}{4} \operatorname{Tr} J^\mu I_4. \tag{72}$$

where $\overset{g}{\nabla}_\mu$ denotes the total covariant differentiation which unifies the action of the g-Levi-Civita connection on spatio-temporal indices and the Yang-Mills covariant differentiation in the sense of internal indices.

It is clear that (72) may be transformed to the following form

$$\chi^{\mu\nu}{}_{;\,\nu} + g\,[\vartheta_\nu, \chi^{\mu\nu}] = gJ^\mu + \frac{q-g}{4} \operatorname{Tr} J^\mu I_4. \tag{73}$$

This is because

$$\overset{g}{\nabla}_\mu \chi^{\mu\nu} = \partial_\nu \chi^{\mu\nu} + g\,[\vartheta_\nu, \chi^{\mu\nu}] = \chi^{\mu\nu}{}_{;\,\nu} + g\,[\vartheta_\nu, \chi^{\mu\nu}]\,. \tag{74}$$

The semicolon symbol denotes the covariant differentiation in the g-Levi-Civita sense. Let us observe that the usual divergence of $\chi^{\mu\nu}$ equals to the Levi-Civita or any other divergence based on the symmetric covariant differentiation. The covariant divergence of skew-symmetric χ with respect to the symmetric affine connection is identical with the usual divergence.

The total Lagrangian is the sum of (67), (68) and some Lagrangian for the metric field $g_{\mu\nu}$. And here we are faced with some problem. Very naively, one can try to postulate for g the usual Hilbert-Einstein model, or rather its modification controlled by two arbitrary constants d, l

$$L_{\mathrm{HE}}(g) = -dR(g)\sqrt{|g|} + l\sqrt{|g|} \tag{75}$$

where $R(g)$ denotes the scalar curvature of g, d is related to the inverse gravitational constant and l has to do with the cosmological constant. The resulting Euler-Lagrange equation would be of course

$$d\left(R(g)^{\mu\nu} - \frac{1}{2}\,R(g)g^{\mu\nu}\right) = \frac{l}{2}\,g^{\mu\nu} + \frac{1}{2}\,T^{\mu\nu} \tag{76}$$

where $R(g)^{\mu\nu}$ is the Ricci tensor and $T^{\mu\nu}$ is the total energy - metric tensor of (Ψ, ϑ)

$$T^{\mu\nu} = T^{\mu\nu}_{\mathrm{m}} + T^{\mu\nu}_{\mathrm{YM}} \tag{77}$$

where the material and Yang-Mills contributions are respectively given by

$$T_{\mathrm{m}}^{\mu\nu} = -\frac{2}{\sqrt{|g|}}\left(\frac{\partial L_{\mathrm{m}}}{\partial g_{\mu\nu}} - \left(\frac{\partial L_{\mathrm{m}}}{\partial g_{\mu\nu,\varkappa}}\right)_{,\varkappa}\right) \tag{78}$$

$$T_{\mathrm{YM}}^{\mu\nu} = -\frac{2}{\sqrt{|g|}}\left(\frac{\partial L_{\mathrm{YM}}}{\partial g_{\mu\nu}} - \left(\frac{\partial L_{\mathrm{YM}}}{\partial g_{\mu\nu,\varkappa}}\right)_{,\varkappa}\right). \tag{79}$$

Nevertheless, the procedure with the metric tensor looks rather naive, even if we put the Palatini values

$$d = 0, \qquad l = 0. \tag{80}$$

It seems that it would be much more reasonable to follow the $\mathrm{SL}(2,\mathbb{C})$-procedure with the metric tensor being a byproduct of something else. Of course, the hypothesis:

$$g_{\mu\nu} :\ = \alpha\,\vartheta^{r}{}_{s\mu}\vartheta^{s}{}_{r\nu} + \beta\,\vartheta^{r}{}_{r\mu}\vartheta^{s}{}_{s\nu} \tag{81}$$

is to be rejected, because it is only globally, but not locally $\mathrm{U}(2,2)$-invariant.

Let us quote some possibilities which are free of this drawback.

1. The complex cotetrad field, i.e., a mapping

$$M \ni x \mapsto W_x \in \mathrm{L}(T_x M, \mathbb{C}^4). \tag{82}$$

   It is analytically represented by the matrix field $W^{r}{}_{\mu}$, but with the homogeneous transformation rule under $\mathrm{U}(2)$, just like the usual co-tetrad

$${}'W^{r}{}_{\mu}(x) = U^{r}{}_{s}(x) W^{s}{}_{\mu}(x). \tag{83}$$

   The following twice covariant tensor field on the manifold M

$$g(W)_{\mu\nu} := \Re\left(\widetilde{W}_{r\mu} W^{r}{}_{\nu}\right) = \Re\left(G_{\bar{s}r}\overline{W}^{\bar{s}}{}_{\mu} W^{r}{}_{\nu}\right). \tag{84}$$

   This object is a locally $\mathrm{U}(2,2)$-invariant metric-like tensor on M. And we just identify this byproduct of W with the metric tensor. The homogeneous rule (83) implies that the exterior differential of W is given by

$$\begin{aligned}\nabla W_{\mu\nu} \ &= \ dW_{\mu\nu} + g\left(\vartheta_{\mu} W_{\nu} - \vartheta_{\nu} W_{\mu}\right) \\ &\quad + \frac{q' - g}{4}\left(\mathrm{Tr}\,\vartheta_{\mu} W_{\nu} - \mathrm{Tr}\,\vartheta_{\nu} W_{\mu}\right).\end{aligned} \tag{85}$$

   Here the coupling constant q' is in a sense, the electric charge of W. Let us mention that when the $\mathrm{SL}(2,\mathbb{C})$-reduction is performed, W represents a particle of spin $\frac{3}{2}$. The analogy to the supersymmetric idea of gravitino is readable. The corresponding g-Lagrangian of W is given by

$$L(W,\vartheta) = \alpha \nabla\widetilde{W}_{\mu\nu} \nabla W_{\varkappa\lambda}\, g^{\mu\varkappa} g^{\nu\lambda}\sqrt{|g|} + \beta\sqrt{|g|}. \tag{86}$$

2. There is also another approach to the problem, namely that the geometric sector is also described by the another $\mathfrak{u}(2,2)$-valued differential form, $M \ni x \mapsto W_x \in \mathrm{L}(T_x M, \mathfrak{u}(2,2))$. In analytic terms, it is described by the system of quantities $W^r{}_{s\mu}$. However, it is assumed to be ruled by the following homogeneous transformation under the pseudo-unitary group

$$'W^r{}_{s\mu}(x) = U^r{}_z \, W^z{}_{t\mu} \, U^{-1t}{}_s. \tag{87}$$

The metric field induced by the object W has the form

$$g(W)_{\mu\nu} = \alpha \, \operatorname{Tr}(W_\mu W_\nu) + \beta \, \operatorname{Tr}(W_\mu) \operatorname{Tr}(W_\nu). \tag{88}$$

The exterior covariant differential of the field W has the form

$$\nabla W_{\mu\nu} = dW_{\mu\nu} + g\,[\vartheta_\mu, W_\nu] - g\,[W_\nu, \vartheta_\mu]. \tag{89}$$

Obviously, W_μ in (88) and later on is an abbreviation for $W^r{}_{s\mu}$.
The Maxwell-Yang-Mills Lagrangian for the field W has the form

$$\begin{aligned} L(W,\vartheta) \; = \; & \alpha \operatorname{Tr}(\nabla W_{\mu\nu} \nabla W_{\varkappa\lambda})\, g^{\mu\varkappa} g^{\nu\lambda} \sqrt{|g|} \\ & + \beta \operatorname{Tr}(\nabla W_{\mu\nu}) \operatorname{Tr}(\nabla W_{\varkappa\lambda})\, g^{\mu\varkappa} g^{\nu\lambda} \sqrt{|g|} + \gamma \sqrt{|g|} \end{aligned} \tag{90}$$

where α, β, γ are constants.

3. But one can also think about another metric model which is insensitive with respect to anything like the a priori metric tensor g or with respect to any auxiliary quantities like W used above. Namely, one can try to substitute for g the following quantities

$$g(\Psi,\vartheta)_{\mu\nu} = a\Re\left(\nabla_\mu \widetilde{\Psi} \nabla_\nu \Psi\right) = a\Re\left(G_{\bar{r}s} \nabla_\mu \overline{\Psi}^{\bar{r}} \nabla_\nu \Psi^s\right). \tag{91}$$

The typical Born-Infeld scheme for the system of (Ψ,ϑ)-fields has the following form

$$L(\Psi,\vartheta) = \sqrt{\left|\det\left[\frac{b}{2}\, g_{\mu\nu} + \frac{a}{4}\, \operatorname{Tr}(F_{\mu\varkappa} F_{\nu\lambda})\, g^{\varkappa\lambda} + \frac{a'}{4}\, \operatorname{Tr} F_{\mu\varkappa}\, \operatorname{Tr} F_{\nu\lambda}\, g^{\varkappa\lambda}\right]\right|} \tag{92}$$

where a, a', b are constants.

5. The $\mathrm{SL}(2,C)$- Correspondence in the Second-Order $\mathrm{SU}(2,2)$ Theory

To discuss solutions of our field equations and their relationships with the usual $\mathrm{SL}(2,\mathbb{C})$-ruled spinor theory we introduce an adapted basis of the Lie algebra

$\mathfrak{u}(2,2)$), or, equivalently, of $\mathrm{Herm}(4,\mathbb{C})$. This basis is induced by the choice of Dirac γ-matrices, i.e., representations of γ^A. Namely we put

$$\begin{aligned} \Sigma^{AB} &= \frac{1}{4}\left[\gamma^A,\gamma^B\right] = \frac{1}{4}\left(\gamma^A\gamma^B - \gamma^B\gamma^A\right) \\ \gamma^5 &= -\gamma_5 = -\gamma^0\gamma^1\gamma^2\gamma^3 \\ {}^A\gamma &= \mathrm{i}\gamma^A\gamma^5 = -\mathrm{i}\gamma^5\gamma^A. \end{aligned} \tag{93}$$

It is clear that the left-indexed Dirac matrices ${}^A\gamma$ obey the Clifford anticommutation rules with the reversed signature

$$\left\{{}^A\gamma, {}^B\gamma\right\} = {}^A\gamma{}^B\gamma + {}^B\gamma{}^A\gamma = -2\,\eta^{AB}I_4. \tag{94}$$

The corresponding basis of the Lie algebra $\mathfrak{u}(2,2)) = \mathfrak{u}(4,G))$ is spanned on the set of matrices

$$\mathrm{i}\gamma^A, \qquad \mathrm{i}\,{}^A\gamma, \qquad \Sigma^{AB}, \qquad \mathrm{i}\gamma^5, \qquad \mathrm{i}I^4. \tag{95}$$

It is convenient to distinguish the following combinations of γ-matrices

$$\tau_A := \frac{1}{2}\left(\gamma_A + {}_A\gamma\right), \qquad \chi^A = \left(\gamma^A - {}^A\gamma\right). \tag{96}$$

It is clear that they span two four-dimensional commutative Lie algebras

$$[\tau_A, \tau_B] = 0, \qquad \left[\chi^A, \chi^B\right] = 0. \tag{97}$$

These algebra generate two four-dimensional Abelian groups. Using the language taken from twistor theory we would say that τ_A generate the group of space-time translation, and χ^A - the group of proper conformal transformations (53), i.e., boosts (accelerations).

Expanding the connection form with respect to the basis (95), we introduce the system of gauge potentials

$$\vartheta_\mu = \frac{1}{2}\,\breve{\Omega}^{AB}{}_\mu\,\Sigma_{AB} + B_\mu\,\frac{1}{\mathrm{i}}\,\gamma_5 + A_\mu\,\mathrm{i}I + e^A{}_\mu\,\mathrm{i}\tau_A + f_{A\mu}\,\mathrm{i}\chi^A. \tag{98}$$

Sometimes one introduces alternative objects

$$\Omega^A{}_{B\mu} = \breve{\Omega}^A{}_{B\mu} + B_\mu\,\delta^A{}_B \tag{99}$$

$$B_\mu = \frac{1}{8}\,\Omega^A{}_{A\mu}, \qquad \breve{\Omega}^A{}_{B\mu} = \Omega^A{}_{B\mu} - \frac{1}{4}\,\Omega^C{}_{C\mu}\delta^A{}_B. \tag{100}$$

So, roughly speaking $\breve{\Omega}^A{}_{B\mu}$ and B_μ are related to the skew-symmetric and trace-like parts of $\Omega^A{}_{B\mu}$. It is clear that ϑ_μ may then be rewritten in the form

$$\vartheta_\mu = \frac{1}{2}\,\Omega^A{}_{B\mu}\left(\Sigma_{AB} + \frac{1}{4}\,\eta_{AB}\,\frac{1}{\mathrm{i}}\,\gamma_5\right) + e^A{}_\mu\,\mathrm{i}\tau_A + f_{A\mu}\,\mathrm{i}\chi^A + A_\mu\,\mathrm{i}I \tag{101}$$

or equivalently

$$\vartheta_\mu = \frac{1}{2g}\,\breve{\Gamma}^{AB}{}_\mu\,\Sigma_{AB} + \frac{1}{4g}\,Q_\mu\,\frac{1}{\mathrm{i}}\,\gamma^5 + \frac{1}{g}\,\varepsilon^A{}_\mu\,\mathrm{i}\tau_A + \frac{1}{g}\,\varphi_{A\mu}\,\mathrm{i}\chi^A + A_\mu\,\mathrm{i}I \qquad (102)$$

or in the form similar to (101)

$$\vartheta_\mu = \frac{1}{2g}\,\Gamma^{AB}{}_\mu\left(\Sigma_{AB} + \frac{1}{4}\,\eta_{AB}\,\frac{1}{\mathrm{i}}\,\gamma_5\right) + \frac{1}{g}\,\varepsilon^A{}_\mu\,\mathrm{i}\tau_A + \frac{1}{g}\,\varphi_{A\mu}\,\mathrm{i}\chi^A + A_\mu\,\mathrm{i}I. \quad (103)$$

The meaning of the symbols used here is as follows

$$\begin{aligned}
&\Gamma^A{}_{B\mu} = g\,\Omega^A{}_{B\mu}, \qquad \breve{\Gamma}^A{}_{B\mu} = \Gamma^A{}_{B\mu} - \frac{1}{4}\,\Gamma^C{}_{C\mu}\delta^A{}_B\\
&Q_\mu = \frac{1}{2}\,\Gamma^A{}_{A\mu} = \frac{g}{2}\,\Omega^A{}_{A\mu} = 4gB_\mu\\
&\varepsilon^A{}_\mu = g\,e^A{}_\mu, \qquad \varphi_{A\mu} = g f_{A\mu}.
\end{aligned} \qquad (104)$$

Let us stress that the action of $U(A)$, $A \in \mathrm{SL}(2,\mathbb{C})$ on $\left[e^B\right]$, $[f_B]$ becomes homogeneous, although the action of unrestricted $U \in \mathrm{U}(2,2)$ on ϑ_μ and its system of coefficients is inhomogeneous. So, roughly speaking $\left[e^B\right]$, $[f_B]$ are two versions of the gravitational co-tetrad, just like in our earlier papers [1–5], [7–24]. And obviously, the action of $\mathrm{GL}(2,\mathbb{C})$ is such that Q_μ is the Weyl covector

$$\nabla_\lambda\, g_{\mu\nu} = -Q_\lambda\, g_{\mu\nu}. \qquad (105)$$

This explains in a very beautiful way the gauge geometric role of cotetrads. They are parts of the $\mathrm{U}(2,2)$ or $\mathrm{SU}(2,2)$ connection form.

The curvature two-form Φ may be also expressed in terms of the gauge potentials and their exterior differentials, namely

$$\Phi = T(e)^A\,\mathrm{i}\tau_A + T(f)_A\,\mathrm{i}\chi^A + \frac{1}{2}\,\widetilde{R}^{AB}\Sigma_{AB} + G\,\frac{1}{\mathrm{i}}\,\gamma^5 + F\,\mathrm{i}I \qquad (106)$$

with the following meaning of coefficients

$$\begin{aligned}
T(e)^A &= \mathrm{d}e^A + g\,\Omega^A{}_B \wedge e^B = \mathrm{d}e^A + \Gamma^A{}_B \wedge e^B\\
T(f)_A &= \mathrm{d}f_A + g f_B \wedge \Omega^B{}_A = \mathrm{d}f^A + f_B \wedge \Gamma^B{}_A\\
\widetilde{R}^A{}_B &= R(\Omega)^A{}_B - \frac{1}{4}\,R(\Omega)^C{}_C\,\delta^A{}_B - 2g\,e^A \wedge f_B + 2g\,\eta^{AC}\eta_{BD}\,e^D \wedge f_C\\
&= \frac{1}{g}\left(R(\Gamma)^A{}_B - \frac{1}{4}\,R(\Gamma)^C{}_C\,\delta^A{}_B - 2g^2\,e^A \wedge f_B + 2g^2\,\eta^{AC}\eta_{BD}\,e^D \wedge f_C\right)\\
G &= \frac{1}{4g}\,\mathrm{d}Q - g\,e^A \wedge f_A = \frac{1}{g}\left(\frac{1}{8}\,R(\Gamma)^A{}_A - g^2\,e^A \wedge f_A\right)\\
F &= \mathrm{d}A.
\end{aligned} \qquad (107)$$

In these equations, $R(\Gamma)$, $R(\Omega)$ are symbols for the usual curvature two-form

$$\begin{aligned} R(\Gamma)^{A}{}_{B} &= \mathrm{d}\Gamma^{A}{}_{B} + \Gamma^{A}{}_{C} \wedge \Gamma^{C}{}_{B} \\ R(\Omega)^{A}{}_{B} &= \mathrm{d}\Omega^{A}{}_{B} + g\,\Omega^{A}{}_{C} \wedge \Omega^{C}{}_{B}. \end{aligned} \tag{108}$$

All those objects give rise to some space-time objects like connections, curvature and torsions. They are analytically described by the following expressions

Affine connections

$$\begin{aligned} \Gamma(e)^{k}{}_{ij} &= e^{k}{}_{A}\,\Gamma^{A}{}_{Bj}\,e^{B}{}_{i} + e^{k}{}_{A}\,e^{A}{}_{i,j} \\ \Gamma(f)^{k}{}_{ij} &= -f_{Ai}\,\Gamma^{A}{}_{Bj}\,f^{kB} + f^{kA}\,f_{Ai,j}. \end{aligned} \tag{109}$$

The torsion tensors

$$\begin{aligned} S(e)^{k}{}_{ij} &= \Gamma(e)^{k}{}_{[ij]} = -\tfrac{1}{2}\,e^{k}{}_{A}\,T(e)^{A}{}_{ij} \\ S(f)^{k}{}_{ij} &= \Gamma(f)^{k}{}_{[ij]} = -\tfrac{1}{2}\,f^{kA}\,T(f)_{Aij}. \end{aligned} \tag{110}$$

The curvature tensors

$$\begin{aligned} R(e)^{m}{}_{kij} &= e^{m}{}_{A}e^{B}{}_{k}R^{A}{}_{Bij} \\ R(f)^{m}{}_{kij} &= -f_{Ak}f^{mB}R^{A}{}_{Bij}. \end{aligned} \tag{111}$$

We do not dare to solve the system of equations (71), (72) and (76) (or instead (76) the system following from (86), (90) and (92)). But instead, we shall try to solve the simplified empty-space solutions with the non-excited matter, $\Psi = 0$ and with the formally substituted Einstein-Dirac metric tensors

$$h(e,\eta)_{\mu\nu} = \eta_{AB}\,e^{A}{}_{\mu}e^{B}{}_{\nu}, \qquad h(f,\eta)_{\mu\nu} = \eta_{AB}f_{A\mu}f_{B\nu}. \tag{112}$$

Therefore, to the mentioned field equations we substitute the following equations

$$\begin{aligned} &\Psi = 0, \qquad f_{A\mu} = k\,\eta_{AB}\,e^{B}{}_{\mu}, \qquad g_{\mu\nu} = ph(e,\eta)_{\mu\nu} \\ &Q_{\mu} = 0, \qquad A_{\mu} = 0, \qquad S(e)^{\lambda}{}_{\mu\nu} = S(f)^{\lambda}{}_{\mu\nu} = 0. \end{aligned} \tag{113}$$

One can show that after the very complicated discussion, the total system of field equations reduces to

$$R^{\mu\nu} - \frac{1}{2}\,R\,g^{\mu\nu} = -12\,\frac{g^{2}k}{p}\,g^{\mu\nu}. \tag{114}$$

Substituting here the very natural values $k = 1$, $p = 1$, we obtain the following system of field equations

$$R^{\mu\nu} - \frac{1}{2}\,R\,g^{\mu\nu} = -12\,g^{2}g^{\mu\nu} \tag{115}$$

As mentioned, the values $k = 1$, $p = 1$ are very natural. If for the metric field $g_{\mu\nu}$ we substitute the Hilbert-Einstein dynamics, then we obtain the equation

$$T_{\mu\nu} = 0 \tag{116}$$

where $T_{\mu\nu}$ denotes the symmetric "energy-momentum tensor" for our fields. But this is quite compatible with (114) (115). Namely, there exist solutions characterizing the space of the constant space-time curvature where

$$R_{\alpha\beta\mu\nu} = \frac{4g^2k}{p}\left(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu}\right) \tag{117}$$

i.e., simply

$$R_{\alpha\beta\mu\nu} = 4g^2\left(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu}\right) \tag{118}$$

if we continue our sticking to the values $k = 1$, $p = 1$.

It is very interesting that the invariance of the theory under the covering group of the conformal group implies the conformal flatness of the space-time manifold expressed by (118). It is much more important and interesting that the global conformal flatness of the space-time manifold is expressed by the microscopic parameter g of the $\mathrm{U}(2,2)$-gauge invariance. The existence of the above solutions to our field equations looks like a miracle!

In the material sector, it seems that there is no discrepancy between our primary second-order differential equation and the first-order Dirac differential equation. So, when we substitute to the equation of matter the above-mentioned choice with $p = 1$, $k = 1$, and assume the space-time background as fixed, then for the material Dirac-Klein-Gordon equation we obtain

$$e^{\mu}{}_{A}\mathrm{i}\gamma^{A}\left(\nabla_{\mu} + S^{\nu}{}_{\nu\mu}I_4\right)\Psi - \frac{4bg^2 - c}{2bg}\,\Psi + \frac{1}{2g}\,\overset{g}{\nabla}_{\mu}\overset{g}{\nabla}_{\nu}\,\Psi = 0. \tag{119}$$

In this equation ∇_{μ} denotes the $\mathrm{SL}(2,\mathbb{C})$-contribution to the covariant differentiation in the gauge sense of $\mathrm{U}(2,2)$. The symbol $\overset{g}{\nabla}_{\mu}$ unifies the ∇_{μ}-differentiation with the $\Gamma^{A}{}_{B}$-covariant differentiation of tensorial objects with the capital indices and with the usual covariant differentiation in the Levi-Civita sense. The first two terms in (119) correspond with the Dirac theory in the space-time of Einstein-Cartan type. But it is well-known that the structure of the general solution of differential equations depends critically on the highest-order differential term. The question appears if the Klein-Gordon term in (119) does not destroy the similarity of the first two terms to the Dirac equation. To discuss this let us consider the specially-relativistic approximation when $e^{A}{}_{\mu} = \delta^{A}{}_{\mu}$, $\Gamma^{A}{}_{B\mu} = 0$, $g_{\mu\nu} = \delta_{\mu\nu}$. In this approximation, (119) becomes

$$\mathrm{i}\gamma^{\mu}\partial_{\mu} - \frac{4bg^2 - c}{2bg}\,\Psi + \frac{1}{2g}\,\eta^{\mu\nu}\partial_{\mu}\partial_{\nu}\Psi = 0. \tag{120}$$

It is clear that its general solution is a linear combination of two Dirac wave amplitudes with two masses $m_{\pm}$ given by

$$m_{\pm}^2 = \frac{c}{b} - 2g^2 \left(1 \pm \sqrt{\frac{c}{bg^2} - 3}\right). \tag{121}$$

The range of Dirac-like behavior is given by $c/b > 3g^2$. For the critical value $c/b = 3g^2$, there is no splitting of mass and $m = |g|$. This is the rigorous Dirac dynamics with one mass. If $c/b = 4g^2$, then one of the masses does vanish, namely $m_- = 0$, and then $m_+ = 2\,|g|$. In general the natural question appears concerning the experimental verification of the two possible mass states. Do they appear in reality or not? And why? There are three a priori possible explanations

1. Perhaps the mass *gap* $m_+ - m_-$ is too large and the creation of the higher mass state is exceeds the experimental abilities.
2. Perhaps conversely, $|g|$ is so small that the mass *gap* $m_+ - m_-$ is also too small to be experimentally observed.
3. The last possibility, in which we believe is that the doubling of mass states not only exists but is also observed in experiments. And one can suspect this to explain the mysterious pairing of fundamental quarks and leptons in the standard model of electroweak interactions. By this we mean their occurrence in pairs like (ν_e, e), (ν_μ, μ), (ν_τ, τ) and (u, d), (c, s), (t, b), respectively for fermions and quarks. In particular, the situation $c/b = 4g^2$ might seem to be a kind of "explanation" of the pairing between heavy leptons and their neutrinos.

6. First-Order $\mathrm{SU}(2,2)$ Field Theory

Let us now go back to our model of first-order differential theory gauge-invariant under $\mathrm{U}(2,2)$. As we remember, it has the following basic variables: matter field Ψ, connection ϑ and "multitetrad" E. The metric field is defined by (65) as a by-product of E. The covariant exterior differentials $F = D\vartheta$, $G = DE$ are given by (60) and (63).

These objects give rise to the pair of affine connections in M. Namely, the metric tensor g gives rise to the Levi-Civita connection ${}^{g}\left\{ \begin{array}{c} \alpha \\ \mu\nu \end{array} \right\}$ based on g. Another one, $\Gamma(E, \vartheta)$ is given by

$$\Gamma^{\lambda}{}_{\mu\nu} := \frac{1}{4}\, \mathrm{Tr}\left(E^{\lambda}\left(E_{\mu,\nu} + g\left[\vartheta_\nu, E_\mu\right]\right)\right). \tag{122}$$

Here E^{λ} with the upper-case index is obviously given by

$$E^{\lambda} := g^{\lambda\mu} E_{\mu}. \tag{123}$$

The definition of $\Gamma(E, \vartheta)$ is based on the assumption that E is parallel under the covariant differentiation based on ϑ (bispinor indices) and Γ (space-time indices)

$$\overset{\vartheta,\Gamma}{\nabla}_{\mu} E^{r}{}_{s\nu} = E^{r}{}_{s\nu,\mu} + g\, \vartheta^{r}{}_{z\mu} E^{z}{}_{s\nu} - g E^{r}{}_{z\nu} \vartheta^{z}{}_{s\mu} - E^{r}{}_{s\lambda} \Gamma^{\lambda}{}_{\nu\mu} = 0. \tag{124}$$

One can check that $\Gamma(E, \vartheta)$ is an Einstein-Cartan connection

$$\nabla g = 0 \tag{125}$$

and its torsion is given by

$$S^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}{}_{[\mu\nu]} = -\frac{1}{8}\,\mathrm{Tr}\left(E^{\lambda} G_{\mu\nu}\right). \tag{126}$$

In analogy to the $\mathrm{SL}(2, \mathbb{C})$-invariant Dirac theory, we have the following form of the matter Lagrangian

$$\begin{aligned} L_{\mathrm{m}}(\Psi; E, \vartheta) &= \frac{\mathrm{i}}{2}\, g^{\mu\nu} E^{r}{}_{s\mu} \left(\widetilde{\Psi}_{r} \nabla_{\nu} \Psi^{s} - \left(\nabla_{\nu} \widetilde{\Psi}_{r}\right) \Psi^{s}\right) \sqrt{|g|} - m\, \widetilde{\Psi}_{r} \Psi^{r} \sqrt{|g|} \\ &= \frac{\mathrm{i}}{2}\, g^{\mu\nu} E_{\bar{r}s\mu} \left(\overline{\Psi}^{\bar{r}} \nabla_{\nu} \Psi^{s} - \left(\nabla_{\nu} \overline{\Psi}^{\bar{r}}\right) \Psi^{s}\right) \sqrt{|g|} - m\, G_{\bar{r}s} \overline{\Psi}^{\bar{r}} \Psi^{s} \sqrt{|g|}. \end{aligned} \tag{127}$$

The wave equation following from L_{m} is given by

$$\mathrm{i} E^{r}{}_{s}{}^{\mu} \left(\nabla^{s}{}_{z\mu} + S^{\nu}{}_{\nu\mu} \delta^{s}{}_{z}\right) \Psi^{z} = m \Psi^{r}. \tag{128}$$

This form is quite analogous to the usual Dirac equation. The Yang-Mills Lagrangian for the ϑ-field has the usual structure

$$\begin{aligned} L_{\mathrm{YM}}(\vartheta; E) \;=\; & \frac{A}{4}\, F^{r}{}_{s\mu\nu} F^{s}{}_{r\varkappa\lambda}\, g^{\mu\varkappa} g^{\nu\lambda} \sqrt{|g|} \\ & + \frac{A'}{4}\, F^{r}{}_{r\mu\nu} F^{s}{}_{s\varkappa\lambda}\, g^{\mu\varkappa} g^{\nu\lambda} \sqrt{|g|}. \end{aligned} \tag{129}$$

As in all such expressions A, A' are constants. The same terms appears in the Yang-Mills Lagrangian for E

$$\begin{aligned} L_{\mathrm{E}}(E; \vartheta) \;=\; & \frac{B}{4}\, G^{r}{}_{s\mu\nu} G^{s}{}_{r\varkappa\lambda}\, g^{\mu\varkappa} g^{\nu\lambda} \sqrt{|g|} \\ & + \frac{B'}{4}\, G^{r}{}_{r\mu\nu} G^{s}{}_{s\varkappa\lambda}\, g^{\mu\varkappa} g^{\nu\lambda} \sqrt{|g|} + C\sqrt{|g|} \end{aligned} \tag{130}$$

where again B, B', C are constants.

This is the main term, however let us notice that in principle one can also add here some other contributions like, e.g., a linear combination of three basic Weitzenböck terms quadratic in S

$$\begin{aligned} L^{\mathrm{torsion}}(E; \vartheta) \;=\; & D_{1}\, g_{\alpha\beta} g^{\mu\varkappa} g^{\nu\lambda}\, S^{\alpha}{}_{\mu\nu} S^{\beta}{}_{\varkappa\lambda} \sqrt{|g|} \\ & + D_{2}\, g^{\mu\nu}\, S^{\alpha}{}_{\beta\mu} S^{\beta}{}_{\alpha\nu} \sqrt{|g|} + D_{3}\, g^{\mu\nu}\, S^{\alpha}{}_{\alpha\mu} S^{\beta}{}_{\beta\nu} \sqrt{|g|}. \end{aligned} \tag{131}$$

One can also think about admitting the Einstein-Cartan expression

$$L^{\mathrm{EC}}(E;\vartheta) := \frac{1}{k}\, g^{\mu\nu} R(\Gamma)^{\alpha}{}_{\mu\alpha\nu}\sqrt{|g|}. \tag{132}$$

However, below we restrict ourselves to the cosmologically modified Yang-Mills term (130). The resulting field equations have the form

$$\begin{aligned} AF^{\mu\nu}{}_{;\nu} + A'\ \mathrm{Tr}\, F^{\mu\nu}{}_{;\nu} I_4 - Ag\,[F^{\mu\nu},\vartheta_\nu] - Bg\,[G^{\mu\nu},E_\nu] + \theta^\mu &= 0 \\ BG^{\mu\nu}{}_{;\nu} + B'\ \mathrm{Tr}\, G^{\mu\nu}{}_{;\nu} I_4 - Bg\,[G^{\mu\nu},\vartheta_\nu] + \tau(\Psi)^\mu + \tau(E)^\mu &= \frac{1}{4}\, T(\vartheta)^{\mu\nu} E_\nu. \end{aligned} \tag{133}$$

The semicolon denotes the Levi-Civita covariant derivative. The source terms in these field equations are shown to be

$$\begin{aligned} \theta^\mu &:= g\,\frac{\mathrm{i}}{2}\left\{\Psi\widetilde{\Psi}, E^\mu\right\} + \frac{q-g}{4}\,\mathrm{i}\widetilde{\Psi}E^\mu\Psi I_4 \\ \tau(\Psi)^\mu &:= \Big((\nabla^\mu\Psi)\,\widetilde{\Psi} - \Psi\left(\nabla^\mu\widetilde{\Psi}\right)\Big) - \frac{1}{2}\, t(\Psi)^{(\mu\varkappa)} E_\varkappa \\ t(\Psi)_{\mu\nu} &:= \frac{\mathrm{i}}{2}\left(\widetilde{\Psi}E_\nu\,(\nabla_\mu\Psi) - \left(\nabla_\mu\widetilde{\Psi}\right)E_\nu\Psi\right). \\ \tau(E)^\mu &:= \frac{B}{2}\left(\mathrm{Tr}\,(G^\mu{}_\varkappa G^{\varkappa\nu}) - \frac{1}{4}\,\mathrm{Tr}\left(G^{\lambda\varkappa}G_{\varkappa\lambda}\right) g^{\mu\nu}\right) E_\nu \\ &\quad + \frac{B'}{4}\left(\mathrm{Tr}\, G^\mu{}_\varkappa\, \mathrm{Tr}\, G^{\varkappa\nu} - \frac{1}{4}\ \mathrm{Tr}\, G^{\lambda\varkappa}\, \mathrm{Tr}\, G_{\varkappa\lambda}\right) E_\nu + \frac{C}{4} E^\nu \\ T(\vartheta)^{\mu\nu} &= -A\left(\mathrm{Tr}\left(F^\mu{}_\lambda F^{\lambda\nu}\right) - \frac{1}{4}\ \mathrm{Tr}\left(F^\alpha{}_\beta F^\beta{}_\alpha\right) g^{\mu\nu}\right) E_\nu \\ &\quad + A'\left(\mathrm{Tr}\, F^\mu{}_\lambda\, \mathrm{Tr}\, F^{\lambda\nu} - \frac{1}{4}\ \mathrm{Tr}\, F^\alpha{}_\beta\, \mathrm{Tr}\, F^\beta{}_\alpha g^{\mu\nu}\right). \end{aligned}$$

Obviously

$$T(\vartheta)^{\mu\nu} = -\frac{2}{\sqrt{|g|}}\frac{\delta L_\vartheta}{\delta g_{\mu\nu}}, \qquad \tau(E)^\mu = -\frac{1}{4}\, T(E)^{\mu\nu} E_\nu$$

and $T(E)$ is constructed of E just like $T(\vartheta)$ of ϑ.

One can notice that θ^μ has the structure similar to that of the Noether current for the Dirac field. The tensor $t(\Psi)_{\mu\nu}$ is similar to the canonical energy-momentum tensor of this field.

We have formally derived the above equations but as yet we were unable to find any their solutions, and the more to interpret them. Because of this it seems that our second-order field equations and their interpretation in terms of pairing between fundamental particles seem to be a good way, certainly more promising than the explicitly first-order equations. The more that surprisingly enough, the Dirac correspondence seems to be inherently contained in them.

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