XIII. Large Ideals on ω_1

§0. Introduction

Here we shall start with κ e.g. supercompact, use semiproper iteration to get results like ($S \subseteq \omega_1$ stationary costationary):

(a) ZFC + GCH + $\mathcal{P}(\omega_1)/(\mathcal{D}_{\omega_1} + S)$ is *layered* + suitable forcing axiom and note that by [FMSh:252] this implies the existence of a uniform ultrafilter on ω_1 such that $\aleph_0^{\omega_1}/D = \aleph_1$ (which is stronger than "D is not regular").

(b) ZFC+GCH+$\mathcal{P}(\omega_1)/(\mathcal{D}_{\omega_1} + S)$ is *Levy* + suitable forcing axiom.

(c) ZFC+GCH+$\mathcal{P}(\omega_1)/(\mathcal{D}_{\omega_1} + S)$ is *Ulam* + suitable forcing axiom.

where (a) Ulam means

$$(\mathcal{D}_{\omega_1} + S)^+ = \{A \subseteq \omega_1 : A \cap S \neq \emptyset \mod \mathcal{D}_{\omega_1}\}$$

is the union of $\aleph_1$, $\aleph_1$-complete filters, hence on $\mathbb{R}$ there are $\aleph_1$ measures such that each $A \subseteq \mathbb{R}$ is measurable for at least one measure

(b) Levy means that, as a Boolean algebra, it is isomorphic to the completion of a Boolean algebra of the Levy collapse Levy$(\aleph_0, < \aleph_2)$

(c) layered means that the Boolean algebra is $\bigcup_{\alpha<\aleph_2} B_\alpha$, where B_α are increasing, continuous, $|B_\alpha| \leq \aleph_1$, and $\mathrm{cf}(\alpha) = \aleph_1 \Rightarrow B_\alpha \lessdot \mathcal{P}(\omega_1)/(\mathcal{D}_{\omega_1} + S)$. We also deal with reflectiveness (see 4.3).

This chapter is a rerepresentation of [Sh:253], we shall give some history later, and now just remark that this work was done (and reclaimed) *after*

[FMSh:240 §1, §2] and [W83] ([W83] starts with "ZFC+DC+ADR+θ *regular*" and forces "ZFC+CH+the club filter on some stationary $S \subseteq \omega_1$ is $\aleph_1$ dense") but *before* Woodin obtained a similar result from a huge cardinal.

In this chapter we got results by semiproper iteration iterating collapses and sealing some maximal antichains of $\mathcal{P}(\omega_1)/\mathcal{D}_{\omega_1}$ up to some large κ. So it is a natural continuation of Chapter X. Our ability to do this to enough chains comes from reflection properties of κ, which is supercompact (or limit of enough supercompacts).

The first section contains preliminaries on semi-stationary sets, relevant reflection properties and what occurs to some such properties when we force. In the second section we deal more specifically with our iterations (S-suitable iterations). In the third section we deal with getting Levy algebra and layeredness, and in the fourth we deal with reflective ideals (see 4.3) and with the Ulam property. Note that for much of the chapter the iteration is of S_3-complete forcing notion, for some (fixed) stationary $S_3 \subseteq \omega_1$, and in this case the iteration is (equivalent to) a CS one; so we will stress less the names of conditions etc.

By Foreman, Magidor and Shelah [FMSh:240], CON(ZFC+κ is supercompact) implies the consistency of ZFC+"$\mathcal{D}_{\omega_1}$ is $\aleph_2$-saturated" [i.e., if $\mathfrak{B}$ is the Boolean algebra $\mathcal{P}(\omega_1)/\mathcal{D}_{\omega_1}$, "$\mathcal{D}_{\omega_1}$ is $\aleph_2$ saturated" means "$\mathfrak{B}$ satisfies the $\aleph_2$-c.c."]. This in fact was deduced from the MM$^+$(=Martin Maximum$^+$) by [FMSh:240] whose consistency was proved by RCS iteration of semiproper forcings (see Chapter X, Chapter XVII §1). Note that [FMSh:240] refutes the thesis: in order to get an elementary embedding j of V with small critical ordinal, into some transitive class M of some generic extension V^P of V, *one should start* with an elementary embedding of j of V' into some M' and then force over V'. Previously, J. Steel and Van Wesep got the same result starting from ZF+AD+AC$_R$ (see [StVW]).

This thesis was quite strongly rooted. Note that it is closely connected to the existence of normal filters D on λ which are λ^+-saturated or at least precipitous (use for P the set of nonzero members of $\mathcal{P}(\lambda)/D$ ordered by inverse inclusion, j the generic ultrapower). See [FMSh:240] for older history.

In fact, it was shortly proved directly that $\mathrm{MM}^+ \equiv \mathrm{SPFA}^+$ and much later it was proved that MM is equivalent to the Semi-Proper Forcing Axiom (in ZFC) (see XVIII §1).

The rsults of [FMSh:240, §1, §2] motivated much activity. Woodin proves from

CON(ZF+ADR+θ regular) the consistency of ZFC+"$\mathfrak{B}\restriction S$ is $\aleph_1$-dense", for some stationary $S \subseteq \omega_1$.

By Shelah and Woodin [ShWd:241], if there is a supercompact cardinal, then every projective set of reals is Lebesgue measurable (etc.). This was obtained by combining (A) and (B) below which were proved simultaneously:

(A) The conclusion holds if there is a weakly compact cardinal κ and a forcing notion P, $|P| = \kappa$, satisfying the κ-c.c., not adding reals and $\Vdash_P$"there is a normal filter D on ω_1, $\mathfrak{B} = \mathcal{P}(\omega_1)/D$ satisfying the $\aleph_2$-c.c."

(B) There is a forcing as required in (A) (see [FMSh:240, §3]).

This was improved for projective sets which are Σ_n using approximately n cardinals κ satisfying:

(*) for every forcing notion $P \in H(\kappa)$ and stationary costationary $S \subseteq \omega_1$ there is semiproper $\underset{\sim}{Q}$, not adding reals, $\Vdash_{P*\underset{\sim}{Q}}$ "$\mathcal{D}_{\omega_1}\restriction S$ is κ-saturated, $\kappa = \aleph_2$" (and $\underset{\sim}{Q}$ is not too large).

A sufficient condition for (*) is $\mathrm{Pr}_a(\kappa) \overset{\mathrm{def}}{=} \kappa$ is strongly inaccessible, and for every $f : \kappa \to \kappa$ there is an elementary embedding $j : V \to M$ (M is a transitive class), κ the critical ordinal of j and $H(j(f)(\kappa)) \subseteq M$. Moreover it suffice (Woodin cardinals) $\mathrm{Pr}_b(\kappa)^\dagger \overset{\mathrm{def}}{=} \kappa$ is strongly inaccessible, and for every $f : \kappa \to \kappa$ there is $\kappa_1 < \kappa$, $(\forall \alpha < \kappa_1), f(\alpha) < \kappa_1$ and for some elementary embedding $j : V \to M$ (M is a transitive class), κ_1 is the critical ordinal of j and $H((j(f))(\kappa_1)) \subseteq M$.

By [Sh:237a] "$2^{\aleph_0} < 2^{\aleph_1} \Rightarrow \mathcal{D}_{\omega_1}$ is not $\aleph_1$-dense", and by [Sh:270] if D is a layered filter on $\lambda = \lambda^{<\lambda}$ then $D^+ = \{A \subseteq \lambda : A \notin D\}$ is the union of λ filters extending D.

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† Later results of Martin, Steel and Woodin clarify the connection between determinacy and large cardinals.

This chapter is a representation of [Sh:253] which was done then, but was mistakenly held as incorrect for quite some time. The main change is that we replace part of the consistency proof of the Ulam statement, ($\mathcal{P}(\omega_1)$ is the union of $\aleph_1$ $\aleph_1$-complete nontrivial measures), by a deduction from a strong variant of layerness. Later Woodin proves from a huge cardinal CON(ZFC+ GCH+$\mathcal{P}(\omega_1)/(\mathcal{D}_{\omega_1} + S)$ is $\aleph_1$-dense).

0.1. Notation and Basic Facts.

(1) $\mathcal{P}$(A) is the power set of A, $\mathcal{S}_{<\lambda}(A) = \{B : B \subseteq A, |B| < \lambda\}$, $<^*_\lambda$ is a well ordering of $H(\lambda)$ which, for simplicity only, we assume is an end extension of $<^*_\mu$ for $\mu < \lambda$.

(2) $\mathcal{D}_\lambda$ is the club filter on a regular $\lambda > \aleph_0$ and $\mathcal{D}_{<\lambda}(A)$ is the club filter on $\mathcal{S}_{<\lambda}(A)$.

(3) (a) $\mathfrak{B}$ is the Boolean Algebra $\mathcal{P}(\omega_1)/\mathcal{D}_{\omega_1}$; we do not distinguish strictly between $A \in \mathcal{P}(\omega_1)$ and $A/\mathcal{D}_{\omega_1}$ and for stationary $S \subseteq \omega_1$, $\mathfrak{B}\restriction S$ is defined naturally.

(b) $\mathfrak{B}$ of course depends on the universe, so we may write $\mathfrak{B}^{V^1}$ or $\mathfrak{B}[V^1]$; instead of $\mathfrak{B}[V^P]$ we may write $\mathfrak{B}^P$ or $\mathfrak{B}[P]$.

(c) If $V^1 \subseteq V^2$, $\omega_1^{V^1} = \omega_1^{V^2}$, then $\mathfrak{B}[V^1]$ is a weak subalgebra of $\mathfrak{B}[V^2]$ (i.e., distinct elements in $\mathfrak{B}[V^1]$ may be identified in $\mathfrak{B}[V^2]$).

(d) If $P \in V$ is a forcing notion preserving stationary subsets of ω_1, then $\mathfrak{B} = \mathfrak{B}[V]$ is a subalgebra of $\mathfrak{B}^P$ (identifying $(A/\mathcal{D}_{\omega_1})^V$ and $(A/\mathcal{D}_{\omega_1})^{V^P}$ for $A \in \mathcal{P}(\omega_1)^V$). If $\bar{Q} = \langle P_i, Q_i : i < \alpha\rangle$ is an iteration (with limit P_α, so $i < j < \alpha \Rightarrow P_i \lessdot P_j$), we let $\mathfrak{B}^{\bar{Q}} = \cup_{i<\alpha}\mathfrak{B}^{P_{i+1}}$.

(4) (a) Let us say, for Boolean algebras B_1 and B_2, that $B_1 \lessdot B_2$ *iff* $B_1 \subseteq B_2$ (i.e., B_1 is a subalgebra of B_2) and every maximal antichain of B_1 is a maximal antichain of B_2.

(b) Note that, for Boolean algebras B_1 and B_2, $B_1 \lessdot B_2$ *iff* $B_1 \subseteq B_2$ and $(\forall x \in B_2 \setminus \{0\})(\exists y \in B_1 \setminus \{0\})(\forall z \in B_1)[z \cap y \neq 0 \to z \cap x \neq 0]$. Hence, if $B_1 \lessdot B_3$ and $B_1 \subseteq B_2 \subseteq B_3$, then $B_1 \lessdot B_2$.

(c) Hence, the satisfaction of "$B_1 \lessdot B_2$" does not depend on the universe of set theory, i.e., if $V \models B_1 \lessdot B_2$ and $V \subseteq V^1$ then $V^1 \models B_1 \lessdot B_2$.

(d) By Solovay and Tennenbaum [ST], $\lessdot$ is transitive, and if $\langle B_i : i < \alpha \rangle$ is $\lessdot$-increasing and continuous then $B_i \lessdot \bigcup_{j<\alpha} B_j$.

(e) Also, if $\langle B_\zeta : \zeta < \xi \rangle$ is a $\subseteq$-increasing sequence of Boolean algebras and $B_0 \lessdot B_\zeta$ for $\zeta < \xi$, then $B_0 \lessdot \bigcup_{\zeta<\xi} B_\zeta$.

(f) If $\langle B_i : i \leq \delta + 1 \rangle$ is an increasing continous sequence of Boolean algebras, $\mathrm{cf}(\delta) > \aleph_0$ and $[i < \delta \Rightarrow \|B_i\| < \delta]$, and $S \overset{\mathrm{def}}{=} \{i < \delta : B_i \lessdot B_{\delta+1}\}$ is a stationary subset of δ then $B_\delta \lessdot B_{\delta+1}$.

[Why? If $x \in B_{\delta+1} \setminus \{0\}$ then by clause (b) for each $\alpha \in S$ for some $y_\alpha \in B_\alpha \setminus \{0\}$ we have

$$(\forall z \in B_\alpha)[z \cap y_\alpha \neq 0 \rightarrow z \cap x \neq 0].$$

So by Fodor lemma for some j,

$$S_1^* \overset{\mathrm{def}}{=} \{\alpha \in S : y_\alpha \in B_j\}$$

is stationary. And so for some y the set $\{\alpha \in S_1^* : y_\alpha = y\}$ is stationary and y is as required.]

(g) For a Boolean algebra B, $X_1 \lessdot X_2$ (in B) *iff* $X_1 \subseteq X_2 \subseteq B \setminus \{0_B\}$ and every predense subset of X_1 is a predense subset of X_2 *where* Y is a predense subset of X if $Y \subseteq X$ & $\forall x \in X \exists y \in Y (\exists z \in X)(z \subseteq_B x \cap y)$. If $0_B \in X_2$ we mean $X_1 \setminus \{0_B\} \lessdot X_2 \setminus \{0_B\}$.

This definition is compatible with the one in clause (a) and the iteration in clause (b) is still true; also clause (c) holds (the others are not needed here).

(5) If in V we have $P_1 \lessdot P_2 \lessdot P_3$, in V^{P_2} we have $\mathfrak{B}^{P_1} \lessdot \mathfrak{B}^{P_2}$, and in $V^{P_3}, \mathfrak{B}^{P_2} \lessdot \mathfrak{B}^{P_3}$, *then* in $V^{P_3}, \mathfrak{B}^{P_1} \lessdot \mathfrak{B}^{P_3}$, [follows by (4)(c), (4)(d)]; similarly for $\mathfrak{B}^{P_i} \restriction S$.

(6) For a set a and forcing notion P, $\underset{\sim}{G}_P$ is the P-name of the generic set and $a[\underset{\sim}{G}_P] = a \cup \{\underset{\sim}{x}[\underset{\sim}{G}_P] : \underset{\sim}{x} \in a$ is a P-name$\}$. So $a[\underset{\sim}{G}_P]$ is a P-name of a set, and for $G \subseteq P$ generic over V its interpretation is $a[G] = a \cup \{\underset{\sim}{x}[G] : \underset{\sim}{x} \in a$ is a P-name$\}$ ($\underset{\sim}{x}[G]$ is the interpretation of the P-name x).

(7) If $\lambda > \aleph_0$ is a cardinal, N a countable elementary submodel of $(H(\lambda), \in), P \in N$ and $G \subseteq P$ is generic over V, then $N[G] \prec (H(\lambda)^{V^P}, \in)$ (as $H(\lambda)^{V^P} = \{\tau[G] : \underset{\sim}{\tau} \in H(\lambda)$ a P-name$\}$ and if $\Vdash_P$ "$(H(\lambda)^{V^P}, \in) \models \exists x \varphi(x, \underset{\sim}{a})$" then for some P-name $\underset{\sim}{\tau} \in H(\lambda)$ we have $\Vdash_P$ "$(H(\lambda)^{V^P}, \in) \models \varphi(\underset{\sim}{\tau}, \underset{\sim}{a})$"). See III 2.11, I 5.17(1).

(8) Also, if some $p \in G$ is (N, P)-generic then $(N, G) \prec (H(\lambda)^V, \in, G)$ (i.e., G is an extra predicate, so you may write $(N, G \cap |N|)$). Also, if R is any relation (or sequence of relations) on $H(\lambda)^V$, $N \prec (H(\lambda)^V, \in, R)$ (and $P \in N, G \subseteq P$ generic over V) and some $p \in G$ is (N, P)-generic *then* $(N, G) \prec (H(\lambda)^V, \in, R, G)$ and even $(N[G], |N|, R^N, G) \prec (H(\lambda)^{V^P}, \in, H(\lambda)^V, R, G)$. Usually we use a well ordering $<^*_\lambda$ of $H(\lambda)$.

(9) Let $N <_\kappa M$ mean $N \subseteq M$ and $N \cap \kappa$ is an initial segment of $M \cap \kappa$ and $N \prec M$; if we use it for sets (rather than models), the last demand is omitted. Note that if $N \prec M \prec (H(\mu), \in)$, $\kappa < \mu$ and $N \cap \kappa = M \cap \kappa$ then $N <_{\kappa^+} M$.

§1. Semi-Stationarity

1.1. Definition.

(1) A forcing notion P is semiproper if: *for every* regular $\lambda > 2^{|P|}$, any countable $N \prec (H(\lambda), \in)$ to which P belongs, and $p \in P \cap N$ *there is* q such that: $p \leq q \in P$ and q is (N, P)-semi-generic (see below).

(2) For a set a, forcing notion P and $q \in P$, we say q is (a, P)-semi-generic *if*: for every P-name $\underset{\sim}{\alpha} \in a$ of a countable ordinal, $q \Vdash_P$ "$\underset{\sim}{\alpha} \in a$" [i.e., if: $q \Vdash$ "$a[\underset{\sim}{G}_P] \cap \omega_1 = a \cap \omega_1$" see 0.1(6); note $a[\underset{\sim}{G}_P] = \{\underset{\sim}{x}[\underset{\sim}{G}_P] : \underset{\sim}{x} \in a$ a P-name$\}$ if a is closed enough, i.e. for $x \in a$ also $\dot{x} \in a$ where $\dot{x}[G]$ is x].

(3) We call $W \subseteq \mathcal{S}_{<\aleph_1}(A)$ (where $\omega_1 \subseteq A$) *semi-stationary* in A (or in $\mathcal{S}_{<\aleph_1}(A)$ or subset of A) *if* for every model M with universe A and countably many relations and functions, there is a countable $N \prec M$, such that $(\exists a \in W)[N \cap \omega_1 \subseteq a \subseteq N]$, [equivalently, $\{a \in \mathcal{S}_{<\aleph_1}(A) : (\exists b \in W)[a \cap \omega_1 \subseteq$

$b \subseteq a]\}$ is a stationary subset of $\mathcal{S}_{<\aleph_1}(A)$ (i.e., $\neq \emptyset \bmod \mathcal{D}_{<\aleph_1}(A)$). As we allow functions in M, we can require only $N \subseteq M$].

1.2. Claim.

(1) If $W \subseteq \mathcal{S}_{<\aleph_1}(A)$ is stationary and $\omega_1 \subseteq A$ *then* W is a semi-stationary subset of A. Also if $\omega_1 \subseteq A, W \subseteq \mathcal{S}_{<\aleph_1}(A)$ is semi-stationary in A, $C \in \mathcal{D}_{<\aleph_1}(A)$ and $[a \in W \;\&\; b \in C \;\&\; b \cap \omega_1 \subseteq a \subseteq b \Rightarrow b \in W]$ *then* W is stationary (subset of $\mathcal{S}_{<\aleph_1}(A)$).

(2) If $\omega_1 \subseteq A \subseteq B$, and $W \subseteq \mathcal{S}_{<\aleph_1}(A)$ *then:* W is semi-stationary in A *iff* W is semi-stationary in B (so we can omit "in A").

(3) If $W_1 \subseteq W_2 \subseteq \mathcal{S}_{<\aleph_1}(A)$, and W_1 is semi-stationary, *then* W_2 is semi-stationary.

(4) If $|A| = \aleph_1$, $\omega_1 \subseteq A$, $A = \cup_{i<\omega_1} a_i$, a_i increasing continuous in i, with a_i countable, *then* $W \subseteq \mathcal{S}_{<\aleph_1}(A)$ is semi-stationary iff $S_W \stackrel{\text{def}}{=} \{i : (\exists b \in W)[i \subseteq b \subseteq a_i]\}$ is stationary (as a subset of ω_1).

(5) If $p \in P$ is (b, P)-semi-generic, $b \cap \omega_1 \subseteq a \subseteq b$ *then* p is (a, P)-semi-generic.

(6) If $W \subseteq \mathcal{S}_{<\aleph_1}(\lambda)$, $\mu > \lambda$, $W \in N$, $N \prec (H(\mu), \in)$ (hence $|W| < \mu$), and for some $a \in W, N \cap \omega_1 \subseteq a \subseteq N$ *then* W is semi-stationary.

(7) Assume A is an uncountable set, $W \subseteq \mathcal{S}_{<\aleph_1}(A)$, f_1, f_2, are one to one functions from ω_1 into A, and $W_\ell \stackrel{\text{def}}{=} \{a \cup \{\alpha < \omega_1 : f_\ell(\alpha) \in a\} : a \in W\} \subseteq \mathcal{S}_{<\aleph_1}(A \cup \omega_1)$. *Then* W_1 is semi-stationary iff W_2 is semi stationary, so in Definition 1.1(3) (of semi stationarity) we can replace "$\omega_1 \subseteq A$" by "A uncountable".

(8) If A_1, A_2 are uncountable sets, f is a one to one function from A_1 to A_2, $W_2 \subseteq \mathcal{S}_{<\aleph_1}(A_2)$, $W_1 \subseteq \mathcal{S}_{<\aleph_1}(A_1)$ and $[a \in W_1 \Rightarrow f''(a) \in W_2]$ and $[b \in W_2 \Rightarrow (\exists a \in W_1) b \cap f''(A_1) = f''(a)]$ *then*: W_1 is semi-stationary iff W_2 is semi-stationary. If f is onto A_2, necessarily $W_1 = \{a \in \mathcal{S}_{<\aleph_1}(A_1) : f''(a) \in W_2\}$.

Proof. (1) - (5), (7), (8) Left to the reader.

(6) If not, some $M = (\lambda, \ldots, F_n, \ldots)$ exemplifies that W is not semi-stationary, so some such M belongs to N, hence $N \cap \lambda$ is a submodel of M (even an elementary submodel of M), a contradiction. $\square_{1.2}$

1.3. Claim. A forcing notion P is semiproper *iff* the set

$$W_P = \{a \in \mathcal{S}_{<\aleph_1}(P \cup {}^P(\omega_1+1)) : \text{for every } p \in P \cap a \text{ there is } q, \text{ such that } p \le q \in P \text{ and } q \text{ is } (a,P)\text{-semi-generic}\}$$

contains a club of $\mathcal{S}_{<\aleph_1}(P \cup {}^P(\omega_1+1))$ where each $h : P \to (\omega_1+1)$ is interpreted as a P-name $\underset{\sim}{\alpha}_h$ with the property that: if

$$\underset{\sim}{\alpha}^0_h[G] = \min\{h(r) : r \in G\},$$

then $\underset{\sim}{\alpha}_h[G]$ is $\underset{\sim}{\alpha}^0_h[G]$ if the latter is $< \omega_1$ and zero otherwise.

Proof. Immediate. $\square_{1.3}$

1.4. Claim. The following are equivalent for a forcing notion P:

(1) P is semiproper.

(2) P preserves semi-stationarity.

(3) P preserves semi-stationarity of subsets of $\mathcal{S}_{<\aleph_1}(2^{|P|})$.

Proof. (1) $\Rightarrow$ (2). Let $\omega_1 \subseteq A$, and $W \subseteq \mathcal{S}_{<\aleph_1}(A)$ be semi-stationary. Suppose $p \in P$ and $p \Vdash_P$ "W is not semi-stationary". So there are P-names of functions $\underset{\sim}{F}_n (n < \omega)$ from A to A, $\underset{\sim}{F}_n$ is n-place, and $p \Vdash$ "if $a \subseteq A$ is countable closed under $\underset{\sim}{F}_n (n<\omega)$ *then* $\neg(\exists b)[a \cap \omega_1 \subseteq b \subseteq a \;\&\; b \in W]$".

Let λ be regular large enough. Let $N \prec (H(\lambda), \in)$ be countable so that $A, \langle \underset{\sim}{F}_n : n < \omega\rangle, p, P$ belong to N and there is $b \in W$ such that $N \cap \omega_1 \subseteq b \subseteq N$ (such N, b exist as W is semi-stationary by 1.2(2)). Let q be (N,P)-semi-generic, $p \le q \in P$. So $q \Vdash_P$ "$N[\underset{\sim}{G}] \cap \omega_1 = N \cap \omega_1$ and $N \subseteq N[\underset{\sim}{G}]$" hence, for the b above,

$$q \Vdash_P \text{"}N[G] \cap \omega_1 \subseteq b \subseteq N[G]\text{"}.$$

Also $q \Vdash_P$ "$N[G] \cap A$ is closed under the $\underset{\sim}{F}_n$'s" (as $N[G] \prec (H(\lambda)[G], \in)$ and $\underset{\sim}{F}_n[G] \in N[G]$, see Basic Fact 0.1(7) in §0), contradicting the choice of the $\underset{\sim}{F}_n$'s.

(2) $\Rightarrow$ (3). Trivial.

$\neg(1) \Rightarrow \neg(3)$. Let $W = \mathcal{S}_{<\aleph_1}(P \cup {}^P(\omega_1+1)) \setminus W_P$ (where W_P is from 1.3). As $\neg(1), W$ is stationary, so for each $a \in W$ choose $p_a \in P \cap a$ which exemplifies $a \notin W_p$, i.e. there is no $q, p \leq q \in P$ and q is (a, P)-semi-generic. By the normality of the filter $\mathcal{D}_{<\aleph_1}(P \cup {}^P(\omega_1+1))$, for some $p(*) \in P$ the set $W_1 = \{a \in W : p_a = p(*)\}$ is stationary. Hence W_1 is semi-stationary (by 1.2(1)). But by the choice of $\langle p_a : a \in W\rangle$ and W_1, easily $p(*) \Vdash$ "W_1 is not semi-stationary". Clearly $|P \cup {}^P(\omega_1+1)| = 2^{|P|}$ (as P is infinite w.l.o.g.), so let f be a one to one function from $2^{|P|}$ onto $P \cup {}^P(\omega_1+1)$ and let $W_2 = \{a \in \mathcal{S}_{<\aleph_1}(2^{|P|}) : f''(a) \in W_1\}$. By 1.3(8) we have W_2 is semi-stationary and $p(*) \Vdash$ "W_2 is not semi-stationary" so (3) fails. $\square_{1.4}$

1.5. Definition.

(1) $\mathrm{Rss}(\kappa, \lambda)$ (reflection for semi-stationarity) is the assertion that for every semi-stationary $W \subseteq \mathcal{S}_{<\aleph_1}(\lambda)$ there is $A \subseteq \lambda$, $\omega_1 \subseteq A$, $|A| < \kappa$ such that $W \cap \mathcal{S}_{<\aleph_1}(A)$ is semi-stationary (in $S_{<\aleph_1}(\lambda)$).

(2) $\mathrm{Rss}(\kappa)$ is $\mathrm{Rss}(\kappa, \lambda)$ for every $\lambda \geq \kappa$.

(3) $\mathrm{Rss}^+(\kappa, \lambda)$ means that for every semiproper P of cardinality $< \kappa$ we have $\Vdash_P$ "$\mathrm{Rss}(\kappa, \lambda)$".

(4) $\mathrm{Rss}^+(\kappa)$ is $\mathrm{Rss}^+(\kappa, \lambda)$ for every $\lambda \geq \kappa$.

1.5A Remark. In 1.5(3), we could strengthen the statement by replacing "semiproper" by "not collapsing $\aleph_1$" with no change below. If we use below forcing notion from a smaller class we could weaken the statement in 1.5(3) accordingly.

1.6. Claim.

(1) In Definition 1.5(1) we can replace λ by B, when $|B| = \lambda$, $\omega_1 \subseteq B$.

(2) If $\kappa \leq \kappa_1 \leq \lambda_1 \leq \lambda$ and $\mathrm{Rss}(\kappa, \lambda)$, *then* $\mathrm{Rss}(\kappa_1, \lambda_1)$. If $\kappa \leq \lambda_1 \leq \lambda$ and $\mathrm{Rss}^+(\kappa, \lambda)$ *then* $\mathrm{Rss}^+(\kappa, \lambda_1)$. Lastly, if $\mathrm{Rss}^+(\kappa_i, \lambda)$ (for $i < \alpha$) *then* $\mathrm{Rss}^+(\sup_{i<\alpha}\kappa_i, \lambda)$.

(3) If κ is a compact cardinal, *then* $\mathrm{Rss}(\kappa)$;

(4) If κ is a compact cardinal *then* $\mathrm{Rss}^+(\kappa)$.

(5) If κ is measurable, $W_i \subseteq \mathcal{S}_{<\aleph_1}(A)$ and $\bigcup_{i<\kappa} W_i$ is semi-stationary *then* for some $\alpha < \kappa$, $\bigcup_{i<\alpha} W_i$ is semi-stationary.

(6) If κ is a limit of compact cardinals, *then* $\mathrm{Rss}^+(\kappa)$.

(7) If κ is λ-compact, $\lambda = \lambda^{\aleph_0} \geq \kappa$ *then* $\mathrm{Rss}(\kappa, \lambda)$ and even $\mathrm{Rss}^+(\kappa, \lambda)$.

Proof. (1) Trivial.

(2) Use 1.2(2).

(3) Let $\kappa \subseteq A, W \subseteq \mathcal{S}_{<\aleph_1}(A)$, and: $W \cap \mathcal{S}_{<\aleph_1}(B)$ is not semi-stationary for every $B \subseteq A, \omega_1 \subseteq B$, with $|B| < \kappa$.

Define the set of sentences Γ:

$$\Gamma = \Gamma^a \cup \Gamma^b \cup \Gamma^c$$

where (each $c \in A$ serves as an individual constant):

$$\Gamma^a = \{c_1 \neq c_2 : c_1, c_2 \text{ are distinct members of } A\},$$

$$\Gamma^b = \{R(c_0, c_1, \dots, c_l, \dots)_{l<\omega} : c_l \in A, \{c_l : l < \omega\} \in W\},$$

Γ^c is the singleton with unique member (F_n is an n-place function symbol, remember $\omega_1 \subseteq A$):

$$(\forall x_0, x_1, \dots, , x_n, \dots)_{n<\omega} \Big[\text{if } \{x_0, x_1, \dots\} \text{ is closed under } F_n (n < \omega), \textit{ then } \neg(\exists y_0, y_1, \dots)\, (R(y_0, \dots, y_n, \dots) \;\&\; \{x_l : l < \omega, \textstyle\bigvee_{i<\omega_1} x_l = i\} \subseteq \{y_l : l < \omega\} \subseteq \{x_m : m < \omega\})\Big].$$

Every subset of Γ of power $< \kappa$ has a model (if it mentions only $c \in B$ where $B \subseteq A$ and $|B| < \kappa$, then use a model witnessing "$W \cap \mathcal{S}_{<\aleph_1}(B \cup \omega_1)$ is not semi-stationary"). A model M of Γ exemplifies "W is not semi-stationary" (in $|M|$, hence in A by 1.2(2)).

(4) As forcing notions of cardinality $< \kappa$ preserve the compactness of κ.

(5) Let Γ^a, Γ^c be as in the proof of 1.6(4), and:

$$\Gamma^b_i = \{R(c_0, c_1, \dots) : c_l \in A, \{c_l : l < \omega\} \in W_i\}.$$

Now $\Gamma^a \cup \Gamma^c \cup \bigcup_{i<\kappa} \Gamma^b_i$ has no model, hence (using the Łoś theorem for L_{ω_1,ω_1} and $\aleph_1$-complete ultrafilters) for some $\alpha < \kappa$, we have: $\Gamma^a \cup \Gamma^c \bigcup_{i<\alpha} \Gamma^b_i$ has no model.

(6) Easy (use last phrase of 1.6(2)).

(7) Same proof as 1.6(3), (4). $\square_{1.6}$

1.7. Claim.

(1) If $\mathrm{Rss}(\kappa, 2^{|P|})$ and P is not semiproper, *then* P destroys the semi-stationarity of some $W \subseteq \mathcal{S}_{<\aleph_1}(A), |A| < \kappa$ (i.e. some $p \in P$ forces this)
[Why? By (1) $\Leftrightarrow$ (3) from 1.4, for some $p \in P$ and semi-stationary $W \subseteq \mathcal{S}_{<\aleph_1}(2^{|P|})$, we have $p \Vdash_P$ "W is not semi-stationary". By the assumption, for some $A \subseteq 2^{|P|}$ we have: $|A| < \kappa$ and $W_1 \stackrel{\text{def}}{=} W \cap \mathcal{S}_{<\aleph_1}(A)$ is semi-stationary. Clearly by 1.2(3) we have $p \Vdash_P$ "W_1 is not semi-stationary", as required].

(2) If P destroys the semi-stationarity of $W \subseteq \mathcal{S}_{<\aleph_1}(A)$, $|A| = \aleph_1$, *then* P destroys the stationarity of $S_W \subseteq \omega_1$ [with S_W as defined in 1.2(4)], which means that S_W is stationary in V but not in V^P.

(3) If $\mathrm{Rss}(\aleph_2, 2^{|P|})$ and P preserves stationarity of subsets of ω_1, *then* P is semiproper
[Why? By parts (1), (2) above].

(4) If $W \subseteq \mathcal{S}_{<\aleph_1}(A)$ exemplifies the failure of $\mathrm{Rss}(\aleph_2, |A|)$, *then* there is a forcing notion P of power $|A|^{\aleph_0}$, not semiproper but not destroying stationarity of subsets of $\aleph_1$
[Why? Let P be $\{\bar{A} : \bar{A} = \langle A_i : i \leq \alpha\rangle$ is an increasing continuous countable sequence of countable subsets of A, each A_i satisfying $\neg(\exists a \in W)(A_i \cap \omega_1 \subseteq a \subseteq A_i)\}$, ordered by being an initial segment. As forcing with P destroy the semi stationarity of W, clearly P is not semiproper; let us prove that forcing with P preserve the stationarity of subsets of ω_1. If $p \in P$ and $p \Vdash$ "S is not stationary" where S is a stationary set of limit ordinals $< \omega_1$, we can find an increasing continuous sequence $\langle N_i : i < \omega_1\rangle$ of countable elementary submodels of $(H(\beth_7^+), \in)$, with $\{W, p, A\} \in N_0$, $N_i \in N_{i+1}$. So $C = \{\delta < \omega_1 : \delta$ a limit ordinal and

$N_\delta \cap \omega_1 = \delta\}$ is a club of ω_1. By the choice of W, for some club $C_1 \subseteq C$ of ω_1, $\delta \in C_1 \Rightarrow \neg(\exists a)(a \in W \cap \delta \subseteq a \subseteq N_\delta)$, hence we can find $\delta \in C_1 \cap S$ and $q \geq p$ which is (N, P)-generic, an easy contradiction.].

(5) $\mathrm{Rss}(\aleph_2)$ is equivalent to the assertion: every forcing notion preserving stationarity of subsets of ω_1 is semiproper.
[By parts (3), (4) above]. $\square_{1.7}$

1.8. Definition. $\langle P_i, \underset{\sim}{Q}_j : i \leq \alpha, j < \alpha\rangle$ is a semiproper iteration if:

(A) it is an RCS iteration [see Ch. X, §1];

(B) if $i < j \leq \alpha$ are non-limit, then $\Vdash_{P_i}$ "P_j/P_i is semiproper";

(C) for every $i < \alpha$ we have, $\Vdash_{P_{i+1}}$ "$(2^{\aleph_1})^{V^{P_i}}$ is collapsed to $\aleph_1$" (we can use another variant instead).

We shall use not only G_{P_i} (or $\underset{\sim}{G}_{P_i}$) but also G_i (or $\underset{\sim}{G}_i$) for the (name of the) generic subset of P_i.

1.9. Theorem. *Suppose λ is measurable, $\langle P_i, \underset{\sim}{Q}_j : i \leq \lambda, j < \lambda\rangle$ is a semiproper iteration, $|P_i| < \lambda$ for $i < \lambda$, and $\{i < \lambda : \underset{\sim}{Q}_i$ is semiproper $\}$ belongs to some normal ultrafilter D on λ. Then in V^{P_λ}, Player II wins $\eth = \eth(\{\aleph_1\}, \omega, \aleph_2)$.*

1.9A. Remarks. On $\eth$ see Ch. XII, Def. 2.1. or see below.

(1) The game lasts ω moves; on the nth move Player I chooses $f_n : \aleph_2 \to \omega_1$ and Player II chooses $\xi_n < \omega_1$. In the end Player II wins if $A \stackrel{\text{def}}{=} \{i < \aleph_2 : \bigwedge_n \bigvee_m f_n(i) < \xi_m\}$ is unbounded in $\aleph_2$.

(2) We can modify the game by requiring $A \neq \emptyset \bmod E$ for a filter E on ω_2. We then denote the game by $\eth(\{\aleph_1\}, \omega, E)$. The result is true for $E = D$.

(3) By XII 2.5(2) we know the following: if Player II wins $\eth(\{\aleph_1\}, \omega, \aleph_2)$, $\lambda > 2^{\aleph_2}$, N a countable elementary submodel of $(H(\lambda), \in, <^*_\lambda)$, *then* for arbitrarily large $i < \omega_2$, there is $N' \prec (H(\lambda), \in, <^*_\lambda), N'$ countable, $N \subseteq N', i \in N'$ and $N \cap \omega_1 = N' \cap \omega_1$ (hence $N <_{\omega_2} N'$; see Basic Fact 0.1(9) in §0).
If Player II wins $\eth(\{\aleph_1\}, \omega, E)$ (where E is a filter on ω_2) then the set of such i is $\neq \emptyset \bmod E$; so we have equivalence.

(4) Can we demand in (3) (on both see XII §2 when we use E) that $N' \cap i = N \cap i$? If $\{\delta < \omega_2 : \mathrm{cf}(\delta) = \aleph_0\} \in E$ the answer is No. If $\{\delta < \omega_2 : \mathrm{cf}(\delta) = \aleph_1\} \in E$ the answer is Yes provided that we can change the game to $\Game'$: Player I is also allowed to choose regressive functions $F_n : \aleph_2 \to \aleph_2$, and Player II in the nth move has to choose also $\xi'_n < \omega_2$, and in the end Player II wins if $S = \{\delta < \aleph_2 :$ for $n < \omega$, we have $\delta \geq \xi'_n$, and $f_n(\delta) < \bigcup_m \xi_m, F_n(\delta) < \bigcup_m \xi'_m\} \neq \emptyset \bmod E$.

(5) If in the theorem $\Vdash_P$ "$\{\delta < \aleph_2 : Q_\delta$ is semiproper and $\mathrm{cf}(\delta)^{V^P} = \aleph_1\} \neq \emptyset \bmod D$" *then* Player II wins also in this variant (from (4) above). The proof of 1.9 still works.

(6) We can replace $\aleph_1$ by any regular $\theta, \aleph_0 < \theta < \lambda$, (as the range of f_n) and use the game $\Game(\{\theta\}, \mu, E)$, E a normal filter on $\lambda, \langle P_i, \underset{\sim}{Q}_i : i < \lambda\rangle$ is a $(< \theta)$-revised support iteration (see Chapter XIV), such that the set of $i < \lambda$ satisfying the following belongs to D: "in V^{P_i} for $p \in P_\lambda/P_i$ in the game $PG^\omega(p, P_\lambda/P_i, \lambda, \theta)$ (see below and Chapter XII, 1.7(3), 1.4), the second player has a winning strategy".

(7) We can replace in the assumption of 1.9, "D is a normal ultrafilter on κ" by "D is a normal filter on κ" and the second player wins in $\Game'$ $(\{\aleph_1\}, \omega, D)$.

(8) If we use the strong preservation version of theorems, we do not need 1.9 (a weaker version is then proved, e.g. for $\alpha < \lambda$, $(P_\kappa/P_\alpha) * \underset{\sim}{\mathrm{N}}\mathrm{m}$ is semi proper) and is really changed.

Proof of 1.9. Let D be a normal ultrafilter on λ (in V), $A \in D$ a set of (strongly) inaccessible cardinals such that: $(\forall \kappa \in A)[(\forall i < \kappa)(|P_i| < \kappa)$ & Q_κ is semiproper (in V^{P_κ})].

For each $\kappa \in A$ the forcing notion P_λ/P_κ (in V^{P_κ}) is a semiproper forcing, hence for each $p \in P_\lambda/P_\kappa$ in the following game, $P\Game^\omega(p, P_\lambda/P_\kappa, \aleph_1)$, Player II has a winning strategy which we call $F_p(P_\lambda/P_\kappa)$ $(\in V^{P_\kappa})$; if $p = \emptyset_{P_\lambda/P_\kappa}$ we omit p [see Chapter XII, 1.7(3), Definition 1.4]: a play of the game lasts ω-moves, in the nth move Player I chooses a P_λ/P_κ-name $\underset{\sim}{\zeta}_n$ of a countable ordinal and

Player II chooses a countable ordinal ξ_n. Player II wins a play if

$$(\exists q)(p \leq q \in P_\lambda/P_\kappa \,\&\, q \Vdash \text{“}\bigwedge_n [\underset{\sim}{\zeta}_n < \bigcup_{m<\omega} \xi_m]\text{”});$$

without loss of generality the ξ_n are strictly increasing.

Let us describe a winning strategy for Player II in $\partial(\{\aleph_1\}, \omega_1, \aleph_2)$ in $V[G_\lambda]$, where $G_\lambda \subseteq P_\lambda$ is generic over V. In the nth move Player I chooses $f_n : \omega_2 \to \omega_1$, Player II, in addition to choosing $\xi_n < \omega_1$, chooses A_n, $\underset{\sim}{f}'_n$, α_n such that:

(0) $\alpha_n < \alpha_{n+1} < \lambda$; in stage n Player II works in $V[G_{\alpha_n}]$, so D is still an ultrafilter (pedantically: generates an ultrafilter);

(1) $A_n \in D, A_{n+1} \subseteq A_n \subseteq A$ and for all $\delta \in A_n$, we have $\alpha_n < \delta$;

(2) $\Vdash_P$ “$\underset{\sim}{f}'_n : \omega_2 \to \omega_1$”;

(3) $\underset{\sim}{f}'_n[G_\lambda] = f_n$; $\underset{\sim}{f}'_n$ is the first such name so $\underset{\sim}{f}'_n$ is from V;

(4) for $\kappa \in A_n$, $\langle\langle \underset{\sim}{f}'_l(\kappa), \xi_l\rangle : l \leq n\rangle$ is (a P_κ-name of) an initial segment of a play of $P\partial^\omega(\emptyset_{P_\lambda}, P_\lambda/G_\kappa, \aleph_1)$ in which Player II uses his winning strategy $F(P_\lambda/G_\kappa)$, i.e. some condition in G_{α_n} forces this.

How can Player II carry out this strategy? Suppose he arrives at stage n and Player I has chosen $f_n \in V^{P_\lambda}, f_n : \lambda \to \omega_1$. Stipulate $\alpha_0 = -1$. Let $B_n = A_{n-1}$ if $n > 0$ and $B_n = A$ if $n = 0$. Player II chooses for $\underset{\sim}{f}'_n \in V$ the first (by $<^*_\chi$, $\chi = (2^\lambda)^+$) P_λ-name $\underset{\sim}{f}_n$ such that $\underset{\sim}{f}'_n[G_\lambda] = f_n$. Now for every $\kappa \in B_n$, working in $V[G_\kappa]$, he continues the play $\langle\langle \underset{\sim}{f}'_l(\kappa), \xi^0_l\rangle : l < n\rangle$ of $P\partial^\omega(\emptyset_P, P_\lambda/G_\kappa, \aleph_1)$, letting the first player play $\underset{\sim}{f}'_n(\kappa)$, and let $\xi^0_n(\kappa)$ be the choice of the second player according to the strategy $F(P_\lambda/G_\kappa)$. So $\xi^0_n(\kappa) = \underset{\sim}{\xi}^0_n(\kappa)$ is a P_κ-name. Now (in $V[G_{\alpha_{n-1}}]$) for every $p \in P_\lambda$ and $\kappa \in B_n$ there is $q_\kappa \in P_\kappa/G_{\alpha_n}$ compatible with p and forcing a value to $\xi^0_n(\kappa)$. But as $B_n \subseteq A$, and by the choice of the set A (and X 1.6) we know that $P_\kappa = \bigcup_{i<\kappa} P_i$, so we can use the normality of D; so for some $\xi < \omega_1$, $A^n_p \in D$, $A^n_p \subseteq B_n$ and q, we have q is compatible with p in $P_\lambda/G_{\alpha_{n-1}}$ and $(\forall \kappa \in A^n_p)[q_\kappa = q$ and $q \Vdash_{P_\kappa}$ “$\underset{\sim}{\xi}^0_n(\kappa) = \xi$”$]$. So there are such $q \in G_\lambda$, and ξ (which we call ξ_n) and a set which we call A_n. It is easy to choose α_n.

We should still prove that this is a winning strategy. We shall consider one play and work in V, so everything is a P_λ-name (as we are using RCS, no problems arise). I.e. we have $p^* \in P_\lambda$ such that $p^* \Vdash_{P_\lambda}$ "$\langle \underset{\sim}{f}_n, \underset{\sim}{\xi}_n : n < \omega\rangle$ is a play of the game with Player II using his strategy, choosing on the side $\langle \underset{\sim}{f}'_n, \underset{\sim}{\alpha}_n, \underset{\sim}{A}_n : n < \omega\rangle$". Now $\underset{\sim}{f}'_n$, $\underset{\sim}{A}_n$, $\underset{\sim}{\alpha}_n$ are P_λ-names of members of V ($\underset{\sim}{f}'_n$ a P_λ-name of a P_λ-name) so there is a maximal antichain $\mathcal{J}_n$ of P_λ of conditions forcing a value to each of $\underset{\sim}{f}_n$, $\underset{\sim}{\xi}_n$, $\underset{\sim}{f}'_n$, $\underset{\sim}{\alpha}_n$, $\underset{\sim}{A}_n$. But P_λ satisfies the λ-c.c., $P_\lambda = \bigcup\limits_{\alpha<\lambda} P_\alpha$ so for some $\alpha(*) < \lambda$, $\bigwedge\limits_{n<\omega} \mathcal{J}_n \subseteq P_{\alpha(*)}$. Also w.l.o.g. $\alpha(*)$ is bigger than every possible value $\underset{\sim}{\alpha}_n$.

Work in $V[G_{\alpha(*)}]$. Now D is (essentially) an ultrafilter (on λ) in $V[G_{\alpha(*)}]$. Each $\underset{\sim}{A}_n$ is a P_λ-name of a member of V so really there are $< \lambda$ candidates so we can find A_ω, such that for each n we have $\Vdash_{P_\lambda/G_{\alpha(*)}}$ "$A_\omega \subseteq \underset{\sim}{A}_n$," $A_\omega \in D$ (alternatively we can compute $\bigcap\limits_{n<\omega} A_n$ in $V[G_{\alpha(*)}]$). Now for $\kappa \in A_\omega$, $\kappa > \alpha(*)$ the sequence $\langle\langle \underset{\sim}{f}_l(\kappa), \xi_l\rangle : l < \omega\rangle$ is a play of $P\eth^\omega(\emptyset_P, P_\lambda/P_\kappa, \aleph_1)$ where Player II uses his winning strategy (this is a P_κ-name, but fortunately $\langle \underset{\sim}{\xi}_l[G_\kappa] : l < \omega\rangle \in V[G_{\alpha(*)}]$). So there is $q_\kappa \in P_\lambda/P_\kappa$ so that

$$q_\kappa \Vdash_{P_\lambda/P_\kappa} \text{“}\bigwedge_l \underset{\sim}{f}_l(\kappa) < \bigcup_n \xi_n\text{”}$$

(more exactly:

$$q_\kappa \Vdash_{(P_\lambda/G_{\alpha(*)})/(P_\kappa/G_{\alpha(*)})} \text{“}\bigwedge_l \underset{\sim}{f}_l(\kappa) < \bigcup_n \xi_n\text{”},$$

actually q_κ is a $P_\kappa/G_{\alpha(*)}$-name of a P_λ/P_κ-condition).

We can consider q_κ as a P_λ-condition with $\mathrm{Dom}(q_\kappa) \subseteq [\kappa, \lambda)$, because we use RCS iteration. Now easily $\langle q_\kappa : \kappa \in A_\omega\rangle \in V[G_{\alpha(*)}]$, and

$$\Vdash_{P_\lambda/G_{\alpha(*)}} \text{“}\{\kappa \in A : q_\kappa \in \underset{\sim}{G}_\lambda\} \text{ is unbounded in } \lambda\text{”}$$

Why? As every $r \in P_\lambda/G_{\alpha(*)}$ has domain bounded in λ, we have: q_κ is compatible with it for κ large enough. This finishes the proof that the strategy works.

$\square_{1.9}$

1.10. Claim. Suppose κ is measurable, $\bar{Q}$ is a semiproper iteration, $\ell g(\bar{Q}) = \kappa, |P_i| < \kappa$ for $i < \kappa$ and $\{i : Q_i$ semiproper $\}$ belongs to some normal ultrafilter on κ (this holds e.g. if $\{i < \kappa :$ if i is strongly inaccessible and $(\forall j < i)[|P_j| < i]$, then $\underset{\sim}{Q}_i$ is semi proper$\} \in \mathcal{D}_\kappa$). *Then:*

(1) $\mathrm{Rss}^+(\kappa, \lambda)$ implies $\Vdash_{P_\kappa}$ "$\mathrm{Rss}(\kappa, \lambda)$".

(2) If $\underset{\sim}{Q}$ is a P_κ-name of a forcing notion, $(P_\kappa / P_{i+1}) * \underset{\sim}{Q}$ is semiproper for each $i < \kappa$ (i.e. this is forced for P_{i+1}) *then* $\Vdash_{P_\kappa}$ "$\underset{\sim}{Q}$ is semiproper".

(3) We can replace measurability of κ by: κ is strongly inaccessible and $\Vdash_{P_\kappa}$ "Player II wins $\eth(\{\aleph_1\}, \omega_1, \aleph_2)$".

Proof. (1) Let $\underset{\sim}{W}$ be a P_κ-name and $p \in P_\kappa$ be such that $p \Vdash_{P_\kappa}$ "$\underset{\sim}{W} \subseteq \mathcal{S}_{<\aleph_1}(\lambda)$ is semi-stationary".

For $i < \kappa$, let $\underset{\sim}{W}_i = \{a : a \in V^{P_i}, a \in \mathcal{S}_{<\aleph_1}(\lambda)$, and for some $q \in \underset{\sim}{G}_{P_i}, q \Vdash_{P_\kappa}$ "$a \in \underset{\sim}{W}$"$\}$. So $\underset{\sim}{W}_i$ is a P_i-name.

Let χ be regular and large enough, and $<^*_\chi$ a well ordering of $H(\chi)^V$.

Let $p \in G = G_\kappa \subseteq P_\kappa$, G generic over V and $G_i = G \cap P_i$ for $i < \kappa$. In $V[G_\kappa]$, as $\underset{\sim}{W}[G_\kappa]$ is semi-stationary, there is a countable $(N, G_\kappa \cap N) \prec (H(\chi)^V, \in, <^*_\chi, G_\kappa)$, such that for some $a \in \underset{\sim}{W}[G_\kappa]$ we have $N \cap \omega_1 \subseteq a \subseteq N \cap \lambda$, and $p, \underset{\sim}{W}, \lambda, \kappa, \bar{Q}$ belong to N (note: G_κ is considered a relation of those models).

So there are $q \in G_\kappa$ and P_κ-names $\underset{\sim}{N}, \underset{\sim}{a}$ such that $q \Vdash_{P_\kappa}$ "$\underset{\sim}{N}, \underset{\sim}{a}$ are as above", and as $p \in G_\kappa$, without loss of generality $p \leq q$. As N and a are countable subsets of $H(\chi)^V$ and λ respectively and $P_\kappa = \bigcup_{i<\kappa} P_i$ satisfies the κ-c.c. (by X 5.3(3)), for some $i < \kappa$ we have $\underset{\sim}{N}, \underset{\sim}{a}$ are P_i-names, $\Vdash_{P_i}$ "$\underset{\sim}{N} \cap \kappa \subseteq i$" and $q \in P_i$. Now by 1.9 + 1.9A(3), in V^{P_κ}, for arbitrarily large ordinal $\theta < \kappa, N^{[\theta]} \cap \omega_1 = N \cap \omega_1$, and Q_θ is semiproper (if not, replace it by $\theta + 1$), where we let:

$$N^{[\theta]} \overset{\text{def}}{=} \text{Skolem Hull } (N \cup \{\theta\})$$

(in $(H(\chi)^V, \in, <^*_\chi, G_\kappa)$, working in the universe $V[G_\kappa]$ such that $q \in G_\kappa$).

Choose such a $\theta > i$. Now $\theta \in N^{[\theta]}$ and $(N^{[\theta]}, G_\theta) \prec (H(\chi)^V, \in, <^*_\chi, G_\theta)$, as $\theta > i$ clearly $\underset{\sim}{a}[G_\theta] \in \underset{\sim}{W}_\theta[G_\theta]$ and $\omega_1 \cap N^{[\theta]} \subseteq \underset{\sim}{a}[G_\theta] \subseteq N^{[\theta]}$. Let $N_{[\theta]}$ be the

Skolem Hull of $N \cup \{\theta\}$ in $(H(\chi)^V, \in, <^*_\chi, G_\theta)$; note as $\underset{\sim}{N}$ is a P_θ-name, in $V[G_\theta]$ we can compute $\underset{\sim}{N}[G_\theta] = \underset{\sim}{N}[G_\kappa] = N$ (and $\underset{\sim}{a}[G_\theta] = \underset{\sim}{a}[G_\kappa] = a$). Clearly $N^{[\theta]} \cap \omega_1 \subseteq \underset{\sim}{a}[G_\theta] \subseteq N \subseteq N_{[\theta]} \subseteq N^{[\theta]}$; hence by 1.2(6), $V[G_\theta] \models$ "$\underset{\sim}{W}_\theta[G_\theta]$ is a semi-stationary subset of $\mathcal{S}_{<\aleph_1}(\lambda)$" (remembering that in $(H(\chi)^V, \in, <^*_\chi, G_\theta)$ we can interprete $(H(\chi)^{V[G_\theta]}, \in H(\chi)^V, <^*_\chi, G))$.

As $\mathrm{Rss}^+(\kappa, \lambda)$ clearly $V[G_\theta] \models \mathrm{Rss}(\kappa, \lambda)$, hence in $V[G_\theta]$ for some $A \subseteq \lambda, |A| < \kappa$ and $\underset{\sim}{W}_\theta[G_\theta] \cap \mathcal{S}_{<\aleph_1}(A)$ is semi-stationary. As P_κ/P_θ is semiproper (by the choice of θ) it preserves the semi-stationary of $\underset{\sim}{W}_\theta[G_\theta] \cap \mathcal{S}_{<\aleph_1}(A)$ (see 1.4), hence $V[G_\kappa] \models$ "$\underset{\sim}{W}_\theta[G_\theta] \cap \mathcal{S}_{<\aleph_1}(A)$ is semi-stationary", but $\underset{\sim}{W}[G_\theta] \subseteq \underset{\sim}{W}[G_\kappa]$ hence $V[G_\kappa] \models$ "$\underset{\sim}{W}[G_\kappa] \cap \mathcal{S}_{<\aleph_0}(A)$ is semi-stationary".

(2) This is similar: suppose $p \Vdash_{P_\kappa}$ "$\underset{\sim}{N} \prec (H(\chi)^V, \in, <^*_\chi, \underset{\sim}{G}_{P_\kappa})$ and $\underset{\sim}{p}' \in \underset{\sim}{Q} \cap \underset{\sim}{N}$ are counterexample to semiproperness of Q".

Let $G_\kappa \subseteq P_\kappa$ be generic over V and $p \in G_\kappa$. Let $\theta < \kappa$, with $\theta > \sup(\underset{\sim}{N}[G_\kappa] \cap \kappa)$, be such that $\underset{\sim}{N}$ is a P_θ-name and $\sup(\underset{\sim}{N}[G] \cap \kappa) < \kappa$ and $\underset{\sim}{N}[G_\kappa]^{[\theta]} \cap \omega_1 = \underset{\sim}{N}[G_\kappa] \cap \omega_1$. Now work in $V[G_\kappa \cap P_{\theta+1}]$ and use: $\Vdash_{P_{\theta+1}}$ "$(P_\kappa/P_{\theta+1}) * \underset{\sim}{Q}$ is semiproper". (Note that if $\mathrm{Rss}^+(\kappa)$ we can get the result by 1.7(3)). Alternatively prove that forcing with $\underset{\sim}{Q}[H_\kappa]$ preserve semi stationarity of sets.

(3) In the proof of (2) we use this only. In the proof of (1) we could have chosen θ to be a successor ordinal (so $\underset{\sim}{Q}_\theta$ is semiproper). So P_κ/G_θ preserves the semi-stationarity of $\underset{\sim}{W}$, hence $V[G_\kappa] \models$ "$\underset{\sim}{W}$ is semi-stationary". $\square_{1.10}$

1.11. Claim. Suppose $\mathrm{Rss}(\kappa, 2^\kappa)$, κ regular and: $\kappa = \aleph_2$ or $(\forall \mu < \kappa)\mu^{\aleph_0} < \kappa$. *Then* for $\lambda > 2^\kappa$ for every countable $N \prec (H(\lambda), \in, <^*_\lambda)$ to which κ belongs, for arbitrarily large $i < \kappa$, letting $N^{[i]} =$ Skolem Hull $(N \cup \{i\})$, we have $N <_{\omega_2} N^{[i]}$ (note that we do not demand $N \cap \kappa \neq N^{[i]} \cap \kappa$).

1.12. Remark. (1) The "$\kappa = \aleph_2 \ldots$" can be omitted if we replace "for arbitrarily large i" by "for some $i < \kappa$ with $i > \sup(N \cap \kappa)$".
(2) We can replace "$\kappa = \aleph_2$, or ..." by

$(*)_1$ "if $\alpha < \kappa$, then there is a closed unbounded $C \subseteq \mathcal{S}_{<\aleph_1}(\alpha)$ of power $< \kappa$" (see the proof).

It even suffices to assume

$(*)_2$ "for every stationary $W \subseteq \mathcal{S}_{<\aleph_1}(\alpha), (\alpha < \kappa)$ there is a semi-stationary $W' \subseteq W$ of cardinality $< \kappa$".

(3) If in the conclusion we want to get $N <_\kappa N^{[i]}$, we have to replace "$(\exists a \in W)(N \cap \omega_1 \subseteq a \subseteq N)$" in the definition of semi-stationary (Definition 1.1) by "$(\exists a \in W)(N \cap \kappa \subseteq a <_\kappa N \cap \kappa)$".

Proof of 1.11. Let

$$W = \{|N| : N \prec (H(\kappa^+), \in, <^*_{\kappa^+}), N \text{ countable and}$$
$$\text{for some } i_N < \kappa, \text{ for no } i \in [i_N, \kappa) \text{ do we have } N <_{\omega_2} N^{[i]}\}.$$

Assume first that W is a stationary subset of $H(\kappa^+)$. So, as $\mathrm{Rss}(\kappa, 2^\kappa)$ holds (and $|H(\kappa^+)| = 2^\kappa$) there is $A \subseteq H(\kappa^+)$, $\omega_1 \subseteq A$, $|A| < \kappa$ such that: $W_A \overset{\text{def}}{=} \{a \in W : a \subseteq A\}$ is a semi-stationary subset of $\mathcal{S}_{<\aleph_1}(A)$. Without loss of generality (see 1.2(2))

$$M \overset{\text{def}}{=} (A, \in \restriction A, <^*_{\kappa^+} \restriction A) \prec (H(\kappa^+), \in, <^*_{\kappa^+})$$

and $A \cap \kappa$ is an ordinal $< \kappa$ (remember κ is regular).

Remembering that (by the definition of W) for countable elementary submodels $N_1 \subseteq N_2$ of $(H(\kappa^+), \in, <^*_{\kappa^+})$, $|N_1| \in W$, $N_1 \cap \omega_1 = N_2 \cap \omega_1$ implies $|N_2| \in W$; by 1.2(1) clearly W_A is stationary (as a subset of $\mathcal{S}_{<\aleph_1}(A)$). We know by assumption that for some closed unbounded $C \subseteq \mathcal{S}_{<\aleph_1}(A), C$ has cardinality $< \kappa$. So

$$\zeta \overset{\text{def}}{=} \sup\{i_N : |N| \in C \cap W_A\} < \kappa.$$

Now for some club $C_1 \subseteq C$, for every $a \in C_1$, the set $a^{[\zeta]}$ = Skolem Hull of $a \cup \{\zeta\}$ (inside $(H(\kappa^+), \in, <^*_{\kappa^+})$), satisfies $a^{[\zeta]} \cap A = a$, hence $a <_{\omega_2} a^{[\zeta]}$. But we can choose $a \in C_1 \cap W_A$, contradiction.

So W is not stationary and let $C^* \subseteq \mathcal{S}_{<\aleph_1}(H(\kappa^+))$ be a club disjoint to W.

Let $\lambda > 2^\kappa$, so $H(\kappa^+)$, $<^*_{\kappa^+}$, $W \in H(\lambda)$, and let N be such that $\kappa \in N \prec (H(\lambda), \in, <^*_\lambda)$ and N is countable. So $H(\kappa^+) \in N$ (and $<^*_{\kappa^+} = <^*_\lambda \restriction H(\kappa^+)$) hence $W \in N$ and without loss of generality $C^* \in N$. Hence $N \cap H(\kappa^+) \in C^*$, and so for arbitrarily large $i < \kappa$ there is N_1^i such that $N \restriction H(\kappa^+) \prec N_1^i \prec (H(\kappa^+), \in, <^*_{\kappa^+}), N \restriction H(\kappa^+) <_{\omega_2} N_1^i$ and $i \in N_1^i$. Let N^i be the Skolem Hull of $N \cup (N_1^i \cap \kappa)$. We can easily check that $N^i \cap \kappa = N_1^i \cap \kappa$, so N^i is as required.

$\square_{1.11}$

§2. S-Suitable Iterations and Sealing Forcing

2.1. Definition. We say $\bar{Q} = \langle P_i, \underset{\sim}{Q}_j, \mathbf{t}_j : i \leq \alpha, j < \alpha \rangle$ is S-suitable (iteration), where $S \subseteq \omega_1$ is stationary, if:

(A) $\bar{Q}$ is an RCS iteration; (i.e. if we remove the $\mathbf{t}_j$'s);

(B) we denote $|\bigcup_{j<i} P_{j+1}| = \kappa_i = \kappa_i^{\bar{Q}}$ so $\kappa_0 = 1, \kappa_i$ increasing continuous. We demand that κ_i is strictly increasing;

(C) for i successor κ_i is strongly inaccessible;

(D) for $i < j \leq \alpha$ non-limit, P_j/P_i is semiproper;

(E) $\underset{\sim}{Q}_i$ satisfies the κ_{i+1}-c.c., $\aleph_2^{V^{P_{i+1}}} = \kappa_{i+1}$;

(F) if $\mathbf{t}_i = 1, i < j \leq \alpha$ and j is a successor, then $\mathfrak{B}^{P_i} \restriction S \lessdot \mathfrak{B}^{P_j} \restriction S$ (see 0.1(3)(a) + (b)).

Remark: We may, but do not, use $\mathbf{t}_\beta$ which are names. Also the demand "Q_i satisfies the κ_{i+1}-c.c." is just for simplicity.

2.1A. Notation. $\alpha^{\bar{Q}} = \alpha, P_i^{\bar{Q}} = P_i, \underset{\sim}{\bar{Q}}_j^{\bar{Q}} = \underset{\sim}{Q}_j, \mathbf{t}_j^{\bar{Q}} = \mathbf{t}_j$ and (remember and recall 0.1(3)(d)): $\mathfrak{B}^{\bar{Q}} = \cup\{\mathfrak{B}^{P_{i+1}} : i < \ell g(\bar{Q})\}$.

2.2. Claim.

(1) Suppose $\bar{Q} = \langle P_i, \underset{\sim}{Q}_j : i \leq \alpha, j < \alpha \rangle$ is a semiproper iteration (see 1.8 for definition). *Then:*

(a) If $i < \alpha$ is non-limit or $\underset{\sim}{Q}_i$ is semiproper or $\underset{\sim}{Q}_i$ preserves stationarity of subsets of ω_1 from V^{P_i} or i is strongly inaccessible and $\bigwedge_{j<i} |P_j| < i$, *then* every stationary subset of ω_1 in V^{P_i} is also stationary in V^{P_α} (i.e., $\mathfrak{B}[P_i]$ is a subalgebra of $\mathfrak{B}[P_\alpha]$).

(b) $\aleph_1^V = \aleph_1^{V^{P_\alpha}}$.

(c) If $\alpha > \aleph_0$ is strongly inaccessible, and $|P_i| < \alpha$ for $i < \alpha$, *then* P_α satisfies the α-c.c. and so

$$\mathcal{P}(\omega_1)^{V^{P_\alpha}} = \bigcup_{i<\alpha} \mathcal{P}(\omega_1)^{V^{P_i}} \text{ and } V^{P_\alpha} \models \text{“}2^{\aleph_1} = \aleph_2\text{”}.$$

(d) If $\omega_1 \backslash S$ is stationary, each $\underset{\sim}{Q}_i$ is $(\omega_1 \setminus S)$-complete [see V §3], *then* so is P_α, hence forcing by P_α preserve the stationarity of $\omega_1 \setminus S$ and even subsets of it and does not add ω-sequences of ordinals, hence $V^{P_\alpha} \models$ “CH”.

(e) If $\bar{Q} \in N_1 \prec N_2 \prec (H(\lambda), \in), N_2$ countable, $N_1 <_\alpha N_2, \alpha$ strongly inaccessible and belongs to $N_1, \alpha > |P_i|$ for $i < \alpha$ and q is (N_1, P_α)-semi-generic and $i = \min(\alpha \cap N_2 \setminus N_1)$ is regular, *then* q is (N_2, P_i)-semi-generic.

(2) Any S-suitable iteration $\bar{Q}$ is a semiproper iteration and $\mathbf{t}_i = 1 \Rightarrow \mathfrak{B}[P_i]\restriction S \lessdot \mathfrak{B}[P_j]\restriction S]$ *when*: $j \geq i$, and j is: successor or strongly inaccessible satisfying $[\gamma < j \Rightarrow |P_\gamma| < j]$.

(3) If (in (1)) $\kappa < \alpha$ is strongly inaccessible, $|P_i| < \kappa$ for $i < \kappa$, and $\Vdash_{P_\kappa}$ “Rss$(\aleph_2)$” *then* $\underset{\sim}{Q}_\kappa$ (and P_j/P_κ when $\kappa \leq j \leq \alpha$) are semiproper.

Proof. Left to the reader. For instance:

(1)(e) Clearly i is a strong limit [as $\{j < \kappa : j \text{ strong limit }\}$ is a club of κ which belongs to N_1, hence i necessarily belongs to it]. Also we have assumed i is regular hence i is strongly inaccessible; similarly $i > \aleph_0$ and $j < i \Rightarrow |P_j| < i$. If $\mathcal{I} \in N_2$ is a maximal antichain of P_i, *then* by X 5.3(3) for some $j < i$ we have $\mathcal{I} \subseteq P_j$, so that consequently there is such j in N_2, and hence $j \in N_1$ and also the rest is easy.

(2) If j is a successor ordinal use clause (F) of Definition 2.1, if j is strong inaccessible use 2.2(1)(c) and 0.1(4)(e).

(3) By 1.7(3) it is enough to prove that forcing with $\underset{\sim}{Q}_\kappa$ does not destroy the stationarity of any $A \subseteq \omega_1$, $A \in V^{P_\kappa}$. However, by 2.2(1)(c) (and 2.2(2)) for some $\beta < \alpha$, $A \in V^{P_\beta}$. Clearly $A \in V^{P_\beta}$ and is a stationary subset of ω_1 in

$V^{P_{\beta+1}}$. As $P_{\kappa+1}/P_{\beta+1}$ is semiproper, A is also stationary in $(V^{P_{\beta+1}})^{P_{\kappa+1}/P_{\beta+1}} = V^{P_{\kappa+1}} = (V^{P_\kappa})^{\underset{\sim}{Q}_\kappa}$, as required. $\square_{2.2}$

2.2A. Remark. It follows that if κ is strongly inaccessible, and $|P_i| < \kappa$ for $i < \kappa$, and A is a stationary subset of ω_1 in V^{P_κ}, *then* A is a stationary subset of ω_1 in V^{P_α} for every large enough $\alpha < \kappa$.

2.3. Claim. Suppose $\bar{Q} = \langle P_j, \underset{\sim}{Q}_i, \mathbf{t}_i : j \leq \alpha, i < \alpha \rangle$ is an RCS iteration, α a limit ordinal and $S \subseteq \omega_1$ is stationary.

(1) If $\bar{Q}\restriction\beta$ is S-suitable for $\beta < \alpha$, *then* $\bar{Q}$ is S-suitable.

(2) If for $\beta < \alpha$, $\bar{Q}\restriction\beta$ is a semiproper iteration, *then* $\bar{Q}$ is a semiproper iteration.

(3) In (2), if $i < \alpha$ and $\underset{\sim}{A}$ is a P_i-name then: $\Vdash_{P_\alpha}$ "$\underset{\sim}{A} \lessdot \mathfrak{B}^{\bar{Q}}\restriction S$" if and only if $\alpha = \sup\{j < \alpha : \Vdash_{P_{j+1}}$ "$\underset{\sim}{A} \lessdot \mathfrak{B}^{P_{j+1}}\restriction S$"$\}$ if and only if for arbitrarily large $j < \alpha$ we have $\Vdash_{P_j}$ "$\underset{\sim}{A} \lessdot \mathfrak{B}^{P_j}$".

(4) In (2), if $\alpha > |P_i|$ for $i < \alpha$, and α is strongly inaccessible, then $\mathfrak{B}^{\bar{Q}} = \mathfrak{B}^{P_\alpha}$.

Proof. (1) For clause (D) from Definition 2.1 use the semiproper iteration lemma. The other clauses are also obvious.

(2), (3), (4) are also easy. $\square_{2.3}$

2.4. Definition. Let $\bar{\mathcal{A}} = \langle \mathcal{A}_\zeta : \zeta < \xi \rangle$ be a sequence of subalgebras or just subsets of $\mathfrak{B}(= \mathfrak{B}^V)$ such that S belongs to each $\mathcal{A}_\zeta$ where $S \subseteq \omega_1$ stationary.

(1) $\mathrm{Sm}(\bar{\mathcal{A}}, S) = \{A \subseteq S :$ for some $\zeta < \xi$, $\{x \in \mathcal{A}_\zeta : x \neq 0 \bmod \mathcal{D}_{\omega_1}$ and $x \cap A = \emptyset \bmod \mathcal{D}_{\omega_1}\}$ is pre-dense in $\mathcal{A}_\zeta\}$ (we should have written $x/\mathcal{D}_{\omega_1} \in \mathcal{A}_\zeta$ for x, $x \subseteq \omega_1$; Ξ is predense in $\mathcal{A}_\zeta$ means that for every $y \in \mathcal{A}_\zeta$, such that $\mathcal{A}_\zeta \models$ "$y \neq 0$" for some $x \in \Xi$ we have $\mathcal{A}_\zeta \models$ "$x \cap y \neq 0$").

(2) For $\Xi \subseteq \mathfrak{B}^V$ let $\mathrm{seal}(\Xi) = \{\langle a_i : i < \alpha \rangle : \alpha$ is a countable ordinal, and letting $a_\alpha = \bigcup\limits_{i<\alpha} a_i$ we have $a_i \in \mathcal{S}_{<\aleph_1}(\Xi \cup \omega_1)$, a_i $(i \leq \alpha)$ is increasing continuous, each a_i countable and $a_i \cap \omega_1$ is an ordinal which belongs to $\bigcup_{A \in \Xi \cap a_i} A\}$, ordered by being an initial segment.

(3) We define the sealing forcing $\mathrm{Seal}(\bar{\mathcal{A}}, S)$ as the product with countable support of $\{$ $\mathrm{seal}(\Xi)$: for some $\zeta < \xi$, Ξ is a pre-dense subset of $\mathcal{A}_\zeta$ and

$\omega_1 \backslash S \in \Xi\}$. Let $\mathrm{Seal}'(\bar{\mathcal{A}}, S) = \{\bar{c} : \bar{c}$ a partial function from $\mathrm{Sm}(\bar{\mathcal{A}}, S)$, with countable domain, and if $A \in \mathrm{Sm}(\bar{\mathcal{A}}, S) \cap \mathrm{Dom}(\bar{c})$, then $\bar{c}_A$ is a continuously increasing function from some countable $\gamma + 1$ to $\omega_1 \setminus A\}$,
the ordering is defined by:

$$\bar{c}^1 \leq \bar{c}^2 \text{ if } A \in \mathrm{Dom}(\bar{c}^1) \text{ implies } A \in \mathrm{Dom}(\bar{c}^2) \text{ and } \bar{c}^1_A \subseteq \bar{c}^2_A.$$

(4) If $\bar{\mathcal{A}} = \langle \mathcal{A} \rangle$ we write $\mathcal{A}$ instead of $\bar{\mathcal{A}}$ in (1), (2) above and (5) below.

(5) For κ ($> \aleph_0$) strongly inaccessible we define the strong sealing forcing $\mathrm{SSeal}(\bar{\mathcal{A}}, S, \kappa)$ as P_κ, where $\langle P_i, \underset{\sim}{Q}_j : i \leq \kappa, j < \kappa \rangle$ is an RCS iteration with $Q_j = \mathrm{Seal}(\bar{\mathcal{A}}, S)^{P_j} \times \mathrm{Levy}(\aleph_1, 2^{\aleph_1})^{V[P_j]}$.

(6) We call $\Xi \subseteq \mathfrak{B}^V$ semiproper *iff* $\mathrm{seal}(\Xi)$ is a semiproper forcing notion.

(7) $\mathrm{WSeal}(S)$ is the product, with countable support, of $\mathrm{seal}(\Xi)$, Ξ semiproper, $\omega_1 \setminus S \in \Xi$.

(8) For κ not strongly inaccessible, but still $\bar{\mathcal{A}}$-inaccessible, which means:

   (*) $(\forall \mu < \kappa)[\mu^{\aleph_0} < \kappa], \kappa = \mathrm{cf}(\kappa), \kappa^{|\mathcal{A}_\zeta|} = \kappa$ for $\zeta < \xi$, and $\xi = \ell g(\bar{\mathcal{A}}) \leq \kappa, \kappa > \aleph_1$,

   we define the strong sealing forcing $\mathrm{SSeal}^*(\bar{\mathcal{A}}, S, \kappa)$ as P_κ where $\langle P_i, \underset{\sim}{Q}_j : i \leq \kappa, j < \kappa \rangle$ is an RCS iteration; $\underset{\sim}{Q}_j = \mathrm{seal}(\underset{\sim}{\Xi}_j, S)^{V^{P_j}}, \underset{\sim}{\Xi}_j$ is a maximal antichain of $\mathcal{A}_{\zeta(j)}$ to which $\omega_1 \setminus S$ belongs for some $\zeta(j) < \xi$ (in V^{P_j}) and every maximal antichain $\underset{\sim}{\Xi}$ of some $\mathcal{A}_\zeta$ from V^{P_κ} is $\underset{\sim}{\Xi}_j$ for some $j < \kappa$. [P_κ is not neccessarily well defined].

(9) If $\boldsymbol{\Xi} \subseteq \{\Xi : \Xi \subseteq \mathfrak{B}\}$ then $\mathrm{seal}(\boldsymbol{\Xi})$ is the product, with countable support, of $\mathrm{seal}(\Xi)$ for $\Xi \in \boldsymbol{\Xi}$.

2.5. Remarks.

(1) We could have used CS iteration for SSeal and SSeal*.

(2) If every maximal antichain of $\mathfrak{B}^V$ is semiproper, the difference between $\mathrm{WSeal}(S) \times \mathrm{Levy}(\aleph_1, 2^{\aleph_1})$ and $\mathrm{Seal}(\mathfrak{B}^V, S)$ defined in 2.4 (7), (3) respectively, is nominal (i.e. they are equivalent, i.e. have isomorphic completions).

(3) If $\mathcal{A}_\zeta \restriction S \lessdot \mathfrak{B}^V \restriction S$ and $|\mathcal{A}_\zeta| \leq \aleph_1$ for $\zeta < \ell g(\bar{\mathcal{A}})$, *then* $\mathrm{Seal}(\bar{\mathcal{A}}, S)$ is equivalent to $\mathrm{Levy}(\aleph_1, 2^{\aleph_1})$.

(4) If $|\mathcal{A}_\zeta| \leq \aleph_1$ (for every $\zeta < \ell g(\mathcal{A})$) *then* the difference between Seal $(\bar{\mathcal{A}}, S)$ and Seal$'(\bar{\mathcal{A}}, S)$ is nominal (i.e. they are equivalent i.e. have isomorphic completions).

(5) We use below mainly SSeal$(\bar{\mathcal{A}}, S)$, we could use SSeal$^*(\bar{\mathcal{A}}, S, \beth_2)$ instead. Also instead SSeal$(\bar{\mathcal{A}}, S)$ we could use SSeal$'(\bar{\mathcal{A}}, S)$ by 0.1(4)(c) (and see 0.1(g)).

(6) For convenience we shall use mostly SSeal$(\bar{\mathcal{A}}, S)$. So in, e.g., 2.11, 2.13 we can deal with SSeal*.

2.6. Notation. We omit κ in SSeal$(\bar{\mathcal{A}}, S, \kappa)$ *when* it is the first strongly inaccessible. We omit S when $S = \omega_1$. We write $\mathcal{A}$ instead of $\langle \mathcal{A} \rangle$.

2.7. Claim. If in V,

$$\bar{\mathcal{A}}_l = \langle \mathcal{A}^l_\zeta : \zeta < \xi_l \rangle \text{ for } l = 1, 2 \text{ and}$$
$$(\forall \zeta_1 < \xi_1)(\exists \zeta_2 < \xi_2)[\mathcal{A}^1_{\zeta_1} \lessdot \mathcal{A}^2_{\zeta_2} \text{ (inside } \mathfrak{B}^V)],$$
$$(\forall \zeta_2 < \xi_2)(\exists \zeta_1 < \xi_1)[\mathcal{A}^2_{\zeta_2} \lessdot \mathcal{A}^1_{\zeta_1} \text{ (inside } \mathfrak{B}^V)]$$

then

$$\mathrm{Sm}(\bar{\mathcal{A}}^1, S) = \mathrm{Sm}(\bar{\mathcal{A}}^2, S),$$
$$\mathrm{Seal}(\bar{\mathcal{A}}^1, S) = \mathrm{Seal}(\bar{\mathcal{A}}^2, S),$$
$$\mathrm{Seal}'(\bar{\mathcal{A}}^1, S) = \mathrm{Seal}'(\bar{\mathcal{A}}^2, S) \text{ and}$$
$$\mathrm{SSeal}(\bar{\mathcal{A}}^1, S, \kappa) = \mathrm{SSeal}(\bar{\mathcal{A}}^2, S, \kappa).$$

Proof. Easy. $\square_{2.7}$

2.8. Claim.

(1) Let $\Xi \subseteq \mathfrak{B}^V$ be pre-dense. Then Ξ is semiproper *iff*: for λ regular large enough and countable $N \prec (H(\lambda), \in)$ with $\Xi \in N$, *there is* a countable $N', N \prec N' \prec (H(\lambda), \in, <^*_\lambda)$, satisfying $N \cap \omega_1 = N' \cap \omega_1 \in \bigcup_{A \in \Xi \cap N'} A$. [Why? For the implication "$\Rightarrow$" let $q \in \mathrm{seal}(\Xi)$ be $(N, \mathrm{seal}(\Xi))$-semi-generic. Let $\underset{\sim}{a}_i[G_{\mathrm{seal}(\Xi)}]$ be a_i for any $\bar{a} = \langle a_j : j \leq \alpha \rangle \in G_{\mathrm{seal}(\Xi)}$ whenever $\alpha > i$ so $\underset{\sim}{C} = \{\underset{\sim}{a}_i \cap \omega_1 : i < \omega_1\}$ is forced to be a club of ω_1. So $\underset{\sim}{C} \in N$, hence

as q is $(N, \mathrm{seal}(\Xi))$- semi-generic, necessarily $q \Vdash$ "$\delta \stackrel{\mathrm{def}}{=} N \cap \omega_1 \in \underset{\sim}{C}$". In fact $\delta = \underset{\sim}{a}_\delta \cap \omega_1 = \bigcup_{i<\delta} \underset{\sim}{a}_i \cap \omega_1$, so possibly increasing q, for some $\langle b_i : i \leq \delta\rangle$ $q \Vdash$ "$\underset{\sim}{a}_i = b_i$ for $i \leq \delta$", so

$$q \Vdash \text{“}\delta = \omega_1 \cap (\text{Skolem hull in } (H(\lambda), \in, <^*_\lambda) \text{ of } |N| \cup b_\delta = |N| \cup \bigcup_{i<\delta} b_i)\text{”}.$$

So this Skolem hull is N' as required. For the implication "$\Leftarrow$" use 2.8(4) below.]

(2) $\Vdash_{\mathrm{seal}(\Xi)}$ "$\Xi \subseteq \mathfrak{B}^{[\mathrm{seal}(\Xi)]}$ is absolutely pre-dense" (absolutely means for extensions not collapsing $\aleph_1$; more specifically in this chapter, there is a list $\langle A_i : i < \omega_1\rangle$ of members of Ξ and a club C of ω_1 such that $\delta \in C \Rightarrow \delta \in \bigcup_{i<\delta} A_i$). [Why? Let $\langle \underset{\sim}{a}_i : i < \omega_1\rangle$ be as in the proof of 2.8(1), so let A_i be such that $\langle A_i : i < \delta\rangle$ lists the member of Ξ in a_δ for limit $\delta < \omega_1$.]

(3) WSeal(S) is semiproper and $\Vdash_{\mathrm{WSeal}(S)}$ "*if* $\Xi \in V$ is semiproper in $\mathfrak{B}^V$ and $(\omega_1 \backslash S) \in \Xi$, *then* Ξ is absolutely pre-dense in $\mathfrak{B}^{[\mathrm{WSeal}(S)]}$". [Why? For semiproperness use 2.8(8) below; for absoluteness use 2.8(2) above.]

(4) seal(Ξ) is A-complete (see V §3) for $A \in \Xi$; so WSeal(S) is $(\omega_1 \setminus S)$-complete. [Why? Think.]

(5) If Ξ is pre-dense in $\mathfrak{B}[V]$, *then* seal(Ξ) preserves stationarity of subsets of ω_1; if $\mathcal{A} \subseteq \mathfrak{B}^V, \Xi$ a pre-dense subset of $\mathcal{A} \setminus \{\emptyset\}$ *then* seal(Ξ) preserves stationary of subsets of ω_1 which belongs to $\mathcal{A}$ or just are not in $\mathrm{Sm}(\mathcal{A}, S)$. [Why? Use 2.8(4) as any A-complete forcing notion surely preserve the stationarity of subsets of A.]

(6) The forcing notion seal(Ξ) forces $|\Xi| \leq \aleph_1$ and has cardinality $\leq (|\Xi| + \aleph_1)^{\aleph_0}$. The forcing notion Seal$(\bar{\mathcal{A}}, S)$ is $(\omega_1 \setminus S)$-complete; SSeal*$(\bar{\mathcal{A}}, S, \kappa)$ and even any initial segment of such iteration of length κ is $(\omega_1 \setminus S)$-complete and if $\kappa > \aleph_0$ is $\bar{\mathcal{A}}$-inaccessible and $S \subseteq \omega_1$ is stationary *then* it satisfies the θ-c.c. if $\theta = \mathrm{cf}(\theta) > |\mathcal{A}_\zeta|$ for $\zeta < \ell g(\bar{\mathcal{A}})$ and $\bigwedge_{\alpha<\kappa} |\alpha|^{\aleph_0} < \theta$. If $\kappa > \aleph_0$ is strongly inaccessible *then* SSeal$(\bar{\mathcal{A}}, S, \kappa)$ satisfies the κ-c.c. and is $(\omega_1 \setminus S)$-complete.

(7) If $\mathrm{Rss}(\aleph_2, \beth_2(\aleph_1))$ *then* for every pre-dense $\Xi \subseteq \mathfrak{B}(V)$, $\mathrm{seal}(\Xi)$ is semiproper.
[Why? By 1.7(3) and 2.8(5).]
In this case $\mathrm{Seal}(\mathfrak{B}^V, S)$, $\mathrm{SSeal}(\mathfrak{B}^V, S, \kappa)$, $\mathrm{SSeal}^*(\mathfrak{B}^V, S, \kappa)$ are semiproper.

(8) For λ regular large enough, and countable $N \prec (H(\lambda), \in, <^*_\lambda)$ *there is* a countable $N', N \prec N' \prec (H(\lambda), \in, <^*_\lambda)$ satisfying: $N \cap \omega_1 = N' \cap \omega_1$ and for every semiproper $\Xi \subseteq \mathfrak{B}^V$ we have: $[\Xi \in N \Rightarrow N' \cap \omega_1 \in \bigcup_{A \in \Xi \cap N'} A$ [use part (1) repeatedly ω-times] and even $\Xi \in N' \Rightarrow N' \cap \omega_1 \in \bigcup_{A \in \Xi \cap N'} A$ [use the previous statement repeatedly ω-times]. $\square_{2.8}$

2.9. Claim. Suppose $\bar{\mathcal{A}} = \langle \mathcal{A}_\zeta : \zeta < \xi \rangle$ is an increasing sequence of subalgebras or just subsets of $\mathfrak{B}$, $\kappa > \aleph_0$ is strongly inaccessible or just $\mathrm{SSeal}^*(\bar{\mathcal{A}}, S, \kappa)$ is well defined. Assume $\langle \bar{\mathcal{A}}, \kappa \rangle \in N \prec (H(\lambda), \in)$, N countable, $P \stackrel{\text{def}}{=} \mathrm{SSeal}(\bar{\mathcal{A}}, S, \kappa)$ or $P = \mathrm{SSeal}^*(\bar{\mathcal{A}}, S, \kappa)$ respectively and

$\oplus^N_{\bar{\mathcal{A}},S}$ if $\Xi \in N$ is a pre-dense subset of $\mathcal{A}_\zeta$ for some $\zeta \in N \cap \xi$ and $\omega_1 \setminus S \in \Xi$, then $N \cap \omega_1 \in \bigcup_{A \in \Xi \cap N} A$.

Then for every $p \in P \cap N$, *there is* $q \in P$, (N, P)-generic, $p \le q, q$ force a value to $\underset{\sim}{G}_P \cap N$ and $q \Vdash$ "$\oplus^{N[\underset{\sim}{G}_P]}_{\mathcal{A},S}$ holds".

Proof. We have to find q, $p \le q \in P$, which is (N, P)-generic. We first show:

(*) if $\underset{\sim}{\zeta}, \underset{\sim}{\Xi} \in N$ are P-names, $\Vdash_P$ "$\underset{\sim}{\Xi}$ is a pre-dense subset of $\mathcal{A}_{\underset{\sim}{\zeta}}$", $p \in N \cap P$, *then* for some p^2, $p \le p^2 \in N \cap P$, and for some A, ζ we have $p^2 \Vdash$ "$\underset{\sim}{\zeta} = \zeta$ and $A \in \underset{\sim}{\Xi} \cap N \cap \mathcal{A}_\zeta$" (so $A \in V$, and $A \in N \cap \mathcal{A}_\zeta$ and $A \in V$) and $N \cap \omega_1 \in A$.

Proof of (*). We can find p^0, $p \le p^0 \in N \cap P$, and ζ such that $p^0 \Vdash$ "$\underset{\sim}{\zeta} = \zeta$" (so necessarily $\zeta \in N$). Next define

$$\Upsilon = \{A \in \mathcal{A}_\zeta : \text{ for some } p^1,\ p \le p^1 \in P, \text{ and } p^1 \Vdash \text{ "} A \in \underset{\sim}{\Xi} \text{"}\}.$$

Clearly $\Upsilon \in N$, $\Upsilon \in V$, and Υ is a pre-dense subset of $\mathcal{A}_\zeta$, $\zeta \in N$.
By $\oplus^N_{\bar{\mathcal{A}},S}$ there is $A \in \Upsilon \cap N$ such that $N \cap \omega_1 \in A$. By the definition of Υ there is p^2, $p^0 \le p^2 \in P$ and $p^2 \Vdash$ "$A \in \underset{\sim}{\Xi}$". As p^0, A and $\underset{\sim}{\Xi}$ are all in N, we

can choose such p^2 in N, thus finishing the proof of $(*)$.

Now we continue with the proof of 2.9. We define p_n for $n < \omega$ such that:

(a) $p_0 = p$, $p_{n+1} \geq p_n$;

(b) $p_n \in P_\kappa \cap N$;

(c) for every dense subset $\mathcal{J}$ of P_κ which belongs to N for some n, $p_{n+1} \in \mathcal{J}$;

(d) *if* $j \in \kappa \cap N, \underset{\sim}{\Xi}, \underset{\sim}{\zeta}$ are P_j-names from N and $\Vdash_{P_j}$ "$\underset{\sim}{\zeta} < \xi$ and $\underset{\sim}{\Xi} \subseteq \mathcal{A}_\zeta$ is pre-dense" *then* for some $n < \omega$ and $B \in \mathfrak{B}^V \cap N$, we have $N \cap \omega_1 \in B$ and

$$p_{n+1} \restriction j \Vdash_{P_j} \text{“}B \in \underset{\sim}{\Xi}\text{”}.$$

This clearly suffices, as (using the notation of Definition 2.4(5)):

(α) for $j \in N \cap \kappa$ we have $(\bigcup_{n<\omega} p_n)(j)$ is in Q_j by (d), and

(β) $\bigcup_{n<\omega} p_n$ is (N, P)-generic by (c).

So we can assign the tasks, and for satisfying (b) and (c) there is no problem. For (d) use $(*)$. $\square_{2.9}$

2.10 Claim. Suppose

(a) $\mathrm{seal}(\Xi)$ is semiproper for every maximal antichain Ξ of $\mathfrak{B}^V$ to which $\omega_1 \setminus S$ belongs, $\bar{\mathcal{A}} = \langle \mathfrak{B}^V \rangle = \langle \mathcal{A}_0 \rangle$

or

(a)′ $\bar{\mathcal{A}} = \langle \bar{\mathcal{A}}_\zeta : \zeta < \xi \rangle$, $\mathcal{A}_\zeta \subseteq \mathfrak{B}^V$, and $\mathrm{seal}(\Xi)$ is semiproper for any predense subset Ξ of $\mathcal{A}_\zeta$, $\zeta < \xi$

and

(b) $\kappa > \aleph_0$ is strongly inaccessible

or at least

(b)′ $\kappa > \aleph_0$ is inaccessible or just $|\mathcal{A}_\zeta|$-inaccessible for $\zeta < \xi$ (see 2.4(5)).

Then $P \overset{\text{def}}{=} \mathrm{SSeal}(\bar{\mathcal{A}}, S, \kappa)$ if (b) or $P \overset{\text{def}}{=} \mathrm{SSeal}^*(\bar{\mathcal{A}}, S, \kappa)$ if (b)′ (both well defined), is semiproper, have the κ-c.c., is $(\omega_1 \setminus S)$-complete and $\Vdash_P$ "$(\mathcal{A}_\zeta \restriction S) \lessdot (\mathfrak{B}^P \restriction S)$" (and in case (b)′ if $\theta = \mathrm{cf}(\theta) > |\mathcal{A}_\zeta|^{\aleph_0}$, θ-c.c.).

Remark. Some points in the proof are repeated in 2.11.

Proof. The $(\omega_1 \setminus S)$-completeness is trivial by the definition of P and Ch. V, Def. 1.1 (and the preservation theorem there i.e. by 2.8(a)).

For semiproperness let λ be regular and large enough, and $N \prec (H(\lambda), \in)$ countable, $P \in N$ and $p \in P \cap N$. Applying repeatedly 2.8(1) (or directly 2.8(8)), there is N', $N \prec N' \prec (H(\lambda), \in)$, $N \cap \omega_1 = N' \cap \omega_1, N'$ countable, and for every maximal antichain $\Xi \subseteq \mathfrak{B}$ (or just pre-dense $\Xi \subseteq \mathfrak{B}^V$ if (a) or predense subset Ξ of $\mathcal{A}_\zeta$ for some $\zeta < \xi$, if (a)$'$):

$$\Xi \in N', N \cap \omega_1 \in S \Rightarrow N \cap \omega_1 = N' \cap \omega_1 \in \bigcup_{A \in \Xi \cap N'} A.$$

Now use 2.9. (with $\langle \mathfrak{B}^V \rangle, N'$ here standing for $\langle \mathcal{A}_\zeta : \zeta < \xi \rangle, N$ there).

So we have proved that P is semiproper and by the present proof and the Δ-system lemma (alternativelly if κ is strongly inaccessible by 2.2(1)(c) or 2.8(6)) P has the κ-c.c., hence $\Vdash_P$ "$(\mathcal{A}_\zeta \restriction S) \lessdot (\mathfrak{B}^P \restriction S)$" follows from the definition of P as every P_κ-name of a subset of some $\mathcal{A}_\zeta$ is a P_j-name for some $j < \kappa$ (as P_κ satisfies the κ-c.c.). $\square_{2.10}$

2.11. Claim. If $\bar{\mathcal{A}} = \langle \mathcal{A}_\zeta : \zeta < \xi \rangle, \mathcal{A}_\zeta \restriction S \lessdot \mathfrak{B}^V \restriction S$ for $\zeta < \xi$, each $\mathcal{A}_\zeta \restriction S$ satisfies the $\aleph_2$-c.c. (e.g. has power $\leq \aleph_1$) and $\kappa > \aleph_0$ is strongly inaccessible, *then*

(1) $P_\kappa \overset{\text{def}}{=} \text{SSeal}(\bar{\mathcal{A}}, S, \kappa)$ is proper;

(2) $\Vdash_{P_\kappa}$ "$\mathcal{A}_\zeta \restriction S \lessdot \mathfrak{B}^{P_\kappa} \restriction S$ for $\zeta < \xi$";

(3) in fact, P_κ is $(\omega_1 \setminus S)$-complete, strongly proper and satisfies the κ-c.c. and $\Vdash_{P_\kappa}$ "$\kappa = \aleph_2 = 2^{\aleph_1}$";

(4) if $\omega_1 \setminus S$ is stationary, P_κ does not add ω-sequences of ordinals.

Proof. (1) Let λ be regular large enough and $N \prec (H(\lambda), \in)$ countable, $\bar{Q} \in N$ (hence $P_\kappa \in N$) and $p \in P_\kappa \cap N$. We want to apply 2.9, so we have (and it suffices) to verify $\oplus$ there, i.e.

(**) if $\mathcal{A}_\zeta \restriction S \lessdot \mathfrak{B}[V] \restriction S$, $\mathcal{A}_\zeta$ satisfies the $\aleph_2$-c.c., $\mathcal{A}_\zeta \in N \prec (H(\chi), \in, <^*_\chi)$, N countable, $\Xi \subseteq \mathcal{A}_\zeta$ is a pre-dense subset of $\mathcal{A}_\zeta$ and $\omega_1 \setminus S \in \Xi$ *then* $N \cap \omega_1 \in \bigcup\{A : A \in \Xi \cap N\}$.

Proof of (**). As $\mathcal{A}_\zeta \restriction S \models$ "$\aleph_2$"-c.c., clearly without loss of generality $|\Xi| \leq \aleph_1$, so let $\Xi = \{A_i : i < \omega_1\}$ (as $\Xi \neq \emptyset$ this is possible) and say $A_0 = \omega_1 \setminus S$.

Since $\mathcal{A}_\zeta \restriction S \lessdot \mathfrak{B}^V \restriction S$, clearly Ξ is pre-dense in $\mathfrak{B}^V$, hence we know $\{\delta : \delta \in \bigcup_{i<\delta} A_i\} \in \mathcal{D}_{\omega_1}$ (otherwise the complement contradicts the pre-density of Ξ in $\mathfrak{B}^V$), so there is a closed unbounded $C \subseteq \omega_1$ such that $C \subseteq \{\delta : \delta \in \bigcup_{i<\delta} A_i\}$. As $\Xi \in N$ without loss of generality $\langle A_i : i < \omega_1 \rangle \in N$ and without loss of generality $C \in N$. As $N \prec (H(\lambda), \in)$ clearly $C \cap N$ is unbounded in $N \cap \omega_1$, hence $N \cap \omega_1 = \sup(C \cap N \cap \omega_1) \in C$, so $N \cap \omega_1 \in \bigcup\{A_i : i \in N \cap \omega_1\}$, so for some $j \in N \cap \omega_1, N \cap \omega_1 \in A_j$. But $\langle A_i : i < \omega_1 \rangle \in N$ so $A_j \in N$, as required.

(2) If $A \in \mathcal{P}(\omega_1)^{V^{P_\kappa}}$ then as P_κ satisfies the κ-c.c. (by 2.10 or as by part (1), $\{p \in P_\kappa : \mathrm{Dom}(p) \text{ is countable}\}$ is dense in P_κ, clearly we can apply the Δ-system lemma) for some $\alpha < \kappa, A \in \mathcal{P}(\omega_1)^{V^{P_\alpha}}$, and so by the definition of $\mathrm{SSeal}(\bar{\mathcal{A}}, S, \kappa)$, if $A/\mathcal{D}_{\omega_1}$ is disjoint to a dense subset of $x \in \mathcal{A}_\zeta, A \subseteq S, \zeta < \xi$ then we "shoot" a club through its completion in the $(\beta+1)$-th iterand in the iteration defining $\mathrm{SSeal}(\bar{\mathcal{A}}, S, \kappa)$ for $\beta \in (\alpha, \kappa)$ large enough. Why? As $V^{P_\kappa} \models$ "$|\mathcal{A}_\zeta| \leq \aleph_1$" (as P_1 collapses $2^{\aleph_1}$ to $\aleph_1$ see 2.4(5)) there is $\beta, \alpha < \beta < \kappa$ such that for every $x \in \mathcal{A}_\zeta$, if $x \cap A$ is not stationary in V^{P_κ}, then it is not stationary in V^{P_β}.

(3) Easy (strong properness hold by the proof of 2.9 and use IX 2.7, 2.7A for preservation of strong properness or prove directly).

(4) By 2.8(4) and V §3. $\square_{2.11}$

2.12. Claim. Let $\bar{Q} = \langle P_i, \underset{\sim}{Q}_j : i \leq \alpha, j < \alpha \rangle$ be a semiproper iteration, and α be a limit ordinal. Suppose $\Vdash_{P_\alpha}$ "$\underset{\sim}{\Xi} \subseteq \mathfrak{B}^{\bar{Q}}$ is pre-dense" and $i < \alpha$. *Then* (a) $\Leftrightarrow$ (b)$^+$ $\Rightarrow$ (b), where:

(a) $(P_\alpha/P_i) * \mathrm{seal}(\underset{\sim}{\Xi})$ is semiproper (in V^{P_i});

(b) If λ is regular large enough, $\bar{Q} \in N \prec (H(\lambda), \in, <^*_\lambda), N$ countable, $\underset{\sim}{\Xi} \in N$, $p \in N \cap P_\alpha$, $i \in N \cap \alpha$, $q \in P_i$ is (N, P_i)-semi-generic, $p \restriction i \leq q$ *then* there are $N^1, p^1, q^1, \underset{\sim}{A}$ and j such that:
  (i) $N \prec N^1 \prec (H(\lambda), \in, <^*_\lambda)$,
  (ii) N^1 is countable, $N^1 \cap \omega_1 = N \cap \omega_1$,
  (iii) $p \leq p^1 \in N^1 \cap P_\alpha$,
  (iv) $i < j < \alpha$, j a non-limit ordinal,

(v) $j \in N^1$,

(vi) $q \leq q^1 \in P_j$,

(vii) q^1 is (N^1, P_j)-semi-generic,

(viii) $p^1 \restriction j \leq q^1$,

(ix) $\underset{\sim}{A} \in N^1$ is a P_j-name,

(x) $q^1 \Vdash$ "$N^1 \cap \omega_1 \in \underset{\sim}{A}$",

(xi) $q^1 \cup p^1 \restriction [j, \alpha) \Vdash_{P_\alpha}$ "$\underset{\sim}{A} \in \underset{\sim}{\Xi}$";

$(b)^+$ Like (b) but N is a P_i-name and N^1 is a P_i-name.

Remark. There is not much difference if in clause (b) (or $(b)^+$) we replace clause (ix) by

(ix) $\underset{\sim}{A} \in N$ is a P_j-name

but then j is allowed to be P_i-name.

Proof. <u>(a) $\Rightarrow$ $(b)^+$</u> Let $\underset{\sim}{Q} \overset{\text{def}}{=} \text{seal}(\underset{\sim}{\Xi})$ and let $q \in G_i \subseteq P_i$, G_i generic over V. In $V[G_i]$, apply the definition of "$(P_\alpha/P_i) * \text{seal}(\underset{\sim}{\Xi})$ is semi porper" to the model $N = \underset{\sim}{N}[G_i]$ and the condition p, and get a condition q^0, so q^0 is $(\underset{\sim}{N}, (P_\alpha/P_i)*\underset{\sim}{Q})$-semi-generic. Let G be such that $q^0 \in G \subseteq P_\alpha * \underset{\sim}{Q}$, $G_i \subseteq G$, and G is generic over V. So by the definition of $\underset{\sim}{Q}$ =seal($\underset{\sim}{\Xi}$) for some $A \in \underset{\sim}{\Xi}[G_\alpha] \cap N[G]$ we have $N \cap \omega_1 = N[G] \cap \omega_1 \in A$. As $A \in \underset{\sim}{\Xi}[G_\alpha] \subseteq \mathfrak{B}^{\overline{Q}} = \bigcup_{j<\alpha} \mathfrak{B}[P_{j+1}]$, for some $j_0 \in \alpha \cap N$, $A \in \mathfrak{B}[P_{j_0+1}]$, and there is a P_{j_0+1}-name $\underset{\sim}{A} \in N[G]$ such that $\underset{\sim}{A}[G] = A$, and without loss of generality q^0 forces this. Now

$$\mathcal{I} = \{r : r \in P_\alpha \text{ and } r \text{ is above } p \text{ or incompatible with } p \text{ and}$$
$$r \Vdash_{P_\alpha} \text{“}\underset{\sim}{A} \in \underset{\sim}{\Xi}\text{” or } r \Vdash_{P_\alpha} \text{“}\underset{\sim}{A} \notin \underset{\sim}{\Xi}\}$$

is a dense subset of P_α and $\underset{\sim}{r}$ = the $<^*_\lambda$-least member of $\mathcal{I}$ which belongs to $\underset{\sim}{G}_\alpha$ is a P_α-name, and $\mathcal{I} \in N$, $\underset{\sim}{r} \in N$. Hence $\underset{\sim}{r}[G] \in N[G]$ and clearly $\underset{\sim}{r}[G]$ is compatible with q^0, $p \leq \underset{\sim}{r}[G]$ and $\underset{\sim}{r}[G] \vDash$ "$\underset{\sim}{A} \in \Xi$", so w.l.o.g. $\underset{\sim}{r}[G] \leq q^0$. Let N^1 be the Skolem hull of $N \cup \{j_0, \underset{\sim}{A}, \underset{\sim}{r}[G]\}$ in $(H(\lambda), \in, <^*_\lambda)$, let $j = j_0 + 1$, $q^1 = q^0 \restriction j$ and $p_1 = \underset{\sim}{r}[G]$.

<u>(a) $\Rightarrow$ (b)</u> Similar proof.

<u>$(b)^+ \Rightarrow$ (a)</u> Use $(b)^+$. Specifically, for $i < \alpha$ let $G_i \subseteq P_i$ be generic over V, $i < \alpha$.

Assume the desired conclusion in clause (a) fails then this is exemplified by some N, $(p, \underset{\sim}{r})$ where $N \prec (H(\lambda)^{V[G_i]}, \in)$ is countable, $(p, \underset{\sim}{r}) \in (P_\alpha/G_i) * \mathrm{seal}(\underset{\sim}{\Xi})$ and $(p, \underset{\sim}{r}) \in N$ (where $N \in V[G_i]$). So for some $q_0 \in G_i$ and $\underset{\sim}{x}$ we have: $\underset{\sim}{x}$ is a P_i-name, $\underset{\sim}{x}[G_i] = N$ and $q_0 \Vdash_{P_i}$ "$\underset{\sim}{x}$ and $(p, \underset{\sim}{r}) \in (P_\alpha/P_i) * \mathrm{seal}(\underset{\sim}{\Xi})$ form a counterexample to semiproperness".

Clause $(\mathrm{b})^+$ applied to $\underset{\sim}{x}$, q_0, p gives $\underset{\sim}{N}^1, p^1, q^1, \underset{\sim}{A}_{q(*)}$ and j as there and w.l.o.g. $q^1 {\restriction} i \in Q_i$ and let $N^1 = \underset{\sim}{N}^1[G_i]$.

As P_α/P_j is semiproper, there is $q^2 \in P_\alpha$ which is (N^1, P_α)-semigeneric, $p \leq q^2$ and $q^1 = q^2 {\restriction} j$ and let G_α be such that $q^2 \in G_\alpha \subseteq P_\alpha$ and G_α is generic over V. By the choice of q_0 (which is $\leq q_1 \leq q^1 \leq q^2 \in G_\alpha$) without loss of generality $G_i = G_\alpha \cap P_i$. So $\underset{\sim}{\Xi}, p, \underset{\sim}{r}, \underset{\sim}{A} \in N[G_\alpha]$, where on $\underset{\sim}{A}$ see clauses (ix), (x), (xi) and as $\mathrm{seal}(\underset{\sim}{\Xi}[G_\alpha])$ is $\underset{\sim}{A}[G_\alpha]$-complete there is $r^2 \in \mathrm{seal}(\underset{\sim}{\Xi}[G_\alpha])$ which is $(N^1[G_\alpha], \mathrm{seal}(\underset{\sim}{\Xi}[G_\alpha])$-semiproper and above $\underset{\sim}{r}[G_\alpha]$. So for some $\underset{\sim}{r}^*$, $q^2 \Vdash_{P_\alpha * \mathrm{seal}(\Xi)}$ " $\underset{\sim}{r}^*$ is above $\underset{\sim}{r}$ and is $(N[\underset{\sim}{G}_\alpha], (P_\alpha/G_i) * \mathrm{seal}(\underset{\sim}{\Xi}))$-semi generic". So $(q^2, \underset{\sim}{r}^*)$ contradict the choice of q_0 and we are done. $\square_{2.12}$

2.13 Claim. Let $\bar{Q} = \langle P_i, \underset{\sim}{Q}_j : i \leq \alpha, j < \alpha \rangle$ be a semiproper iteration and α be a limit ordinal.

(1) If we have P_α-name $\underset{\approx}{\Xi}$ satisfying $\underset{\approx}{\Xi} \subseteq \underset{\approx}{\Xi}^* = \{\underset{\sim}{\Xi} \in V^{P_\alpha} : \underset{\sim}{\Xi}$ is a P_α-name of a maximal antichain or just a pre-dense subset of $\mathfrak{B}^{\bar{Q}}$, such that for every $i < \alpha$, $(P_\alpha/P_{i+1})*$ $\mathrm{seal}(\underset{\sim}{\Xi})$ is semiproper (i.e. this is $\Vdash_{P_{i+1}})\}$ *then* $(P_\alpha/P_{i+1})*$ $\mathrm{Seal}(\underset{\approx}{\Xi})$ is semiproper for every $i < \alpha$.

(2) If

(*) $(P_\alpha/P_{i+1})*$ $\mathrm{seal}(\underset{\sim}{\Xi})$ is semiproper for every $i < \alpha$ and maximal antichain (or just a pre-dense subset) $\underset{\sim}{\Xi}$ of $\mathfrak{B}^{\bar{Q}}$ (from V^{P_α}) to which $\omega_1 \setminus S$ belongs,

then for every $i < \alpha$, $(P_\alpha/P_{i+1}) * \mathrm{Seal}(\mathfrak{B}^{\bar{Q}}, S)$ is semiproper and for $\kappa > |P_\alpha|$ strongly inaccessible $(P_\alpha/P_{i+1}) * \mathrm{SSeal}(\mathfrak{B}^{\bar{Q}}, S, \kappa)$ is semiproper with κ-c.c.

(3) The hypothesis (*) of (2) holds if for arbitrarily large $i < \alpha$:

$\underset{\sim}{Q}_i$ is semiproper and $\Vdash_{P_i}$ "$\mathrm{Rss}(\aleph_2)$".

(4) If ($\underset{\sim}{\mathcal{A}}$ is a P_α-name and it is forced for P_α that) $\underset{\sim}{\Xi}$ is a predense subset of $\underset{\sim}{\mathcal{A}}$, $\bigvee_{i<\alpha} \underset{\sim}{\mathcal{A}} \subseteq \mathfrak{B}^{P_{i+1}}$, and $\underset{\sim}{\mathcal{A}} \lessdot \mathfrak{B}^{\bar{Q}}$ (for this $\alpha = \sup\{i : \underset{\sim}{\mathcal{A}} \lessdot \mathfrak{B}^{P_{i+1}}\}$ suffice), *then* $\underset{\sim}{\Xi} \in \underset{\sim}{\Xi}^*$ ($\underset{\sim}{\Xi}^*$ from part (1)).

(5) Assume

(**) $\bar{\mathcal{A}} = \langle \bar{\mathcal{A}}_\beta : \beta < \beta^* \rangle$ and for $\beta < \beta^*$ we have: $\Vdash_{P_\alpha}$ "$\mathcal{A}_\beta \subseteq \mathfrak{B}^{P_{i+1}}$ for some $i < \alpha$" and if $i < \alpha$ and $\underset{\sim}{\Xi}$ is a P_α-name of a pre-dense subset of $\mathcal{A}_\beta$ to which $\omega_1 \setminus S$ belongs *then* $\Vdash_{P_{i+1}}$ " if $\underset{\sim}{\mathcal{A}}_\beta \subseteq \mathfrak{B}^{P_{i+1}}$ then $(P_\alpha/P_{i+1}) * \mathrm{seal}(\underset{\sim}{\Xi})$ is semiproper".

Then for every $i < \alpha$, $(P_\alpha/P_{i+1}) *$ Seal $(\bar{\mathcal{A}}, S)$ is semiproper and if $\kappa > |P_\alpha|$ is strongly inaccessible then $(P_\alpha/P_{i+1}) * \mathrm{SSeal}(\bar{\mathcal{A}}, S, \kappa)$ is also semiproper, satisfies the κ-c.c., has cardinality κ, forces $\kappa = \aleph_2$ and forces $\underset{\sim}{\mathcal{A}}_\beta \lessdot \mathfrak{B}^{V[P_\alpha * \mathrm{SSeal}(\bar{\mathcal{A}}, S, \kappa)]}$.

Proof. (1) Use Claim 2.12 ω times and the definition of RCS (note that in 2.12(b) we do not get $q^1 \restriction i = q$, but we can replace q by any q', $q \leq q' \in P_i$).

(2) For the first phrase use 2.13(1). For the SSeal case, use also 2.9 with $\bar{\mathcal{A}} = \langle \mathfrak{B}^{\bar{Q}} \rangle$ (so $\xi = 1$), where the assumption of 2.9 can be gotten by the first phrase; the κ-c.c. is proved as in 2.11(3) using models N as in 2.9.

(3) By 1.7(5) the statement $\mathrm{Rss}(\aleph_2)$ implies that semiproperness and preserving stationarity of subsets of ω_1 are equivalent. Suppose $i < \alpha, Q_i$ is semiproper and $\Vdash_{P_i}$ "$\mathrm{Rss}(\aleph_2)$". As by 2.8(5), $\mathrm{seal}(\underset{\sim}{\Xi})$ (for $\Xi \subseteq \mathfrak{B}^{\bar{Q}}$ a maximal antichain) preserves stationarity of subsets of ω_1 from V^{P_i} which are stationary in V^{P_α} (and this property is preserved by composition (though not by limit)) and $P_\alpha/P_i = Q_i * (P_\alpha/P_{i+1})$ is semiproper hence preserve stationarity of subsets of ω_1, we get that $(P_\alpha/P_i)*$ $\mathrm{seal}(\underset{\sim}{\Xi})$ preserves stationarity of subsets of ω_1 hence is semiproper (in V^{P_i} of course). This holds for arbitrarily large $i < \alpha$, hence (by the composition of semiproperness) for every non-limit i, which is the demand (*) of (2).

4) As in the proof of (**) from the proof of 2.11(1), it suffices to prove clause $(\mathrm{b})^+$ of 2.12 for successor $i < \alpha$, so let $\underset{\sim}{\mathcal{A}}$, $\underset{\sim}{\Xi}$ be as in the assumption of 2.13(4), $q \Vdash$ "$\{\underset{\sim}{\mathcal{A}}, \underset{\sim}{\Xi}\} \in \underset{\sim}{N}$" and $\underset{\sim}{N}, i, p, q$ be as in the assumption of $2.12(\mathrm{b})^+$. We know that for some $i_0 \geq i$ we have $\underset{\sim}{\mathcal{A}} \subseteq \mathfrak{B}^{P_{i_0+1}}$, so without loss of generality

(possibly increasing p and q) for some i_0, $p \Vdash$ "$\underset{\sim}{\mathcal{A}} \subseteq \mathfrak{B}_{i_0+1}$", by the preservation of semiproperness by composition without loss of generality $i_0 = i+1$. Let G_i be such that $q \in G_i \subseteq P_i$, G_i generaic over V and $N = \underset{\sim}{N}[G_i]$; in $V[G_i]$ we define $\Upsilon \stackrel{\text{def}}{=} \{A \in \mathfrak{B}^{P_{i_0+1}} : p \nVdash_{P_\alpha/G_i}$ "$A \notin \underset{\sim}{\Xi}$"$\}$. So $\Upsilon \in N[G_i]$ and $V[G_i] \models$ "$|\Upsilon| \leq \aleph_1$", $\Upsilon \neq \emptyset$ so let, in $V[G_i]$, $\Upsilon = \{A_\zeta : \zeta < \omega_1\}$ and without loss of generality $\langle A_\zeta : \zeta < \omega_1\rangle \in N[G_i]$. Let $B = \{\delta < \omega_1$: δ limit and $\delta \notin \bigcup_{i<\delta} A_i\}$, so $B \subseteq \omega_1$, $B \in V[G_i]$, and (in $V[G_i]$) we have: $B \cap A_\zeta = \emptyset \bmod \mathcal{D}_{\omega_1}$. So in V^{P_α}, B cannot be stationary (as $B \in \mathfrak{B}^{\bar{Q}}$, $\underset{\sim}{\mathcal{A}} \lessdot \mathfrak{B}^{\bar{Q}}$) so as P_α/G_i is semiproper also in $V[G_i]$ we know that B is not stationary, and we finish as in the proof of (**) from the proof of 2.11(1).

5) The proof of 2.13(2) (and see 2.16). $\square_{2.13}$

2.14 Claim. Suppose $\bar{Q} = \langle P_i, \underset{\sim}{Q}_j, \mathbf{t}_j : i \leq \alpha+1, j < \alpha+1\rangle$ is an RCS iteration, $\bar{Q}\restriction\alpha$ is S-suitable, and $\kappa > |P_\alpha|$ is strongly inaccessible.

(1) If $\mathbf{t}_\alpha = 0, \underset{\sim}{Q}_\alpha = \mathrm{SSeal}(\langle \mathfrak{B}[P_j] : j < \alpha, \mathbf{t}_j = 1\rangle, S, \kappa)$ *then* $\bar{Q}$ is S-suitable and also: for α successor or $\alpha = \mathrm{cf}(\alpha) > |P_i|$ for $i < \alpha$ even Q_α is proper.

(2) If α is a limit ordinal, $\bar{\mathcal{A}} = \langle \underset{\sim}{\mathcal{A}}_\zeta : \zeta < \xi\rangle$ is a sequence of (P_α-names of) subalgebras of $\mathfrak{B}^{\overline{Q}\restriction\alpha}$ with $\bigwedge_\zeta \bigvee_{i<\alpha} \underset{\sim}{\mathcal{A}}_\zeta \subseteq \mathfrak{B}^{P_{i+1}}$, and for every $\zeta < \xi, \Vdash_{P_\alpha}$ "for $\zeta < \xi$ the set $\{i < \alpha : \underset{\sim}{\mathcal{A}}_\zeta\restriction S \lessdot \mathfrak{B}[P_{i+1}]\restriction S\}$ is unbounded below α", and $\Vdash_{P_\alpha}$ "for every $j < \alpha$ satisfying $\mathbf{t}_j = 1$ for some $\zeta, \mathfrak{B}^{P_j}\restriction S \lessdot \mathcal{A}_\zeta\restriction S$" and $\mathbf{t}_\alpha = 0$, and $Q_\alpha = \mathrm{SSeal}(\bar{\mathcal{A}}, S, \kappa)$ *then* $\bar{Q}$ is S-suitable.

Proof. (1) First assume α is non-limit or $\alpha = \mathrm{cf}(\alpha) > |P_i|$ for $i < \alpha$. We have to check clauses (A) – (F) of Definition 2.1. Clause (D) holds by Claim 2.11(1); clause (E) holds by Claim 2.11(3); clause (F) holds by 2.11(2); the other parts of Definition 2.1 hold trivially. Lastly the conclusion concerning "Q_α is proper" holds by 2.11(3).

If α is limit, then this follows from 2.14(2) which is proved below.

2) Let $\underset{\sim}{\boldsymbol{\Xi}} = \{\underset{\sim}{\Xi} : \underset{\sim}{\Xi}$ is a P_α-name of a pre-dense subset of $\mathfrak{B}[P_{i+1}]$ to which $\omega_1 \setminus S$ belongs for some $i < \alpha$ and $(P_\alpha/P_{j+1}) * \mathrm{seal}(\underset{\sim}{\Xi})$ is semiproper for every $j < \alpha\}$. By 2.13(4) above: if $\underset{\sim}{\Xi}$ is a P_α-name of a maximal antichain of $\mathcal{A}_\zeta (\zeta < \xi)$ *then*

$\underset{\sim}{\Xi} \in \underset{\sim}{\Xi}$. So by 2.13(5) clauses (D),(E),(F) of Definition 2.1 hold (the others are trivial). $\square_{2.14}$

2.15 Claim. 1) Suppose $\bar{\mathcal{A}} = \langle \mathcal{A}_\zeta : \zeta < \xi \rangle$ is an increasing sequence of subalgebras (or just subsets) of $\mathfrak{B}, \chi$ regular, N a countable elementary submodel of $(H(\chi), \in, <^*_\chi)$ and $\oplus^N_{\bar{\mathcal{A}},S}$ from 2.9(1) holds, i.e.

$\oplus^N_{\bar{\mathcal{A}},S}$ if $\zeta \in \xi \cap N$ and $\Xi \in N$ is a pre-dense subset of $\mathcal{A}_\zeta$ and $\omega_1 \setminus S \in \Xi$ then $N \cap \omega_1 \in \bigcup_{A \in N \cap \Xi} A$.

If $Q \in N$ is a strongly proper forcing notion, $p \in Q \cap N$ *then* there is $q \in Q, p \leq q, q$ is (N, Q)-generic and $q \Vdash$ "$\oplus^{N[G_P]}_{\bar{\mathcal{A}},S}$".

2) In 2.10 we can conclude also that for a strongly proper Q which is $(\omega_1 \setminus S)$-complite and satisfies $|Q| < \kappa$, the forcing notion $Q*$ $\mathrm{SS\underset{\sim}{e}al}(\mathfrak{B}^V, S, \kappa)^{V^Q}$ is semiproper $(\omega_1 \setminus S)$-complete.

3) Parallel strengthenings of 2.11, 2.13 (see mainly 2.13(1)) and 2.14 hold.

2.15A Remark. This claim can be used in §3, §4 to get appropriate axioms: it gives a comprehensive family of forcing notions which we can use quite freely in the iterations, without making problems for what is already accomplished there.

For a more general property: see 4.6.

Proof. Straightforward (reread the proof of 2.11). $\square_{2.15}$

2.16 Claim. Assume $\bar{\mathcal{A}} = \langle \mathcal{A}_\zeta : \zeta < \xi \rangle$, $\mathcal{A}_\zeta \restriction \zeta \subseteq \mathfrak{B}^V \restriction S$,

$$W = W_{\bar{\mathcal{A}}} = \Big\{ a : a \subseteq H(\beth_2(\aleph_1)), a \text{ is countable}, a \cap \omega_1 \text{ is an ordinal and: }$$
$$\text{if } \zeta < \xi, \Xi \subseteq \mathcal{A}_\zeta, \Xi \text{ is a pre-dense subset of } \mathcal{A}_\zeta,$$
$$\omega_1 \setminus S \text{ belongs to } \Xi \text{ and } \{\zeta, \Xi\} \in a$$
$$\text{then } a \cap \omega_1 \in \bigcup \{A : A \in \Xi \cap a\} \Big\}$$

is a stationary subset of $H(\beth_2(\aleph_1))$ and $\kappa > \aleph_0$ is strongly inaccessible. *Then*

(1) $P_\kappa \overset{\text{def}}{=} \mathrm{SSeal}(\bar{A}, S, \kappa)$ is W-proper.

(2) $\Vdash_{P_\kappa}$ "$\mathcal{A}_\zeta \restriction S \lessdot B^V \restriction S$" for $\zeta < \xi$.

(3) In fact, P_κ is $(\omega_1 \setminus S)$-complete strongly W-proper and satisfies the κ-c.c.

(4) If $\omega_1 \setminus S$ is stationary *then* P_κ does not add ω-sequences of ordinals.

(5) If $\xi = \zeta + 1$, $\mathcal{A}_\zeta = \mathfrak{B}^V$ and $\mathrm{Rss}(\aleph_2)$ *then* P_κ is semiproper.

(6) If $\lambda > \kappa$, $\mathrm{Rss}^+(\kappa, \lambda)$ *then* $V^{P_\kappa} \models \mathrm{Rss}(\aleph_2, \lambda)$.

Proof. 1) W-properness is proved as in the proof of 2.11(1) (and 2.9) restricting ourselves to models N such that $N \cap H(\beth_2(\aleph_1)) \in W$.

2), 3), 4) As in the proof of 2.11(2), (3), (4).

5) W-properness implies semiproperness by 2.8(7), (8), (note: we can ignore $\mathcal{A}_\varepsilon$ when $\varepsilon + 1 < \xi$ as $W_{\langle \mathcal{A}_{\xi-1} \rangle} = W_{\bar{\mathcal{A}}}$).

6) Should be clear. $\square_{2.16}$

2.17 Claim.

Assume $\bar{Q} = \langle P_i, \underset{\sim}{Q}_j : i \le \kappa, j < \kappa \rangle$ is an RCS-iteration, κ is strongly inaccessible ($i < \kappa \Rightarrow |P_i| < \kappa$) and, for stationarily many $i < \kappa$, for arbitrarily large $j \in (i, \kappa)$, $\mathfrak{B}^{\bar{Q} \restriction i} \lessdot \mathfrak{B}^{P_j}$. *Then* in V^{P_κ}, for $\bar{\mathcal{A}} = \langle B[P_\kappa] \rangle$, $W = W_{\bar{\mathcal{A}}}$ contains a club of $\mathcal{S}_{<\aleph_1}(H(\beth_2(\aleph_1)))$.

Proof. By Fodor's Lemma, $\mathfrak{B}^{P_\kappa}$ satisfies the $\aleph_2$-c.c., hence we can apply 2.11. $\square_{2.17}$

§3. On $\mathcal{P}(\omega_1)/\mathcal{D}_{\omega_1}$ Being Layered or the Levy Algebra

On layered ideals see [Sh:237a], Foreman Magidor Shelah [FMSh:252] and [Sh:270]. A reader can read sepeately 3.1 – 3.3, 3.4 – 3.8, 3.4 – 3.10. Here in 3.1, 3.2, 3.3 we deal with "$\mathfrak{B} \restriction S$ being layered"; in 3.4, 3.5, 3.6 we prepared the ground for "$\mathfrak{B} \restriction S$ being the Levy algebra" and in 3.7, 3.9 we deal with "$\mathfrak{B} \restriction S$ being the Levy algebra". We deal also with getting forcing axioms and try to present some approaches (rather than saving in consistency strength around "ZFC+ there is a supercompact cardinal").

3.1. Theorem. Suppose κ is supercompact. *Then* for some forcing notion P:

(i) P satisfies the κ-c.c., has cardinality κ, does not collapse $\aleph_1$, but collapses every $\lambda \in (\aleph_1, \kappa)$ and $\Vdash_{P_\kappa}$ "$\kappa = \aleph_2$, and $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$",

(ii) $\mathfrak{B}[P]$ is S^*-layered (see 3.1A(4) below), for some stationary $S^* \subseteq \{\delta < \kappa : \mathrm{cf}(\delta) = \aleph_1 \text{ (in } V^P)\}$,

(iii) in V^P, $Ax^+\Big[Q$ semiproper collapsing $\aleph_2$ and $\mathfrak{B}[(V^P)] \lessdot \mathfrak{B}[(V^P)^Q]\Big]$.

3.1A Remark. 1) In (iii), of course if we have Ax rather than Ax^+, we can replace the condition on the forcing Q by:

for some $R, Q \lessdot R$, R is semiproper, $\mathfrak{B}[V^P] \lessdot \mathfrak{B}[(V^P)^R]$ and R collapses $\aleph_2$ (of V^P).

2) Note for 3.1(iii) that, in V^P, we have $|\mathfrak{B}| = 2^{\aleph_1} = \aleph_2$.

3) In (iii) of 3.1 we can replace Ax^+ by Ax_{ω_1}; similarly in 3.2, 3.3(1)(iii).

4) A Boolean algebra B of regular cardinality λ is S^*-layered (for $S^* \subseteq \lambda$) if: letting $B = \bigcup_{i<\lambda} B_i$, B_i increasing continuous in i, $|B_i| < \lambda$, we have $\{\delta < \lambda : \delta \in S^* \Rightarrow B_\delta \lessdot B\} \in \mathcal{D}_\lambda$.

5) We say that a filter $\mathcal{D}$ on a set A is S-layered if $\mathcal{P}(A)/D$ is S-layered.

Proof. Let $S = \omega_1$ and let $h : \kappa \to H(\kappa)$ be a Laver diamond (see Definition VII2.8; later we may say: repeat this proof for other stationary $S \subseteq \omega_1$ and $h : \kappa \to H(\kappa)$). By induction on $i < \kappa$ we define $P_i, Q_i, \mathbf{t}_i$ such that:

(A) $\bar{Q}^\alpha = \langle P_i, \underset{\sim}{Q}_j, \mathbf{t}_j : i \le \alpha, j < \alpha \rangle$ is an S-suitable iteration.

(B) Q_α is defined by cases:

CASE a: Assume $(*)_1 + (*)_2$ where

$(*)_1$ α is measurable and $\bigwedge_{i<\alpha}[|P_i| < \alpha]$ and $\big[i < \alpha \;\&\; \mathbf{t}_i = 1 \Rightarrow \mathfrak{B}[P_i] \lessdot \mathfrak{B}[P_\kappa]\big]$, and $\mathrm{Rss}^+(\alpha, 2^\alpha)$ and

$(*)_2$ $h(\alpha)$ is a P_α-name of a semiproper forcing notion, and $\Vdash_{P_\alpha * h(\alpha)}$ "$\mathfrak{B}[P_\alpha] \lessdot \mathfrak{B}[P_\alpha * h(\alpha)]$ and $\alpha = \aleph_2^{V[P_\alpha]}$ is collapsed".

Then $\mathbf{t}_\alpha = 1$ and $\underset{\sim}{Q}_\alpha = h(\alpha) * \mathrm{SSeal}^{V[P_\alpha * h(\alpha)]}(\langle \mathfrak{B}[P_i] : i \le \alpha,\ \mathbf{t}_i = 1\rangle, S)$.

CASE b: Assume $(*)_1$ but not $(*)_2$,

then $\mathbf{t}_\alpha = 1$ and $Q_\alpha = \mathrm{SSeal}\ (\langle \mathfrak{B}[P_i] : i \le \alpha,\ \mathbf{t}_i = 1\rangle, S)$.

CASE c: Assume not $(*)_1$.

Then $\mathbf{t}_\alpha = 0$ and $Q_\alpha = \mathrm{SSeal}^{V[P_\alpha]}(\langle \mathfrak{B}[P_i] : i < \alpha, \mathbf{t}_i = 1\rangle, S)$.

3.1B Observation. $\bar{Q}$ is S-suitable and $\beta < \kappa \Rightarrow \bar{Q}\restriction\beta \in H(\kappa)$ and: $\underset{\sim}{Q}_\beta$ is semiproper when $\beta = \mathrm{cf}(\beta) > |P_i|$ for $i < \beta$ (or β successor).

Proof of 3.1B. We prove by induction on $\beta \leq \kappa$ that $\bar{Q}\restriction\beta$ is S-suitable and when $\beta < \kappa$, then $\bar{Q}\restriction\beta$ belongs to $H(\kappa)$ and if $\beta = \alpha + 1$, $\alpha = \mathrm{cf}(\beta) > |P_i|$ for $i < \alpha$ then $\underset{\sim}{Q}_\alpha$ is semiproper.

<u>For $\beta = 0$</u>: trivial.

<u>For β limit</u>: by 2.3(1).

<u>For $\beta = \alpha + 1$ and for α, $(*)_1$ above fails</u>: By the induction hypotheses $\bar{Q}\restriction\alpha = \langle P_i, \underset{\sim}{Q}_j, \mathbf{t}_j : i \leq \alpha, j < \alpha\rangle$ is S-suitable, hence it is a semiproper iteration and by our choice $\bar{Q}\restriction\beta = \bar{Q}\restriction(\alpha+1)$ is an RCS iteration and letting κ_α be the first strongly inaccessible $> |P_\alpha|$, we have $Q_\alpha = \mathrm{SSeal}\,(\langle \mathfrak{B}[P_i] : i < \alpha, \mathbf{t}_i = 1\rangle, S, \kappa_\alpha)$.

Now by 2.14(1) we are done (in particular Q_α is semiproper if: α is a successor or $\alpha = \mathrm{cf}(\alpha) > |P_i|$ for $i < \alpha$).

<u>For $\beta = \alpha + 1$ and for α, $(*)_1$ above holds but $(*)_2$ fails</u>

By the induction hypothesis $\bar{Q}\restriction\alpha = \langle P_i, \underset{\sim}{Q}_j : i \leq \alpha,\ j < \alpha\rangle$ is S-suitable, hence a semiproper iteration and by our choice $\bar{Q}\restriction\beta = \bar{Q}\restriction(\alpha+1)$ is an RCS-iteration and letting κ_α be the first strongly inaccessible $> |P_\alpha|$, we have:

$\mathbf{t}_\alpha = 1$ and in V^{P_α} we have $Q_\alpha = \mathrm{SSeal}(\langle \mathfrak{B}[P_i] : i \leq \alpha, \mathbf{t}_i = 1\rangle, S, \kappa_\alpha)$.
Note that, as $(*)_1$ holds, α is measurable so $\{\gamma < \alpha$: case (c) applies and $\gamma = \mathrm{cf}(\gamma) > |P_i|$ for $i < \gamma\}$ includes all strongly inaccessible non-measurable cardinals in C, for some club C of α. It is well known that there is a normal ultrafilter on α to which this set belongs so 1.10 applies.

By 1.10(1) and, as $V \vDash$ "$\mathrm{Rss}^+(\alpha, 2^\alpha)$" holds by $(*)_1$, we know that in V^{P_α}, $\mathrm{Rss}(\aleph_2, 2^{\aleph_2})$ holds. So by 2.8(7) every maximal antichain Ξ of $\mathfrak{B}[P_\alpha]$ (in V^{P_α}) is semiproper. Hence by 2.10 $\mathrm{SSeal}(\mathfrak{B}, S, \kappa_\alpha)$ is semiproper. Now $\mathfrak{B}^{\bar{Q}\restriction\alpha} = \mathfrak{B}^{P_\alpha}$ [as α is (by $(*)_1$) strongly inaccessible, $\bigwedge_{i<\alpha} |P_i| < \kappa$, now use

2.2(1)(c)], and $\mathrm{SSeal}(\mathfrak{B}^{\bar{Q}\restriction\alpha}, S, \kappa_\alpha) = \mathrm{SSeal}(\langle \mathfrak{B}[P_i] : i \le \alpha, \mathbf{t}_i = 1\rangle, S, \kappa_\alpha)$ by claim 2.7, as $\mathbf{t}_\alpha = 1$ and $[i < \alpha \ \& \ \mathbf{t}_i = 1 \Rightarrow \mathfrak{B}[P_i]\restriction S \lessdot \mathfrak{B}[P_\alpha]]$. Together, Q_α is semiproper and we can check that $\bar{Q}\restriction\beta$ is S-suitable.

<u>For $\beta = \alpha + 1$, and for α, $(*)_1 + (*)_2$ above holds.</u>

Similar to the previous case, but now we use the statement in $(*)_2$ to note that $h(\alpha)$ is (in V^{P_α}) a semiproper forcing. Now by $(*)_2$ we know that $\mathfrak{B}[P_\alpha] \lessdot \mathfrak{B}[P_\alpha * h(\alpha)]$ and $V^{P_\alpha * h(\alpha)} \models$ "$\mathfrak{B}[P_\alpha]$ has cardinality $\aleph_1$" hence we can use 2.11 to show that $\mathrm{SSeal}^{V[P_\alpha * h(\alpha)]}(\mathfrak{B}^{P_\alpha}, S)$ is semiproper. $\square_{3.1B}$

Remark. Note that we could use only semiproper Q_α's (so demand in $(*)_2$ that $h(\alpha)$ is semiproper).

3.1C. Observation. (1) If $\alpha < \kappa$, and $\mathbf{t}_\alpha = 1$ (equivalently $(*)_1$ holds) *then* in V^{P_κ} we have $\mathfrak{B}[P_\alpha]\restriction S \lessdot \mathfrak{B}[P_\kappa]\restriction S$.
2) If $\mathcal{D}$ is a normal ultrafilter on $\mathcal{S}_{<\kappa}(H(\beth_8(\kappa))$, then $\{a : a \in \mathcal{S}_{<\kappa}(H(\beth_8(\kappa)))$ and $(*)_1$ is satisfied by $a \cap \kappa\} \in \mathcal{D}$.

Proof of 3.1C. Should be clear. $\square_{3.1C}$

Letting $P = P_\kappa$ and $S^* = \{\alpha < \kappa : (*)_1 + \neg(*)_2$ holds for α or at least $(*)_1 + V^{P_\kappa} \models$ "$\mathrm{cf}(\alpha) = \aleph_1$"$\}$, we easily finish, note that for $\alpha \in S^* \cup \{\kappa\}$: $\mathfrak{B}[P_\alpha] = \bigcup_{i<\alpha} \mathfrak{B}[P_i]$ and for $\alpha \in S^*$: $\mathbf{t}_\alpha = 1$. As κ is supercompact S^* is a stationary subset of κ (by 3.1C) and forcing with P_κ preserves it (as P_κ satisfies the κ-c.c.) and $\alpha \in S^* \Rightarrow \Vdash_{P_\kappa}$ "$\mathrm{cf}(\alpha) = \aleph_1$" (check Q_α). Also the other requirement causes no problems. $\square_{3.1}$

3.2 Theorem. 1) In 3.1 we can weaken "P satisfies the κ-c.c" to "P does not collapse $\aleph_2$ and has cardinality κ" but add that we have $\mathfrak{B}[P]$ is layered, which means it is S^*-layered for $S^* \stackrel{\mathrm{def}}{=} \{\delta < \aleph_2 : \mathrm{cf}(\delta) = \aleph_1$ (in V^P)$\}$.
2) In 3.1 we can add to the conclusion ($P = P_\kappa$, $\bar{Q} = \langle P_i, \underset{\sim}{Q}_j, \mathbf{t}_j : i \le \kappa, j < \kappa, \rangle$ is S suitable and):

(iv) In V^P, Ax [Q is semiproper and $i < \kappa \ \& \ \mathbf{t}_i = 1 \Rightarrow \mathfrak{B}^{P_i} \lessdot \mathfrak{B}[(V^P)^Q]$].

3) In 3.2(1) we can add to the conclusion ($P = P_\kappa$, $\bar{Q}$ as above and):

(iv)$^-$ In V^P we have $\mathrm{Ax}^+\Big[Q$ is semiproper changing the cofinality of $\aleph_2$ to $\aleph_0$, and $i<\kappa$ & $\mathbf{t}_i = 1 \Rightarrow \mathfrak{B}^{P_i} \lessdot \mathfrak{B}[(V^P)^Q]\Big]$.

Proof. 1) Force as in 3.1, and then let $P = P_\kappa * \underset{\sim}{Q}_\kappa$ where in $V^{P_\kappa}, Q_\kappa =$ club$(S^* \cup \{\delta : \mathrm{cf}^{V^P}(\delta) = \aleph_0\})$, where for S, club$(S) = \{h : h$ a strictly increasing continuous function h from some $\gamma+1 < \sup(S)$ to $S\}$.

As, in V^{P_κ}, the set $S^* = \{\alpha < \kappa : (*)_1 + \neg(*)_2$ from the proof of 3.1 hold$\} \subseteq \{\delta < \aleph_2 : \mathrm{cf}(\delta) = \aleph_1\}$ is stationary, moreover $\alpha \in S^*$ implies: there is, in V^{P_κ}, a subset b_α of α of order type ω_1 such that $\gamma < \alpha \Rightarrow b_\alpha \cap \gamma \in \bigcup\limits_{\beta<\alpha} V^{P_\beta}$. As $\langle \bigcup\limits_{\beta<\alpha} (\mathcal{P}(\beta) \cap V^{P_\beta}) : \alpha < \kappa\rangle$ is increasing and continuous and $V^{P_\kappa} \models$ "$|\mathcal{P}(\beta) \cap V^{P_\beta}| = \aleph_1$", clearly Q_κ adds no bounded subsets to κ and $\kappa = \aleph_2^{V[P_\kappa]}$, so $\mathfrak{B}[P_\kappa] = \mathfrak{B}[P_\kappa * \underset{\sim}{Q}_\kappa]$ and $\Vdash_{Q_\kappa}$ "$\{\delta < \kappa : \mathrm{cf}(\delta) = \aleph_1$ but not $\mathfrak{B}[P_\delta]\restriction S \lessdot \mathfrak{B}[P_\kappa]\restriction S\}$ is not stationary.

Why does (iii) of 3.1 continue to hold? Suppose, in $V^{P*\underset{\sim}{Q}_\kappa}$, R is a semiproper forcing collapsing $\aleph_2$ such that $(V^P)^{Q_\kappa} \models \big[\Vdash_R$ "$\mathfrak{B} \lessdot \mathfrak{B}[R]$"$\big]$. Let $\underset{\sim}{R}$ be a $P_\kappa * \underset{\sim}{Q}_\kappa$-name of such a forcing notion and $(p, \underset{\sim}{q}) \in P_\kappa * Q_\kappa$. Apply (iii) to $Q_\kappa * \underset{\sim}{R}$ in $V[P_\kappa]$ (strictly speaking, its proof). I.e. by the properties of the Laver diamond, for some χ, $2^{|Q_\kappa * \underset{\sim}{R}|} < \chi$, and $M \prec (H(\chi), \in, <^*_\chi)$ to which $\bar{Q}$, $\underset{\sim}{Q}_\kappa$, $\underset{\sim}{R}$, and $(p, \underset{\sim}{q})$ belong and M isomorphic to some $(H(\chi_1), \in, <^*_{\chi_1})$, by the Mostowski collapsing isomorphism g, taking P_κ to P_{κ_1} where $\kappa_1 = M \cap \kappa$, and $h(\kappa_1) = g(Q_\kappa * \underset{\sim}{R})$. Clearly κ_1 satisfies $(*)_1$ and without loss of generality also $(*)_2$, hence $\mathbf{t}_{\kappa_1} = 1$. So we could have increased $(p, \underset{\sim}{q})$ to guarantee the existence of the generic enough subset of R (i.e. we use the generic subset of $g(Q_\kappa)$ to increase $\underset{\sim}{q}$).

(2) In the proof of 3.1, case b is now divided into subcases b_1 and b_2;

<u>case b$_1$</u>: $(*)_1$, not $(*)_2$ but

$(*)_{1.5}$ $h(\alpha)$ is a P_α-name of a semiproper forcing notion such that $i < \alpha$, $\mathbf{t}_i = 1 \Rightarrow \mathfrak{B}^{P_\alpha} \lessdot \mathfrak{B}^{P_\alpha * h(\alpha)}$.

Then we let $\mathbf{t}_\alpha = 0, \underset{\sim}{Q}_\alpha = h(\alpha) *$ SSeal$^{V[P_\alpha * h(\alpha)]}(\langle \mathfrak{B}[P_i] : i \le \alpha, \mathbf{t}_i = 1\rangle, S, \kappa_\alpha)$, where κ_α is the first strongly inaccessible $> |P_\alpha * h(\alpha)|$.

<u>case b$_2$</u>: $(*)_1$, not $(*)_2$ and not $(*)_{1.5}$.

Then (as in the old case b) $\mathbf{t}_\alpha = 1, Q_\alpha = \mathrm{SSeal}^{V[P_\alpha]}(\langle \mathfrak{B}[P_i] : i \leq \alpha, \mathbf{t}_i = 1\rangle, S)$.

3) Should be clear. $\square_{3.2}$

3.3 Theorem. In 3.1, 3.2 we can add, as a parameter (from V), $\bar{S} = \langle S_1, S_2, S_3\rangle$ a partition of ω_1, $S = S_1$ is a stationary and restrict ourselves to pseudo $(*, S_3)$-complete forcing, see X 3.10, (so if S_3 is not stationary this is not a restriction) so if S_3 is stationary the forcing notions will not be adding reals; i.e.

1) There is a forcing notion P such that:

(i) P satisfies the κ-c.c., does not collapse $\aleph_1$, but collapses every $\lambda \in (\aleph_1, \kappa)$, $\Vdash_P$ "$\kappa = \aleph_2$ and $2^{\aleph_0} \leq \aleph_2, 2^{\aleph_1} = \aleph_2$ and if $S_3 = \emptyset \mod \mathcal{D}_{\omega_1}$ then $2^{\aleph_0} = \aleph_2$" and P is pseudo $(*, S_3)$-complete,

(ii) $\mathfrak{B}[P]\restriction S_1$ is S^*-layered, for some stationary $S^* \subseteq \{\delta < \kappa : \mathrm{cf}(\delta) = \aleph_1$ (in V^{P_κ})$\}$,

(iii) in V^P, $Ax^+[Q$ semiproper, pseudo $(*, S_3)$-complete collapsing $\aleph_2$ and $\mathfrak{B}[(V^{P_\kappa})] \lessdot \mathfrak{B}[(V^{P_\kappa})^Q]]$,

(iv) if S_3 is stationary, the forcing P adds no new reals (so $V^P \models CH$).

2) In 3.3(1) we can replace "P satisfies the κ-c.c." by "P does not collapse κ" and have $\mathfrak{B}[P_1]$ is layered, i.e. $S^* = \{\delta < \aleph_2 : \mathrm{cf}(\delta) = \aleph_1\}$ (in V^P).

3) We can add in 3.3(1): ($P = P_\kappa$, $\bar{Q} = \langle P_i, \underset{\sim}{Q}_j, \mathbf{t}_j : i \leq \kappa, j < \kappa\rangle$ is S_1-suitable and)

(v) in V^P, $Ax[Q$ semiproper, pseudo $(*, S_3)$-complete and $i < \kappa$ & $\mathbf{t}_i = 1 \Rightarrow \mathfrak{B}[V^{P_i}] \lessdot \mathfrak{B}[(V^P)^Q]]$.

4) Actually in (3) it suffices "for $i < \kappa$, $(P_\kappa/P_{i+1}) * \underset{\sim}{Q}$ is semiproper, pseudo $(*, S_3)$-complete and: $j \leq i$ & $\mathbf{t}_j = 1 \Rightarrow \mathfrak{B}^{P_j} \lessdot V^{P_\kappa \times \underset{\sim}{Q}}$".

3.3A. Remark. 1) In 3.2(2)(iv) and in 3.3(3)(v), if we deal with Ax (Ax^+) it is enough that $Q \lessdot Q'$, Q' as there, or more directly, for each $i < \kappa$, there are enough models N as in 2.9.

2) The "solution" of x/3.3(3),(4) = 3.2(3)/3.2(2) holds.

Proof. 1) Like the proof of 3.1 but we seal only $\mathfrak{B}^{V[P_i]}\restriction S_1$ when $\mathbf{t}_i = 1$ and in $(*)_2$ we add "$h(\alpha)$ is pseudo $(*, S_3)$-complete", but we have to check that all forcing notions Q_α are pseudo $(*, S_3)$-complete (and use the iteration lemma X 3.11). Now all the sealing forcing notions which we use satisfies this trivially.

2), 3), 4) Similar. $\square_{3.3}$

3.4. Claim. Suppose $\bar{Q} = \langle P_i, \underset{\sim}{Q}_j : i \leq \kappa,\ j < \kappa \rangle$ is a semiproper iteration, κ strongly inaccessible with $\kappa > |P_i|$ for $i < \kappa$, and lastly $S \subseteq \omega_1$ is stationary. Suppose further

$(*)$ (a) for $i < \kappa$, in $V^{P_{i+1}}$, Player II wins $\partial(\{\aleph_1\}, \omega, \mathcal{D}_\kappa + E_i^+)$ where $E_i^+ = \{\delta < \kappa : \delta > i, \delta$ strongly inaccessible, $(\forall \alpha < \delta)[|P_\alpha| < \delta]$ and $\Vdash_{P_\delta/P_{i+1}}$ "Q_δ is semiproper"$\}$; (for a definition of the game see 1.9A(2)) so we are assuming $E_i^+ \neq \emptyset \mod \mathcal{D}_\kappa$ in $V^{P_{i+1}}$ for each $i < \kappa$; and let $E^+ = E_0^+$.

(b) $E^* = \{i < \kappa : \Vdash_{P_i}$ "Rss$(\aleph_2)$ and $\underset{\sim}{Q}_i$ semiproper"$\}$ is unbounded in κ.

Then $R_{i+1} \overset{\text{def}}{=} (P_\kappa/P_{i+1}) * \underset{\sim}{\text{Nm}} * \text{S}\underset{\sim}{\text{eal}}(\mathfrak{B}[P_\kappa], S)$ is (in the universe $V^{P_{i+1}}$, Nm in V^{P_κ}, SSeal in $V^{P_\kappa * Nm}$ of course) is semiproper for every $i < \kappa$.

3.4A. Remark. (1) Remember that $\text{Nm} = \{T : T \subseteq {}^{\omega>}(\aleph_2)$ is closed under initial segments, is nonempty, and for every $\eta \in T$ we have $|\{\nu : \eta \trianglelefteq \nu \in T\}| = \aleph_2\}$; ordered by the inverse of inclusion. Clearly $\{T :$ for $\eta \in T$, $\text{Suc}_T(\eta)$ is a singleton or has power $\aleph_2\}$ is a dense subset, so usually we restrict ourselves to it. For such T_1 the trunk is the $\eta \in T$ of minimal length such that $|\text{Suc}_T(\eta')| > 1$.
(2) We can use $\text{Nm}(D)$ instead of Nm and even Nm', $\text{Nm}'(D)$.
(3) We can replace Nm by any forcing notion satisfying, e.g. pseudo $(*, S)$-completeness (see X 3.9, 10) or the $\mathbb{I}$-condition (see Chapter XI) where $\mathbb{I} \in V$ is a family of κ-complete normal ideals or even $UP(\mathbb{I})$, see Chapter XV.
(4) Instead of $(*)$(b) we can have "largeness" demands on κ. We need it to make $(P_\kappa/P_j)*$ seal$(\underset{\sim}{\Xi})$ semiproper for $j \in E^+, \underset{\sim}{\Xi}$ a maximal antichain of $\mathfrak{B}$ from V^{P_κ}.
(5) Note that $\Vdash_{P_\kappa}$ "cf$(\delta) = \aleph_0$" is not forbidden in the definition of E_i^+; we can in clause (a) of $(*)$ of 3.4 in the game allow pressing down functions (see 1.9A(4)), add $\Vdash_{P_{\delta+1}}$ "cf$(\delta) = \aleph_1$"; in the proof below we strengthen the

definition of $j \in E^0_\eta$ by $j = \min(N_{\eta,j} \cap \kappa \setminus N_\eta)$ and demand E^0_η to be stationary and this somewhat simplify the proof.

Proof. We work in $V^{P_{i+1}}$ so let $G_{i+1} \subseteq P_{i+1}$ be generic over V. Let λ be regular and large enough, $N \prec (H(\lambda)[G_{i+1}], \in, <^*_\lambda)$ countable, $i \in N$, $\kappa \in N$, $\kappa \in N$, $\bar{Q} \in N$ and $(p^a, \underset{\sim}{p}^b, \underset{\sim}{p}^c) \in R_{i+1} \cap N$.

We shall choose below $q_{\langle\,\rangle} \in P_\kappa/P_{i+1}$ which is $(N, P_\kappa/P_{i+1})$-semi-generic, $p^a \leq q_{\langle\,\rangle}$ and $G_\kappa \subseteq P_\kappa$ generic over V containing $G_{i+1} \cup \{q_{\langle\rangle}\}$.

We now, in $V[G_\kappa]$ (but G_κ is defined only during the definition for $n = 0$) define by induction on n, T_n, N_η $(\eta \in T_n)$ such that:

(A) $T_n \subseteq {}^{n\geq}\kappa$,

(B) $T_0 = \{\langle\ \rangle\}$,

(C) $(\forall \nu \in T_{n+1})[\nu \restriction n \in T_n]$ and $T_{n+1} \cap {}^{n\geq}\kappa = T_n$,

(D) $(\forall \eta \in T_n)[\{i : \eta^\wedge\langle i\rangle \in T_{n+1}\}$ has power $\kappa]$,

(E) $N[G_\kappa] \cap H(\lambda)[G_{i+1}] <_{\omega_2} N_{\langle\rangle} \prec (H(\lambda)[G_{i+1}], \in, <^*_\lambda, G_\kappa)$ and $N_{\langle\rangle}$ is countable and $(p^a, \underset{\sim}{p}^b, \underset{\sim}{p}^c) \in N_{\langle\rangle}$ and $\bar{Q} \in N_{\langle\rangle}$, (note, abusing notation we do not distinguish strictly between $N_{\langle\rangle}$ and $(N_{\langle\rangle}, G_\kappa \cap N_{\langle\rangle})$ and similarly for N_η)

(F) for $\eta \in T_{n+1}$ the model $N_\eta \prec \Big(H(\lambda)[G_{i+1}], \in, <^*_\lambda, G_\kappa\Big)$ is countable, extends $N_{\eta\restriction n}$, and $N_{\eta\restriction n} <_\kappa N_\eta$,

(G) $\eta \in N_\eta$,

(H) If $\underset{\sim}{\Xi}$ is a P_κ/P_{i+1}-name of a dense subset of $\mathfrak{B}(P_\kappa)$, $\underset{\sim}{\Xi} \in N_\eta$ and $\eta \in T_n$, *then* for some natural number $k = k(\underset{\sim}{\Xi}, \eta)$ and every ν : if $\eta \trianglelefteq \nu \in T_{n+k}$ then:

$$(\exists \underset{\sim}{A} \in N_\nu)\Big[\underset{\sim}{A} \in \underset{\sim}{\Xi}\ \&\ \underset{\sim}{A} \text{ a } (P_\kappa/P_{i+1}) - \text{name } \&\ N \cap \omega_1 \in \underset{\sim}{A}[G_\kappa]\Big],$$

(I) E^0_η is a stationary subset of κ, where

$$E^0_\eta \stackrel{\rm def}{=} \{j < \kappa : N_\eta <_\kappa N_{\eta,j} \text{ where } N_{\eta,j} \text{ is the Skolem Hull of } N_\eta \cup \{j\} \text{ in } \left(H(\lambda)[G_{i+1}], \in, <^*_\lambda, G_\kappa\right) \text{ and } j \text{ is strongly inaccessible in } V \text{ and } (\forall i < j)[|P_i| < j] \text{ and } \Vdash_{P_j} \text{“}\underset{\sim}{Q}_j[G_j] \text{ is semiproper}\}\text{”}.$$

Now in carrying out the definition, (H) involves standard bookkeeping.

For $n = 0$ (we start to work in $V[G_{i+1}]$) our main problem is satisfying (I). We shall now define $q_{\langle\rangle}$. For $j < \kappa$, let N_j be the Skolem Hull of $N \cap \{j\}$ in $\left(H(\lambda)[G_{i+1}], \in, <^*_\lambda\right)$. By (*)(a) and XII 2.6.

$$E^1 = \{j < \kappa : N <_{\omega_2} N_j, j \text{ strongly inaccessible, } |P_i| < j \text{ for every } i < j \text{ and } \Vdash_{P_j/P_{i+1}} \text{“}\underset{\sim}{Q}_j \text{ is semiproper”}\}$$

is a stationary subset of κ. So by the Fodor lemma [as $\delta \in E^1 \Rightarrow \mathrm{cf}(\delta) > \aleph_0$ in $V[G_{i+1}]$ and $\mu < \kappa \Rightarrow \mu^{\aleph_0} < \kappa$] we know that for some stationary $E^2 \subseteq E^1$, $\langle N_j : j \in E^2\rangle$ form a Δ-system; let $\cap\{N_j : j \in E^2\}$ be $N'_{\langle\,\rangle}$. For $j \in E^2$ let $q_j \in P_\kappa/P_{i+1}$ be $(N_j, P_\kappa/P_{i+1})$-semi-generic and above p^a. Now we know that $P_j = \bigcup_{\zeta<j} P_\zeta$, hence by the Fodor Lemma w.l.o.g. $q_j{\restriction}j$ is constant, so let this constant value be called $q_{\langle\,\rangle}$. Clearly $q_{\langle\,\rangle}$ is $(N'_{\langle\rangle}, P_\kappa/P_{i+1})$-semi-generic and it is the $q_{\langle\,\rangle}$ which we promised. Now we actually choose G_κ i.e. a subset of P_κ generic over V and including $G_{i+1} \cup \{q_{\langle\rangle}\}$. Let $N_{\langle\,\rangle} = N'_{\langle\,\rangle}[G_\kappa] \cap H(\lambda)[G_{i+1}]$. So $N_{\langle\,\rangle} \prec \left(H(\lambda)[G_{i+1}], \in, <^*_\lambda\right)$, moreover $(N_{\langle\rangle}, G_\kappa \cap N_{\langle\rangle}) \prec (H(\lambda)[G_{i+1}], \in, <^*_\lambda, G_\kappa)$ so $N_{\langle\rangle})$ is as required in clause (E). As for clause (I), by the genericity of G_κ we have $\{j \in E^2 : q_j \in G_\kappa\}$ is unbounded in κ (even stationary) and it include E^0_η (think).

For $n > 0$ assume N_η are defined, $\ell g(\eta) = n - 1$. Clearly, as P_κ satisfies the κ-c.c., for some $\varepsilon_\eta < \kappa$ we have $\langle N_{\eta\restriction\ell} : \ell \le \ell g(\eta)\rangle$ belongs to $V[G_{\varepsilon_\eta}]$ and ε_η is a successor ordinal $> \sup(N_\eta \cap \kappa)$. By (I), E^0_η is a stationary subset of κ, and we shall define $E^1_\eta \supseteq E^0_\eta$ stationary and will let

$$T_{\ell g(\eta)+1} \cap \{\nu : \eta \lhd \nu \in {}^{(n+1)}\kappa\} = \{\eta\hat{\ }\langle j\rangle : j \in E^1_\eta\}.$$

So $T_{\ell g(\eta)+1}$ will really be constructed as required.

Actually E^0_η is the interpretation of some $P_\kappa/G_{\varepsilon_\eta}$-name $\underset{\sim}{E}^0_\eta$ forced to be as above: just read the definition in clause (I). W.l.o.g. some member of G_{ε_η} force ($\Vdash_{P_\kappa}$) that N, $\underset{\sim}{E}^0_\eta$ are as above.

In $V[G_\kappa]$ for each $\gamma \in E^0_\eta = \underset{\sim}{E}^0_\eta[G_\kappa/G_{\varepsilon_\eta}]$ there is $q^1_{\eta,\gamma} \in G_\kappa/G_{\varepsilon_\eta}$ such that $q^1_{\eta,\gamma} \Vdash$ "$\gamma \in \underset{\sim}{E}^0_\eta$". So in $V[G_\kappa]$, for some $q^2_\eta \in G_\kappa$ we have $\{\gamma \in E^0_\eta : q^1_{\eta,\gamma}\restriction\gamma = q^2_n\}$ is stationary. As we can increase ε_η w.l.o.g. $q^2_\eta \in G_{\varepsilon_\eta}$. In $V[G_{\varepsilon_\eta}]$ we define

$$\begin{aligned} E^1_\eta = \{\gamma : {} & \text{there is } q = q^3_{\eta,\gamma} \text{ such that } q^3_{\eta,\gamma}\restriction\gamma = q^1_\eta \text{ and} \\ & q^3_{\eta,\gamma} \Vdash_{P_\kappa/G_{\varepsilon_\eta}} \text{“}\gamma \in \underset{\sim}{E}^0_\eta\text{”}\}, \end{aligned}$$

so $E^1_\eta \in V[G_{\varepsilon_\eta}]$, $E^1_\eta \supseteq E^0_\eta$ hence E^1_η is stationary.

So, in $V[G_\kappa]$, $N^0_{\eta,\gamma}$ = the Skolem Hull of $N_\eta \cup \{\gamma\}$ in $(H(\lambda)[G_{i+1}], \in, <^*_\lambda, G_\kappa)$, clearly $N^0_{\eta,\gamma} \subseteq N_{\eta,\gamma}$ (as G_γ is definable from G_κ and γ) hence $N_\eta <_\kappa N^0_{\eta,\gamma}$. Also for every $x \in N^0_{\eta,\gamma}$ for some function $f \in N_\eta$, $\mathrm{Dom}(f) = \kappa$, $f(\gamma)$ is a P_γ-name of a member of $H(\lambda)$ and $x = f(\gamma)[G_\gamma]$.

But P_γ satisfies the γ-c.c., hence $f(\gamma)$ is a P_β-name for some $\beta < \gamma$ and let $h_f(\gamma) < \gamma$ be minimal such β so $h_f(\gamma) \in N^0_{\eta,\gamma}$, but as $N_\eta <_\kappa N^0_{\eta,\gamma}$, it follows that $h_f(\gamma) \in N_\eta$, so $\sup_{f \in N_\eta}(h_f(\gamma)) < \gamma$, hence, increasing ε_η and decreasing E^1_η (preserving their properties) w.l.o.g. we have $N^0_{\eta,\gamma} \in V[G_{\varepsilon_\eta}]$.

For $\gamma \in E^1_\eta$, choose any $G'_{\gamma+1}$ such that $q^3_{\eta,\gamma}\restriction(\gamma+1) \in G'_{\gamma+1}$, $G_\gamma \subseteq G'_{\gamma+1}$ and $G'_{\gamma+1}$ is generic over V. Let our bookkeeping give us $\underset{\sim}{\Xi}_\eta \in N_\eta \subseteq N^0_{\eta,\gamma}$, a P_κ-name of a pre-dense subset of $\mathfrak{B}[P_\kappa]$.

We shall now prove that condition (a) of 2.12 holds for the iteration

$$\langle P_j/G'_{\gamma+1}, \underset{\sim}{Q}_j : \gamma + 1 \le j < \kappa \rangle$$

and any non-limit ordinal (denoted by i in 2.12) in the universe $V[G'_{\gamma+1}]$.

Let $\xi \in [\gamma+1, \kappa)$ be a non-limit (or just $\underset{\sim}{Q}_\xi$ semiproper). By (*)(b) from the assumptions of 3.4 we can find $\gamma(\xi) \in E^*, \xi < \gamma(\xi) < \kappa$, such that:

$$\Vdash_{P_{\gamma(\xi)}} \text{“}\mathrm{Rss}(\aleph_2^{V^{P_{\gamma(\xi)}}}) \text{ and } Q_{\gamma(\xi)} \text{ is semiproper”}.$$

Now $(P_\kappa/P_{\gamma(\xi)}) *$ seal($\underset{\sim}{\Xi}$) does not destroy stationary subsets of ω_1 (as $P_\kappa/P_{\gamma(\xi)}$ is semiproper and $\underset{\sim}{\Xi}$ is pre-dense so that seal($\underset{\sim}{\Xi}$) preserves stationary subsets of ω_1); so because $\gamma(\xi) \in E^*$ this forcing is semiproper by 1.7(3). As Q_ξ is semiproper, $P_{\gamma(\xi)}/P_\xi$ is semiproper. Hence $(P_\kappa/P_\xi)*$seal($\underset{\sim}{\Xi}$) is semiproper. So condition (a) of 2.12 holds, hence condition (b) of 2.12 holds. Let $N^1_{\eta,\gamma}$ be the Skolem hull of $N^0_{\eta,\gamma}$ in $(H(\lambda)^{V[G_{i+1}]}, \in, <^*_\lambda, G'_{\gamma+1})$. Note that $q^3_{\gamma,\eta} \Vdash$ "$N^0_{\eta,\gamma} \prec N^1_{\eta,\gamma} \subseteq N_{\eta,\gamma}$", hence $q^3_{\eta,\gamma} \Vdash$ "$N_\eta \prec N^0_{\eta,\gamma} \prec N^1_{\eta,\gamma} \prec N_{\eta,\gamma}$ and $N_\eta \cap \kappa = N^0_{\eta,\gamma} \cap \gamma = N^1_{\eta,\gamma} \cap \sup(N_\eta \cap \kappa) \subseteq \kappa$".

Now by 2.12(b) applied in $V[G'_{\gamma+1}]$, there is countable model $N^2_{\eta,\gamma}$ satisfying $N^2_{\eta,\gamma} \prec \left(H(\lambda)^{V[G_{i+1}]}, \in, <^*_\lambda, G'_{\gamma+1}\right)$ such that $N^1_{\eta,\gamma} <_{\gamma+1} N^2_{\eta,\gamma}$ (remember 0.1(9), and $\Vdash_{P_{\gamma+1}}$ "$\gamma < \aleph_2$") and $q^4_{\eta,\gamma} \in P_\kappa/G_{\gamma+1}$ and $j_{\eta,\gamma} < \kappa$ successor such that:

(i) $q^4_{\eta,\gamma} \dot{\in} P_{j_{\eta,\gamma}}/G'_{\gamma+1}$, $\gamma < j_{\eta,\gamma} \in N^2_{\eta,\gamma}$,

(ii) $q^4_{\eta,\gamma} \geq q^3_{\eta,\gamma}$,

(iii) $q^4_{\eta,\gamma}$ is $(N^2_{\eta,\gamma}, P_{j_{\eta,\gamma}})$-semi-generic, and

(iv) $q^4_{\eta,\gamma} \Vdash_{P_{j_{\eta,\gamma}}/P_{\gamma+1}}$ "for some $\underset{\sim}{A} \in N^2_{\eta,\gamma}$ we have: $\underset{\sim}{A} \in \underset{\sim}{\Xi}_\eta$ and $N \cap \omega_1 \in \underset{\sim}{A}$" and $\underset{\sim}{A}$ is a $P_{j_{\eta,\gamma}}$-name.

Also by $(*)$(a) of the assumption of 3.4, there is $\xi_{\eta,\gamma} > \sup(N^2_{\eta,\gamma} \cap \kappa) > \gamma$ strongly inaccessible, such that $\bigwedge_{\xi<\xi_{\eta,\gamma}} |P_\xi| < \xi_{\eta,\gamma}$, and $N^3_{\eta,\gamma} =$ Skolem Hull of $N^2_{\eta,\gamma} \cup \{\xi_{\eta,\gamma}\}$ in $\left(H(\lambda)[G_{\gamma+1}], \in, G'_{\gamma+1} <^*_\lambda\right)$ satisfyies $N^2_{\eta,\gamma}[G'_{\gamma+1}] <_\kappa N^3_{\eta,\gamma}$ and $q^4_{\eta,\gamma} \in P_{\xi_{\eta,\gamma}}$. Back to $V[G_\gamma]$, let $N^4_{\eta,\gamma}$ be the Skolem Hull of $N^1_{\eta,\gamma} \cup \{\gamma, \xi_{\eta,\gamma}\}$ in $(H(\lambda)[G_{i+1}], \in, <^*_\lambda, G_{\varepsilon_\eta})$, and $q^5_{\eta,\gamma} \in P_{\xi_{\eta,\gamma}}/G_{\varepsilon_\eta}$ forces all the above and in particular is above q^3_η and $q^4_{\eta,\gamma}$. In addition $q^5_{\eta,\gamma}\restriction[\gamma+1,\kappa) = q^4_{\eta,\gamma}\restriction[\gamma+1,\kappa) = q^4_{\eta,\gamma}\restriction[\gamma+1,\xi_{\eta,\gamma})$, and $q^5_{\eta,\gamma}\restriction(\gamma+1) \in G'_{\gamma+1}$, so $q^4_{\eta,\gamma}\restriction\varepsilon_\eta \in G_{\varepsilon_\eta}$ and $N_\eta \prec N^4_{\eta,\gamma}$, $\gamma \in N^4_{\eta,\gamma}$, $N_\eta \cap \kappa = N_\eta \cap \gamma = N^4_{\eta,\gamma} \cap \sup(N_\eta \cap \kappa)$, and

$$q^4_{\eta,\gamma} \Vdash \text{"the Skolem Hull } N^5_{\eta,\gamma} \text{ of } N^4_{\eta,\gamma} \text{ in } (H(\lambda)^{V[G_{i+1}]}, \in, <^*_\lambda, \underset{\sim}{G}_{\gamma+1})$$
$$\text{satisfies } N^5_{\eta,\gamma} \cap \sup(N_\eta \cap \kappa) = N_\eta \cap \kappa\text{"},$$

hence

$$q^5_{\eta,\gamma} \Vdash \text{"the Skolem Hull } N^6_{\eta,\gamma} \text{ of } N^4_{\gamma,\gamma} \text{ in } (H(\lambda)^{V[G_{i+1}]}, \in, <^*_\lambda, \underset{\sim}{G}_{\gamma+1})$$
$$\text{satisfies } N^6_{\eta,\gamma} \cap \sup(N_\eta \cap \kappa) = N_\eta \cap \kappa \text{ (as } V[G_{\gamma+1}] \models |\gamma| = \aleph_1)\text{"}.$$

(looking at the definition of $\underset{\sim}{E}^0_\eta$ in clause (I) above). As we can increase ε_η and decrease E^2, w.l.o.g. $q^5_\eta \restriction \gamma \in G_{\varepsilon_\eta}$ and $q^5_\eta \restriction \gamma$ is the same for all $\gamma \in E^2$.

Now as $q^5_{\eta,\gamma} \in P_\kappa / G_{\varepsilon_\eta}$ and $q^5_{\eta,\gamma} \restriction \gamma \in G_{\varepsilon_\eta}$, easily $\Vdash_{P_\kappa/G_{\varepsilon_\eta}}$ "$\underset{\sim}{E}^1_\eta \stackrel{\text{def}}{=} \{\gamma : q^5_{\eta,\gamma} \in \underset{\sim}{G}_\kappa\}$ is a stationary subset of κ", so we have defined at least $\underset{\sim}{E}^1_\eta$. Now in $V[G_\kappa]$, if $\gamma \in \underset{\sim}{E}^1_\eta[G_\kappa]$ then $\gamma \in E^0_\eta$ (see above). We still have to define $N_{\eta \frown \langle\gamma\rangle}$ and $E^0_{\eta \frown \langle\gamma\rangle}$ (for $\gamma \in \underset{\sim}{E}^1_\eta[G_\kappa]$). For each such γ we repeat the proof in the case $n = 0$ with universe $V[G_{\xi_{\eta,\gamma}}]$ and Skolem Hull of $N^4_{\eta,\gamma}$ in $(H(\lambda)^{V[G_{i+1}]}, \in, <^*_\lambda, G_{\xi_{\eta,\eta}})$ here standing for $V[G_{i+1}]$, N there.

We have carried out the construction.

We now define by induction on n, for every $\eta \in T \cap {}^n\kappa$, a condition $p^b_\eta \in \mathrm{Nm}$ and $m_\eta < \omega$ such that (note $N_\eta[G_\kappa] \prec (H(\lambda)[G_\kappa], \in, <^*_\kappa)$, $N_\eta[G_\kappa] \cap H(\lambda)[G_{i+1}] = N_\eta$):

(a) $p^b_\eta \in N_\eta[G_\kappa]$, $m_\eta < \omega$ and $p^b_{\langle\rangle} = \underset{\sim}{p}^b[G_\kappa]$,

(b) $p^b_\eta \in \mathrm{Nm}$, and $\mathrm{tr}(p^b_\eta)$ (the trunk of p^b_η) has length $\geq \ell g(\eta)$ (and has κ immediate successors in p^b_η),

(c) $p^b_{\eta\restriction\ell} \leq p^b_\eta$ and $m_{\eta\restriction\ell} \leq m_\eta$ when $\ell \leq \ell g(\eta)$; and if $p^b_{\eta\restriction\ell}$ has a trunk of length $> \ell g(\eta)$ or $m_{\eta\restriction\ell} > \ell g(\eta)$ *then*: $p^b_\eta = p^b_{\eta\restriction\ell}$ & $m_\eta = m_{\eta\restriction\ell}$,

(d) if $\eta \in T_n$, $\underset{\sim}{\alpha}$ is a Nm-name for a countable ordinal, $\underset{\sim}{\alpha} \in N_\eta[G]$, *then* for some $k = k^1(\underset{\sim}{\alpha}, \eta)$, and every $\nu \in \bigcup_{m<\omega} T_m$, for some ordinal $\beta = \beta(\underset{\sim}{\alpha}, \nu) \in N_\nu$ we have

$$k{+}1 = |\{\ell < \ell g(\nu) : m_{\nu\restriction\ell} < m_{\nu\restriction(\ell+1)}\}| \;\&\; \eta \trianglelefteq \nu \Rightarrow p^b_\nu \Vdash_{\mathrm{Nm}} \text{“}\underset{\sim}{\alpha} = \beta(\underset{\sim}{\alpha}, \nu)\text{”},$$

(e) if $\eta \in T_n$ and $\underset{\sim}{\Xi}$ is a Nm-name for a pre-dense subset of $\mathfrak{B}[P_\kappa]$ and $\underset{\sim}{\Xi} \in N_\eta[G_\kappa]$, *then* for some $k = k^2(\underset{\sim}{\Xi}, \eta)$, for every $\nu \in \bigcup_{m<\omega} T_m$ we have:

$$[k{+}1 \leq |\{\ell < \ell g(\nu) : m_{\nu\restriction\ell} < m_{\nu\restriction(\ell+1)}\}| \;\&\; \eta \trianglelefteq \nu]$$
$$\Rightarrow [\text{for some } A \in N_\nu[\underset{\sim}{G}_{N_m}] \text{ we have } N \cap \omega_1 \in A \;\&\; p^b_\nu \Vdash_{\mathrm{Nm}} \text{“}A \in \underset{\sim}{\Xi}\text{”}].$$

(f) if p^b_η has a trunk of length $\leq \ell g(\eta)$, say ν_η, and $m_\eta \leq \ell g(\eta)$ and if h_η is a one-to-one function from κ onto $\{j < \kappa : \nu_\eta {}^\frown \langle j\rangle \in p^b_\eta\}$, $h_\eta \in N_\eta[G_\kappa]$, then for $\eta^\wedge\langle i\rangle \in \bigcup_n T_n$ we have:

$$(\forall \rho \in p^b_{\eta^\frown\langle i\rangle})[\ell g(\rho) > \ell g(\eta)) \Rightarrow \rho(\ell g(\eta)) = h_\eta(i)],$$

(g) for $\eta \in T_n$ we have: the sequences $\langle k^1(\underset{\sim}{\alpha}, \eta) : \underset{\sim}{\alpha} \in N_\eta[G_\kappa]$ is a Nm-name of a countable ordinal$\rangle$ and $\langle k^2(\underset{\sim}{\Xi}, \eta) : \underset{\sim}{\Xi} \in N_\eta[G_\kappa]$ is a Nm-name of a predense subset of $\mathfrak{B}[P_\kappa]\rangle$ are with no repetitions, with disjoint ranges whose union is a co-infinite subset of ω [Why the m_η's? just as below Υ depend on p^b_η].

There is no problem to do this. [For (e), when we come to deal with $\underset{\sim}{\Xi}$, say at η, where p^b_η has trunk of length $\leq \ell g(\eta)$ and $m_\eta \leq \ell g(\eta)$, we let

$$\Upsilon = \{A : (\exists p)(p^b_\eta \leq p \in \text{Nm} \;\&\; p \Vdash_{\text{Nm}} \text{“}A \in \underset{\sim}{\Xi}\text{”})\}.$$

So $\Upsilon \in N_\eta[G_\kappa]$ is a pre-dense subset of $\mathfrak{B}(P_\kappa)$, and by (H) above there is $k(\underset{\sim}{\Upsilon}, \rho)$ as there, choose it as $k^2(\underset{\sim}{\Xi}, \eta)$, so we shall have $p^b_\nu = p^b_\rho$ if $\nu \lhd \rho \in T'_{\ell g(\rho)}, \ell g(\rho) \leq \ell g(\nu) + k^2(\Xi, \eta)$.]

Now in V^{P_κ} let:

$$q^b = \{\rho \in {}^{\omega>}\kappa : \rho \in \underset{\sim}{p}^b[G_\kappa] \text{ and for some } \eta \in \bigcup_n T_n, \rho \text{ is an initial segment of the trunk of } p^b_\eta\}.$$

We can easily see that $p^b \leq q^b \in \text{Nm}$ (in $V[G_\kappa]$). Also (in $V[G_\kappa]$) q^b is $(N[G_\kappa]$,Nm)-semi-generic and moreover

$$q^b \Vdash_{\text{Nm}} \text{“}\kappa \cap N[G_\kappa][\underset{\sim}{G}_{\text{Nm}}] = \kappa \cap \bigcup_{l<\omega} N_{\underset{\sim}{\nu}\restriction l}[G_\kappa]\text{”},$$

where $\underset{\sim}{G}_{\text{Nm}}$ is the (canonical name of the) generic subset of Nm and $\underset{\sim}{\eta}$ is the Nm-name of the ω-sequence in ${}^\omega\kappa$ which it defines naturally and $\underset{\sim}{\nu}$ is the Nm-name of the ω-sequence in ${}^\omega\kappa$ such that $\nu\restriction\eta \in T_n$ and the trunk of $p^b_{\nu\restriction\eta}$ is $\lhd\eta$. [Remember that if $N_1, N_2 \prec (H(\lambda), \in), N_1 \cap \omega_1 = N_2 \cap \omega_1$, and $i \in N_1 \cap N_2$, $i < \aleph_2$, then $N_1 \cap i = N_2 \cap i$.] Hence $q^b \Vdash_{N_m}$ “$\mathcal{P}(\omega_1)^{V[G_\kappa]} \cap N[G_\kappa][\underset{\sim}{G}_{\text{Nm}}] = \mathcal{P}(\omega_1)^{V[G_\kappa]} \cap \bigcup_{l<\omega} N_{\underset{\sim}{\nu}\restriction l}[G_\kappa]$”.

Now clearly by the above and (e) we have

$$q^b \Vdash_{\mathrm{Nm}} \text{"for every pre-dense subset } \Xi \text{ of } \mathfrak{B}[P_\kappa] \text{ in } N[G_\kappa][\underset{\sim}{G}_{\mathrm{Nm}}],$$

$$N \cap \omega_1 \in \bigcup_{A \in \Xi} \{A : A \in \Xi \cap N[G_\kappa][\underset{\sim}{G}_{\mathrm{Nm}}]\}\text{"}.$$

So we can apply Claim 2.9 to get q^c, which is $(N[G_\kappa][G_{\mathrm{Nm}}], \mathrm{SSeal}(\mathfrak{B}[P_\kappa], S))$-semi-generic $\geq \underset{\sim}{p}^c[G_\kappa][\underset{\sim}{G}_{\mathrm{Nm}}]$. Let $q^a = q_{\langle\,\rangle}$ so we are assuming just $q^a \in G_\kappa \subseteq P_\kappa, G_\kappa$ generic over V and so for some P_κ-name $\underset{\sim}{q}^b$, we have: $q^a \Vdash_{P_\kappa}$ "$\underset{\sim}{q}^b$ is as above ". Similarly for some $\underset{\sim}{q}^c$, $(q^a, \underset{\sim}{q}^b) \Vdash_{(P_\kappa/P_{i+1})*\underset{\sim}{\mathrm{Nm}}}$ "$\underset{\sim}{q}^c$ is as above". Now $(q^a, \underset{\sim}{q}^b, \underset{\sim}{q}^c)$ is as required (i.e., (R_{i+1}, N)-semi-generic). $\square_{3.4}$

3.4B Remark. It seems that we can weaken clause (a) of $(*)$ of 3.1 to
(a)$'$ for $i < \kappa$ in $V^{P_{i+1}}$, player II wins in the game $\eth(\{\aleph_1\}, \omega, \kappa)$.

See [Sh:311]

3.5 Claim. Suppose $\bar{Q} = \langle P_i, \underset{\sim}{Q}_j : i \leq \kappa,\ j < \kappa\rangle$ is a semiproper iteration, $\kappa > |P_i|$ for $i < \kappa$ and $S \subseteq \omega_1$ is stationary. Suppose further that

$(*)$ (a) for $i < \kappa$, in V^{P_i}, Player II wins in $\eth(\{\aleph_1\}, \omega, \mathcal{D}_\kappa + E_i^+)$ where $E_i^+ = \{\delta < \kappa : \delta > i,\ \delta$ strongly inaccessible, $\Vdash_{P_\delta/P_i}$ "$\underset{\sim}{Q}_\delta$ is semiproper"$\}$,

(b) $E = \{i < \kappa : \Vdash_{P_i}$ "$\mathrm{Rss}(\aleph_2)$ and $\underset{\sim}{Q}_i$-semiproper"$\}$ is unbounded,

(c) It is forced ($\Vdash_{P_\kappa}$) that $\underset{\sim}{W} \subseteq \{\delta < \kappa : V^{P_\kappa} \models$ "$\mathrm{cf}(\delta) = \aleph_0$" $\}$ is stationary ($\underset{\sim}{W}$ a P_κ-name).

Then $(P_\kappa/P_{i+1})*\mathrm{club}_{\aleph_1}(\underset{\sim}{W})*\mathrm{SSeal}\ (\mathfrak{B}(P_\kappa), S)$ is semiproper for $i < \kappa$ where $\mathrm{club}_\mu(W) \overset{\text{def}}{=} \{f$: for some non-limit $\gamma < \mu, f$ is an increasing continuous function from γ into $W\}$.

Proof. Like the previous claim, only after defining N_η for a set $G_\kappa \subseteq P_\kappa$ generic over $V, q_{\langle\,\rangle} \in G_\kappa$, in $V[G_\kappa]$ there is $\eta \in {}^\omega\kappa$, $\bigwedge_n(\eta{\restriction}n \in T_n)$ such that $\eta(\ell) > \sup(N_{\eta\restriction\ell} \cap \kappa)$ and $\sup\{\eta(\ell) : \ell < \omega\}$ belong to $\underset{\sim}{W}[G_\kappa]$ and then in $V[G_\kappa]$ continue with $\bigcup_\ell N_{\eta\restriction\ell}[G]$. $\square_{3.5}$

3.5A Remark. 3.5, 3.4 are cases of a more general theorem, see XV.

3.6 Claim. In 3.4, 3.5, if we add to the hypothesis:

$(*)$ player II wins in $V^{P_i}(i < \kappa)$, for $\mathcal{D}_\kappa$ in the game of "divide and choose" i.e. X 4.9 for $S = \{2, \aleph_0, \aleph_1\}, \alpha = \omega$,

$(*)'$ for $i < j < \kappa$ non-limit, P_j/P_i is pseudo $(*, \omega_1 \setminus S^*)$-complete,

then $(P_\kappa/P_{i+1}) * \underset{\sim}{\mathrm{Nm}}$ and $(P_\kappa/P_{i+1}) * \underset{\sim}{\mathrm{club}}_{\aleph_1}(\underset{\sim}{W})$ are pseudo $(*, \omega_1 \setminus S^*)$-complete.

Proof. Left to the reader.

3.7 Theorem. 1) Suppose $\{\mu < \kappa : \mu \text{ supercompact}\}$ is unbounded below κ and κ is 3-Mahlo.

If $\langle S_1, S_2, S_3\rangle$ is a partition of ω_1 with S_1 stationary, *then* for some semiproper pseudo $(*, S_3)$-complete forcing notion P satisfying the κ-c.c., we have:

$\Vdash_P$ "$\mathfrak{B}[P_\kappa]\restriction S_1$ has a dense subset which is (up to isomorphism) Levy$(\aleph_0, < \aleph_2)$".

Proof. We define by induction on i, P_i, $\underset{\sim}{Q}_i$, $\mathbf{t}_i$ such that

(A) $\bar{Q}^\alpha = \langle P_i, \underset{\sim}{Q}_j, \mathbf{t}_j : i \le \alpha,\ j < \alpha\rangle$ is S_1-suitable,

(B) there is no strongly inaccessible Mahlo λ, $i < \lambda \le |P_i|$,

(C) if i is a singular ordinal or $(\exists j < i)[|P_j| > i]$ or i inaccessible not a limit of supercompacts or i inaccessible not Mahlo *then* $\mathbf{t}_i = 0$, $\underset{\sim}{Q}_i =$ SSeal$(\langle \mathfrak{B}[P_j] : j \le i,\ \mathbf{t}_j = 1\rangle, S_1)$ (as defined in V^{P_i}, of course),

(D) if i is supercompact, not limit of supercompacts *then* $\mathbf{t}_i = 1, \underset{\sim}{Q}_i =$ SSeal$(\mathfrak{B}[P_i], S_1)$,

(E) if $(\forall j < i)[|P_j| < i], i$ limit of supercompacts and i is inaccessible 1-Mahlo but not 2-Mahlo, we let $\mathbf{t}_i = 1$, $\underset{\sim}{Q}_i = \underset{\sim}{\mathrm{Nm}} *$ SSeal$(\mathfrak{B}[P_i])$ (the SSeal in $V^{P_i * \underset{\sim}{\mathrm{Nm}}}$ of course),

(F) if $(\forall j < i)[|P_j| < i], i$ is 2-Mahlo and a limit of supercompacts then $W_i \overset{\text{def}}{=} \{\delta < i : \delta = \mathrm{cf}(\delta)$ is Mahlo and a limit of supercompacts and $(\forall j < \delta)[|P_j| < \delta]\}$ is a stationary subset of i, then we let:

$$\mathbf{t}_i = 1, \quad \underset{\sim}{Q}_i = \ \mathrm{club}_{\aleph_1}(W_i) * \ \underset{\sim}{\mathrm{SSeal}}(\mathfrak{B}[P_i], S_1).$$

Why is $\bar{Q}$ S_1-suitable? We shall prove by induction on i that $\bar{Q}\restriction i$ is S_1-suitable.

Note that the use of SSeal guarantees (F) of Definition 2.1, as well as (E) (see 2.11(3), 2.13(2)). Remembering 2.3(2), it suffices to show by induction on i that $j < i \Rightarrow (P_i/P_{j+1}) * \underset{\sim}{Q}_i$ is semiproper (actually the only problematic case is when i is inaccessible limit of supercompacts, but then for arbitrarily large $j < i$ we have $\mathrm{Rss}^+(j)$ (by 1.10, 1.6(2), 1.6(4)), so in V^{P_j}, every forcing notion preserving stationary sets is semiproper, but we check by cases:

For $i = 0$: trivial.

For $i+1$, and i satisfies clause (C) above (in the definition of $\bar{Q}$) the result follows by Claim 2.14.

For $i+1$, and i satisfies (D) above: first note that $|P_j| < i$ for $j < i$, hence $j < i$ & $\mathbf{t}_j = 1 \Rightarrow \mathfrak{B}[P_j] \lessdot \mathfrak{B}[P_i]$, hence by Claim 2.7 we have $Q_i = \mathrm{SSeal}(\mathfrak{B}^{P_i}, S_1) = \mathrm{SSeal}(\langle \mathfrak{B}^{P_j} : j \le i, \mathbf{t}_j = 1\rangle, S_1)$. Now for a club C of i, $j \in C$ & $j = \mathrm{cf}(j) \Rightarrow Q_j$ is semi proper (see the previous case), so by 1.6(4)+1.10 we have $\Vdash_{P_i}$ "$\mathrm{Rss}(\kappa)$ i.e. $\mathrm{Rss}(\aleph_2)$". Hence by claim 2.10, $\mathrm{SSeal}(\mathfrak{B}[P_\kappa], S_1)$ is semiproper in V^{P_κ}.

For $i+1$, and i satisfying clause (E) above: we shall apply 3.4 with i here standing for κ there. Note that condition $(*)$(a) (of 3.4) holds for $E^+_{j,i} \overset{\text{def}}{=} \{\delta < i : \delta > j, \delta$ strongly inaccessible, not Mahlo, $\delta > |P_\zeta|$ for $\zeta < \delta$, Q_δ semiproper$\}$. Why does the second player win $\Game(\{\aleph_1\}, \omega, \mathcal{D}_i + E^+_{j,i})$ in the universe $V^{P_{j+1}}$? By 1.6(6) clearly for $j < i, V^{P_j} \models$ "$\mathrm{Rss}^+(i)$" and use 1.11, (and 1.9A(3), i.e. XII, 2.5(2)) but this give just winning in $\Game(\{\aleph_1\}, \omega, \kappa)$. However for $\mu < i$, there is a μ-complete filter on i containing the clubs of i and $E^+_{j,i}$, so winning the game is easy, and lastly if $j < i$ is strongly inaccessible not Mahlo and $(\forall \varepsilon < j)(|P_\varepsilon| < j)$ then Q_j is even proper by 2.11. Condition $(*)$(b) of 3.4

holds by the definition of case (E): if $\lambda < i$ is supercompact then $\mathrm{Rss}^+(\lambda), Q_\lambda$ semiproper by the induction hypothesis (see previous case) so any $\lambda < i$ which is supercompact satisfies the requirement on E^*.

For $i+1$, and i satisfying clause (F) above: similar to the previous case by replacing 3.4 by Claim 3.5 (and remember 0.1(5) of the Notation).

Also each Q_i is pseudo $(*, S_3)$-complete (by 3.6), hence P_κ is pseudo $(*, S_3)$-complete so when S_3 is stationary,

$$\Vdash_{P_\kappa} \text{“}2^{\aleph_0} = \aleph_1, 2^{\aleph_1} = \aleph_2\text{”}$$

and in any case $\Vdash_{P_\kappa}$ “$2^{\aleph_1} = \kappa = \aleph_2^{(V^{P_\kappa})}$”.

Let $\mathfrak{B}_i = \mathfrak{B}[P_i]$, so $\mathbf{t}_i = 1 \Rightarrow \mathfrak{B}_i \restriction S_1 \lessdot \mathfrak{B}[P_\kappa] \restriction S_1$. Let

$$W^* \stackrel{\text{def}}{=} \{i < \kappa : \mathfrak{B}_i \restriction S_1 \lessdot \mathfrak{B}[P_\kappa] \restriction S_1\}.$$

So in V^{P_κ} (as case (F) occurs stationarily often),

$$W^{**} \stackrel{\text{def}}{=} \{\delta \in W^* : \mathrm{cf}(\delta) = \aleph_1 \text{ and } W^* \text{ contains a club of } \delta\}$$

is stationary. Hence it is well known that in V^{P_κ},

$$\begin{aligned} \mathrm{club}_\kappa(W^*) = \{h : & h \text{ an increasing continuous function} \\ & \text{from some } \alpha + 1 < \kappa \text{ to } W^*\} \end{aligned}$$

does not add bounded subsets to $\kappa (= \aleph_2)$. (More exactly, if CH holds this is straightforward. If CH fails, this holds if we can find $\bar{\mathcal{P}} = \langle \mathcal{P}_\alpha : \alpha < \kappa \rangle$, $\mathcal{P}_\alpha \subseteq \mathcal{S}_{<\aleph_1}(\alpha)$, $|\mathcal{P}_\alpha| \leq \aleph_1$ ($\bar{\mathcal{P}} \in V^{P_\kappa}$ of course) such that $\{\delta \in W^{**}$: for some unbounded C of δ we have that $C \subseteq W^*$, $\mathrm{otp}(C) = \omega_1$ and $\alpha \in C \Rightarrow C \cap \alpha \in \bigcup_{\beta < \alpha} \mathcal{P}_\beta\}$ and this holds (with $\mathcal{P}_\alpha = (\mathcal{S}_{<\aleph_1}(\alpha))^{V^{P_\alpha}}$ in fact $\alpha \in C \Rightarrow C \cap \alpha \in P_\alpha$).)
So forcing will give us a universe as required. $\square_{3.7}$

3.8 Remarks. The proof of 3.1, 3.7 exemplifies two constructions which we may interchange. Another variation is 3.9 below.

3.9 Theorem. Suppose $\{\mu < \kappa : \mu \text{ supercompact}\}$ is unbounded below κ, κ is strongly inaccessible, $h : \kappa \to H(\kappa)$, and $\langle S_1, S_2, S_3 \rangle$ is a partition of ω_1, and S_1 is stationary. *Then* for some forcing notion P:

(i) P satisfies the κ-c.c., is pseudo $(*, S_3)$-complete, has cardinality κ, does not collapse $\aleph_1$ and κ but collapses every $\lambda \in (\aleph_1, \kappa)$ and in $V^{P_\kappa}, \kappa = \aleph_2, 2^{\aleph_1} = \aleph_2$, and $2^{\aleph_0} = \aleph_1 \iff S_3$ stationary,

(ii) $\mathfrak{B}[P]{\restriction}S_1$ has a dense subset isomorphic to Levy $(\aleph_0, < \aleph_2)$,

(iii) in V^P, an axiom holds as strong as h is a diamond, i.e.

(a) If h is a Laver diamond for $x \in H(2^\lambda)$ *then* in $V^P, Ax[Q$ is a pseudo $(*, S_3)$-complete, semiproper$^{*[S_1]}$ (see definition below), $Q \in H(\lambda)]$ (see 3.9A below) and $Ax^+[Q$ is pseudo $(*, S_3)$-complete, semiproper$^{*[S_1]}, Q \in H(\lambda)$ and $\mathfrak{B}[V^P] \lessdot \mathfrak{B}[(V^P)^Q]$.

(b) When $\lambda = \kappa$, then we can weaken the demand on h to: for every $x \subseteq \kappa$ satisfying a Σ^1_1-sentence ψ (i.e. $(\exists z \subseteq \mathcal{P}(\kappa)$ such that $\dots))$ *then* $\{i < \kappa : h(i) = x \cap i, (H(i), \in, x \cap i) \models \psi\}$ is stationary. *Then* a conclusion similar to the one in clause (a) holds for $Q \subseteq H(\kappa)$

where

3.9A Definition. Let $\bar{\mathcal{A}} = \langle \mathcal{A}_\zeta : \zeta < \xi \rangle, \mathcal{A}_\zeta \lessdot \mathfrak{B}[V]$.

1) A forcing notion Q is semiproper$^{*[\bar{\mathcal{A}}]}$ *if* χ regular large enough, $N \prec (H(\chi), \in, <^*_\chi)$ is countable, $Q \in N, \bar{\mathcal{A}} \in N, p \in Q \cap N$ satisfies "$(\forall \Xi, \zeta)[\Xi \in N$ a pre-dense subset of $\mathcal{A}_\zeta$ & $\zeta \in N \cap \xi \Rightarrow N \cap \omega_1 \in \bigcup_{A \in \Xi} A]$" (if $\underset{\sim}{\mathcal{A}}_\zeta$ satisfies the $\aleph_2$-c.c. this always holds) *then* there is $q \in Q$ which is (N, Q)-semi-generic and $q \Vdash$ "if $\zeta \in \xi \cap N[G_Q]$ and $\Xi \in N[G_Q]$ is a pre-dense subset of $\mathcal{A}_\zeta$, *then* $N \cap \omega_1 \in \cup\{A : A \in N[G_Q]\}$".

2) If $\xi = 1, \mathcal{A}_\zeta = \{A \subseteq \omega_1 : A \cap (\omega_1 \setminus S) \in \{\emptyset, \omega_1 \setminus S\}\}$, write $*[S]$ instead $*[\bar{\mathcal{A}}]$. We do not strictly distinguish between $\mathfrak{B}[V]{\restriction}S$ and $\{A \in \mathfrak{B}[V] : A \cap (\omega_1 \setminus S) \in \{\emptyset, \omega_1 \setminus S\}\}$.

Proof. We define by induction on $\alpha < \kappa, P_i, \underset{\sim}{Q}_i, \mathbf{t}_i$ for $i < \alpha$ such that:

(A) $\bar{Q}^\alpha = \langle P_i, \underset{\sim}{Q}_j, \mathbf{t}_j : i \le \alpha, j < \alpha \rangle$ is S_1-suitable, $|P_i| < \kappa$ for $i < \kappa$, and for $\alpha < \kappa, \bar{Q}^\alpha \in H(\kappa)$ and $\mathbf{t}_i = 1 \Leftrightarrow$ (i successor *or* i strongly inaccessible

& $\bigwedge_{j<i} |P_j| < i$), (note that for i limit we are trying to get $\mathfrak{B}^{\bar{Q}\restriction i} \lessdot \mathfrak{B}^{P_\kappa}$, not $\mathfrak{B}^{P_i} \lessdot \mathfrak{B}^{P_\kappa}$). Let $\underset{\sim}{\mathcal{A}}_j$ be the following P_j-name: if $j=0$ we let $\underset{\sim}{\mathcal{A}}_j$ be trivial, if $j>0$ we let it be $\mathfrak{B}^{\bar{Q}\restriction j} = \bigcup_{\beta<j} \mathfrak{B}[P_{\beta+1}]$.

(B) For i non-limit, let κ_i be the first supercompact $> |P_i|$,

$$\text{if } i=0, \text{ let } Q_i = \mathrm{Levy}(\aleph_1, <\kappa_0),$$

$$\text{if } i>0, \text{ let } Q_i = \mathrm{SSeal}(\langle \underset{\sim}{\mathcal{A}}_j : j \leq i\rangle, S_1, \kappa_i).$$

(C) For i limit $< \kappa$ such that $h(i)$ is a P_i-name of a pseudo $(*, S_3)$-complete semiproper$^{*[\bar{\mathcal{A}}^i]}$, where $\bar{\mathcal{A}}^i \overset{\text{def}}{=} \langle \mathcal{A}_j : j \leq i\rangle$, remember $\mathcal{A}_i = \mathfrak{B}^{\bar{Q}\restriction i}$.

Let $\kappa_{i+1} < \kappa$ be such that $h(i) \in H(\kappa_{i+1})$, κ_{i+1} supercompact and $Q_i = h(i) * \underset{\sim}{\mathrm{SSeal}}(\langle \bar{\mathcal{A}}^i, S_1, \kappa_{i+1}\rangle)$.

(D) For i limit, but (C) does not hold, let $Q_i = \mathrm{SSeal}(\langle \underset{\sim}{\mathcal{A}}_j : j \leq i\rangle, S_1, \kappa_{i+1})$, κ_{i+1} as before.

We can prove by induction on α that $\bar{Q}^\alpha$ is S_1-suitable and Q_α is semiproper, and if $i < \kappa$ is successor, then $\Vdash_{P_i}$ "$\mathrm{Rss}(\aleph_2)$". If α is limit ordinal use 2.3(1) and for $\alpha = 0$ this should be clear. If $\alpha = \beta + 1$, β not limit by 2.11 we can see that $\bar{Q}^\alpha$ is S_1-suitable, i.e. the first phrase holds. For the second, clearly by 2.7 we have $\underset{\sim}{Q}_\beta = \mathrm{SSeal}(\mathfrak{B}[P_\beta], S_1, \kappa_{\beta+1})$, and by the induction hypothesis $V^{P_\beta} \models$ "$\mathrm{Rss}(\aleph_2), \aleph_2 = \kappa_{\beta+1}$", hence by 2.8(7), $\Vdash_{P_\beta}$ "$\underset{\sim}{Q}_\beta$ is semiproper". Moreover in V^{P_β}, Q_β is an iteration (see Definition 2.4(5)) $\langle P^\beta_i, \underset{\sim}{Q}^\beta_j : i \leq \kappa_{\beta+1}, j < \kappa_{\beta+1}\rangle$ and for every strongly inaccessible $j < \kappa_\beta$, Q^β_j and even $P^\beta_{\kappa_\beta}/P^\beta_j$ are proper by 2.11. So by 1.10 we have $\Vdash_{P_\alpha}$ "$\mathrm{Rss}(\kappa_\beta)$". For $\alpha = \beta+1$, β limit use 4.9 from the next section and 2.11 for the first phrase (if clause (D) apply then use 2.13), the second is proved as in the previous case. Remembering strong preservation of pseudo $(*, S_3)$-completeness we have no problems. $\square_{3.9}$

3.9B Remark. We can wave in the proof some $\mathbf{t}_i = 1$, more acurately some $\underset{\sim}{\mathcal{A}}_\zeta$'s and then get stronger forcing axioms.

§4. $\mathcal{P}(\omega_1)/(\mathcal{D}_{\omega_1} + S)$ is Reflective or Ulam

In 4.3 we deal with reflectiveness: if $A_i \subseteq S \subseteq \omega_1$ is stationary for $i < \aleph_2$ then for some $W \subseteq \aleph_2$ of cardinality $\aleph_2$, $[w \subseteq W \ \& \ |w| \leq \aleph_0 \Rightarrow \bigcap_{i \in w} A_i$ is stationary]. Claims 4.1, 4.2 prepare the ground. In 4.4 we deal with the Ulam property, for this we prove in ZFC a sufficient condition for a filter to satisfy the Ulam property (see 4.5A – 4.5F, Definition 4.6 and the proof of the consistency of the Ulam property (i.e. 4.4) in 4.7). The rest of the section deal with the forcing.

4.1 Claim. Suppose $S \subseteq \omega_1$ is stationary $\bar{Q} = \langle P_i, \underset{\sim}{Q}_j : i \leq \alpha,\ j < \alpha \rangle$ is a semiproper iteration, $\mu < \alpha$ ($\mu = 0$ is allowed), and $\Vdash_{P_\mu}$ "Rss($\aleph_2[V^{P_\mu}]$)" (e.g., if μ is supercompact, $[i < \mu \Rightarrow |P_i| < \mu]$ and $\{i < \mu : \underset{\sim}{Q}_i$ is semiproper (i.e. $\Vdash_{P_i}$ "$\underset{\sim}{Q}_i$ is semiproper")$\}$ belongs to some normal ultrafilter on μ); note that $\Vdash_{P_\mu}$ "$\mu = \aleph_2$" if μ is strongly inaccessible, $|P_i| < \mu$ for $i < \mu$.

Let $\underset{\sim}{A}$ be a P_α-name for a subset of S and $\underset{\sim}{B}$ a P_α-name for a member of $\mathfrak{B}[P_\mu]$ such that:

$$\Vdash_{P_\alpha} \text{“}(\forall X \in \mathfrak{B}^{P_\mu})[0 < X \leq \underset{\sim}{B} \Rightarrow X \cap \underset{\sim}{A} \neq 0 \text{ (in } \mathfrak{B}^{P_\alpha})]\text{”}.$$

Then

$\otimes$ *if* λ is regular and large enough, $N \prec (H(\lambda), \in, <^*_\lambda)$ is countable, and $\bar{Q}$, λ, p, $\underset{\sim}{A}$, $\underset{\sim}{B}$ and μ belong to N, $p \in P_\alpha \cap N$ and $q \in P_\mu$ is (N, P_μ)-generic, $p{\restriction}\mu \leq q$ and $q \cup p{\restriction}[\mu, \alpha) \Vdash_{P_\alpha}$ "$N \cap \omega_1 \in \underset{\sim}{B}$" (if $\underset{\sim}{B}$ is a P_μ-name this means $q \Vdash_{P_\mu}$ "$N \cap \omega_1 \in \underset{\sim}{B}$"), *then* there is a (N, P_α)-semi generic condition $q' \in P_\alpha$ satisfying $q'{\restriction}\mu = q$ such that $q' \Vdash_{P_\alpha}$ "$N \cap \omega_1 \in \underset{\sim}{A}$".

4.1A Remark. (1) If $\bar{Q}$ is S-suitable, $\mathbf{t}_\mu = 1$, and $\underset{\sim}{A} \neq \emptyset$ mod $\mathcal{D}_{\omega_1}$, $\underset{\sim}{A}$ is a P_β-name for some $\beta < \alpha$, then we know that such $\underset{\sim}{B}$ exists as $\mathbf{t}_\mu = 1$ (by definition 2.1).

(2) Note, e.g., for S-suitable $\bar{Q}, \ell g(\bar{Q}) = \alpha = \bigcup_{n<\omega} \alpha_n, \alpha_n < \alpha_{n+1}, \mathbf{t}_{\alpha_n} = 1$, we can use $Q_\alpha =$ SSeal$(\mathfrak{B}^{\bar{Q}}, S)$ and not only SSeal$(\mathfrak{B}^{P_\alpha}, S)$ [because in 2.13 we had demanded "$(P_\alpha/P_{i+1})*$ seal (Ξ) is semiproper"].

Proof. As we can increase p, without loss of generality p forces $\underset{\sim}{B}$ to be equal to some P_μ-name, so without loss of generality $\underset{\sim}{B}$ is a P_μ-name.

Let us fix p, $\underset{\sim}{A}$, $\underset{\sim}{B}$, μ and work in $V[G_\mu]$, $G_\mu \subseteq P_\mu$ generic over V such that $q \in G_\mu$. Let

$$
\begin{aligned}
W_\lambda \stackrel{\text{def}}{=} \{N \prec (H(\lambda)^{V[G_\mu]}, \in, <^*_\lambda) : \; & N \text{ is countable and } N \cap \omega_1 \in \underset{\sim}{B}[G_\mu], \text{ but} \\
& \text{there is no } r \in P_\alpha/G_\mu \text{ such that:} \\
& r \text{ is } (N, P_\alpha/G_\mu)\text{-semi-generic, } p{\restriction}[\mu,\alpha) \le r \text{ and} \\
& \quad r \Vdash_{P_\alpha/G_\mu} \text{"}N \cap \omega_1 \in \underset{\sim}{A}\text{"}\}.
\end{aligned}
$$

If $W_\lambda = \emptyset \bmod \mathcal{D}_{<\aleph_1}(H(\lambda)^{V[P_\mu]})$, we can easily get the desired result (as in the proof of 1.11)): let λ_1 be such minimal that $2^{\lambda_1} < \lambda$, and $\bar{Q} \in H(\lambda_1)$. Clearly also $W_{\lambda_1} = \emptyset \bmod \mathcal{D}_{<\aleph_1}(H(\lambda)^{V[P_\mu]})$ and let $W'_{\lambda_1} \subseteq \mathcal{S}_{<\aleph_1}(H(\lambda_1)^{V[P_\mu]}$ be closed unbound disjoint to it. So if N is as in the assumption of $\otimes$, then necessarily $\lambda_1 \in N$ hence $W_{\lambda_1} \in N$ and w.l.o.g. $W'_{\lambda_1} \in N$. Then clearly $N \cap H(\lambda_1) \in W'_{\lambda_1}$, hence $N \cap H(\lambda_1) \notin W_{\lambda_1}$, hence $N \notin W_\lambda$, which suffices. So (in $V[G_\mu]$) the set W is a stationary subset of $\mathcal{S}_{<\aleph_1}(H(\lambda))$, hence semi-stationary. As $V[G_\mu] \models$ "Rss($\aleph_2$)" there is $u \subseteq H(\lambda)$ such that $\omega_1 \subseteq u$, $|u| < \aleph_2$ (in $V[G_\mu]$) and $W \cap \mathcal{S}_{<\aleph_1}(u)$ is semi-stationary; now by 1.2(2) without loss of generality $(u, \in, <^*_\lambda {\restriction} u) \prec (H(\lambda), \in, <^*_\lambda)$. Let $u = \bigcup_{\zeta<\omega_1} u_\zeta$, with each u_ζ countable and u_ζ is increasing and continuous. So

$$B_1 = \{\zeta < \omega_1 : (\exists N \in W)(\omega_1 \cap u_\zeta \subseteq N \subseteq u_\zeta)\}$$

is a stationary subset of ω_1 (see 1.2(4)) which belongs to $\mathfrak{B}[P_\mu]$, and obviously:

$$(*) \qquad p \Vdash_{P_\alpha/G_\mu} \text{"}\underset{\sim}{A} \cap B_1 \text{ is not stationary"}.$$

[Why? For $\zeta \in B_1$ let $\omega_1 \cap u_\zeta \subseteq N_\zeta \subseteq u_\zeta, N_\zeta \in W$ and for $\xi < \omega_1$ let N'_ξ be the Skolem Hull in $(H(\lambda)^{V[G_\mu]}, \in, <^*_\lambda)$ of $\{\zeta : \zeta < \xi\} \cup \{p, \langle u_\zeta, N_\zeta : \zeta \in B_1, \zeta < \xi\rangle\}$, and

$$\underset{\sim}{C} = \{\xi < \omega_1 : N'_\xi[\underset{\sim}{G}_{P_\alpha}] \cap \omega_1 = \xi \text{ and } N'_\xi[\underset{\sim}{G}_{P_\alpha}] \cap u = u_\xi\}.$$

As $\langle N'_\xi[\underset{\sim}{G}_{P_\alpha}] : \xi < \omega_1\rangle$ is increasing continuous, clearly $\underset{\sim}{C}$ is a P_α/G_μ-name of a club of ω_1. Now $\underset{\sim}{C} \cap \underset{\sim}{A}$ is necessarily disjoint to B_1 by the definition of W: if $\zeta < \omega_1, q \in P_\alpha/G_\mu$, and $q \Vdash_{P_\alpha/G_\mu}$ "$\zeta \in \underset{\sim}{C} \cap \underset{\sim}{A} \cap B_1$", then $N_\zeta \in W$ is defined (because $\zeta \in B_1$) and q_α is $(N_\zeta, P_\alpha/G_\mu)$-semi-generic, and $q_\alpha \Vdash_{P_\alpha/G_\mu}$ "$N_\zeta \cap \omega_1 \in \underset{\sim}{A}$", contradicting "$N_\zeta \in W$" so $(*)$ holds]. Also

$$(**) \qquad\qquad B_1 \subseteq B$$

by the clause "$N \cap \omega_1 \in \underset{\sim}{B}[G_\mu]$" in the definition of W.

Of course $B_1 \in V^{P_\mu}$ and as said after the definition of B_1, it is stationary so we get a contradiction to an assumption on $\underset{\sim}{A}, \underset{\sim}{B}$.

$\square_{4.1}$

4.2 Claim. (1) Suppose $\overline{Q} = \langle P_i, \underset{\sim}{Q}_j : i \leq \alpha, j < \alpha\rangle$ is a semiproper iteration, $\langle \mu_\zeta : \zeta < \xi\rangle$ an increasing sequence of strongly inaccessible cardinals $\leq \alpha$, $\bigwedge_{\zeta<\xi}\left[(\forall i < \mu_\zeta)(|P_i| < \mu_\zeta) \text{ and } \Vdash_{P_{\mu_\zeta}} \text{“Rss}(\mu_\zeta)\text{”}\right]$ and
$(*)$ every countable set of ordinals from V^{P_α} is included in a countable set of ordinals from V.

Suppose further that $\underset{\sim}{B}$ is a P_{μ_0}-name of a subset of ω_1, $\underset{\sim}{A}_\zeta$ is a $P_{\mu_{\zeta+1}}$-name of a subset of ω_1 (if $\zeta + 1 = \xi$ we stipulate $\mu_{\zeta+1} = \alpha$) and $p \in P$ satisfies:

$p{\restriction}\mu_0 \Vdash_{P_{\mu_0}}$ "$\underset{\sim}{B}$ is stationary",

$p{\restriction}\mu_{\zeta+1} \Vdash_{P_{\mu_{\zeta+1}}}$ " for every $X \in \mathfrak{B}[P_{\mu_\zeta}] \setminus \{0\}$, if $X \subseteq \underset{\sim}{B}$ *then* $\underset{\sim}{A}_\zeta \cap X$ is stationary ".

Then $p \Vdash_{P_\alpha}$ "the intersection of any countable subset of $\{A_\zeta : \zeta < \xi\}$ is stationary".

(2) In 4.2(1) we can replace the assumption $(*)$ by:

$(*)^-$ if $\delta \in (\mu_0, \alpha)$ is strongly inaccessible and $[i < \delta \Rightarrow |P_i| < \delta]$, *then* $\Vdash_{P_\alpha}$ "$\mathrm{cf}(\delta) > \aleph_0$".

Proof. 1) Let $\underset{\sim}{w}$ be a P_α-name for a countable subset of ξ. So without loss of generality $\underset{\sim}{w} = w$ and let $w = \{\zeta(n) : n < \omega\}$. Let Y be the closure of $\{\mu_\zeta : \zeta < \xi\} \cup \{\alpha\}$ (in the order topology on the ordinals). If the conclusion of 4.2

fails then (as we can increase p) without loss of generality $p \Vdash_{P_\alpha}$ "$\bigcap_{n<\omega} \underset{\sim}{A}_{\zeta(n)}$ is disjoint to $\underset{\sim}{C}$ where $\underset{\sim}{C}$ is a club of ω_1".

We now prove by induction on $j \in Y$:

$\otimes_j$ *if* $\mu_0 \leq i < j$, both in Y, λ regular and large enough, $N \prec (H(\lambda), \in, <^*_\lambda)$ countable, $\underset{\sim}{C} \in N$, $\underset{\sim}{B} \in N, \langle \mu_\zeta, \underset{\sim}{A}_\zeta : \zeta < \xi \rangle \in N$ and $\{i,\ j,\ \bar{Q}\} \in N$, $p \leq p' \in N \cap P_\alpha$ and $q \in P_i$ is (N, P_i)-semi-generic, $p' \restriction i \leq q$, and $q \Vdash_{P_i}$ "$N \cap \omega_1 \in \underset{\sim}{B}$ and for $n < \omega$ we have $[\mu_{\zeta(n)} \leq i \Rightarrow N \cap \omega_1 \in \underset{\sim}{A}_{\zeta(n)}]$", *then* there is $q' \in P_j, (N, P_j)$-semi-generic, $p' \restriction j \leq q', q' \restriction i = q$ and $q' \Vdash_{P_j}$ "$N \cap \omega_1 \in \underset{\sim}{B}$ and for $n < \omega$ we have $[\mu_{\zeta(n)} \leq j \Rightarrow N \cap \omega_1 \in \underset{\sim}{A}_{\zeta(n)}]$".

Clearly this is enough (apply it with $p' = p,\ i = \mu_0,\ j = \alpha$, and there are $N,\ q$ as required and $\underset{\sim}{B}$ is a P_{μ_0}-name of a stationary subset of $\subseteq \omega_1$).

Case 1. $j = \mu_0$. Trivial.

Case 2. j is an accumulation point of Y (hence is of countable cofinality). As in the proof of the iteration lemma for semiproperness.

Case 3. $j = \mu_{\zeta+1}$.

Apply the previous claim 4.1 (for $\bar{Q} \restriction \mu_{\zeta+1}$ and μ_ζ).

2) The proof is similar but $\underset{\sim}{w}$ is P_α-name of a countable subset of ζ, and for $j \in [\mu_0, \alpha]$ the statement $\otimes_j$ is now for every $\underset{\sim}{w}$ which is a P_j-name (not P_α-name) of a countable subset of $\{\zeta : \mu_\zeta < j\}$. So proving it we increase $p \restriction [i, j]$ also for this purpose and $i \in [\mu_0, j)$. Cases 1, 3 remain as before. Note that we can replace $\underset{\sim}{w}$ by a larger set

Case 2A. $j > \sup[j \cap \{\mu_\zeta : \zeta < \xi\}]$

Trivial

Case 2B. $j = \sup[j \cap \{\mu_\zeta : \zeta < \xi\}]$.

W.l.o.g. p force a value to $\sup \underset{\sim}{w} \cap \{\mu_\zeta : \zeta < \xi\}$, call it ξ^*.

Subcase α. $\xi^* < j$: the proof is as in case 2A, as increasing $\underset{\sim}{w}$ w.l.o.g. it is P_{ξ^*+1}-name.

Subcase β. $\xi^* = j$: for some $i_1 < j$, $p \restriction i_1 \nVdash_{P_{i_1}}$ "cf$(j) = j$" is easy too. W.l.o.g. $i_1 < i$ (by the induction hypothesis), $p \restriction i_1 \Vdash$ "cf$(j) < j$". So in V^{P_i} we know cf(j), and it is $\aleph_0$ or $\aleph_1$. Now $\aleph_1$ is impossible (as $\xi^* = j$) and if it is $\aleph_0$ act as in the old case 2.

But by $(*)^-$ of 4.2(2), one of the subcases occurs. $\square_{4.2}$

4.3 Theorem. Suppose $\{\mu < \kappa : \mu \text{ supercompact }\}$ is a stationary subset of κ, $\langle S_1, S_2, S_3\rangle$ is a partition of ω_1 with each S_i stationary. Let $h : \kappa \to H(\chi)$ and assume $\{\mu < \kappa : \mu \text{ supercompact}, h(\mu) = 0\}$ is stationary. *Then* for some forcing notion P:

(i) $P = P_\kappa$ ($= R\mathrm{Lim}\bar{Q}$ for some S_1-suitable $\bar{Q}$) is S_3-complete, or at least S_3-proper and satisfies the κ-c.c.

(ii) In V^{P_κ}, from any $\aleph_2$ stationary subsets of $S_1 \subseteq \omega_1$, there are $\aleph_2$ of them such that the intersection of any countably many of them is stationary (and $\mathfrak{B}^{P_\kappa}$ is layered, of course). We then call $\mathfrak{B}[V^{P_\kappa}]$ reflective.

(iii) A forcing axiom as strong as h holds (see the proof and 3.9).

4.3A Remark. 1) We really use a weaker assumption

(a) $\{\mu < \kappa : \mu \text{ measurable}\}$ is stationary;

(b) $\{\mu < \kappa$: for $\chi < \kappa, \mu$ is χ-compact$\}$ is unbounded; use 1.6(2), 1.6(3), 1.10(1). See more in XVI§2.

2) The situation is similar in 4.4, where we get better bound (i.e. using smaller large cardinals) for a stronger result (but lose in forcing axiom.)

3) We can demand only "S_1 is stationary" etc. if we use 4.1(2) instead of 4.1(1), but then we should satisfy $(*)^-$ of 4.1(2).

Proof. We define by induction on $\alpha \leq \kappa$ the iteration $\bar{Q}^\alpha = \langle P_i, \underset{\sim}{Q}_i, \mathbf{t}_j : i \leq \alpha, j < \alpha\rangle$ such that:

(A) $\bar{Q}^\alpha$ is S_1-suitable.

(B) Each Q_i is S_3-proper.

(C) $\bar{Q}^\alpha$, $P_\alpha \in H(\kappa)$ when $\alpha < \kappa$.

(D) If $h(i) = \langle \mathbf{t}, \underset{\sim}{R}\rangle$, i measurable, $\mathbf{t}$ a truth value, $\underset{\sim}{R}$ a P_i-name and $\left[\mathbf{t} = 1 \Rightarrow \mathfrak{B}[P_i]\restriction S_1 \lessdot \mathfrak{B}[P_i * \underset{\sim}{R}]\restriction S_1\right]$ and $\left[j < i \ \&\ \mathbf{t}_j = 1 \Rightarrow \mathfrak{B}[P_j]\restriction S_1 \lessdot \mathfrak{B}[P_i * \underset{\sim}{R}]\restriction S_1\right]$ and $(P_i/P_{j+1}) * \underset{\sim}{R}$ is semiproper and S_3-proper for $j < i$ *then* $\mathbf{t}_i = \mathbf{t}, \underset{\sim}{Q}_i = \underset{\sim}{R} *$ SSeal $(\langle \mathfrak{B}[P_j] : j \leq i, \mathbf{t}_j = 1\rangle)$.

(E) If not (D), i is inaccessible, $|P_j| < i$ for $j < i$, $h(i) = 0$, and $\Vdash_{P_i}$ "$\mathrm{Rss}(\aleph_2)$," then $\mathbf{t}_i = 1$ and $\underset{\sim}{Q}_i = \mathrm{SSeal}(\langle \mathfrak{B}[P_j] : j \leq i,\ \mathbf{t}_j = 1\rangle, S)$.

(F) If neither (D) nor (E) then $\mathbf{t}_i = 0$, and

$$Q_i = \text{ SSeal}(\langle \mathfrak{B}[P_j] : j \leq i, \mathbf{t}_j = 1\rangle, S).$$

We can carry out the construction and prove by induction on α that $\bar{Q}^\alpha$ is S_1-suitable.

$\alpha = 0$. Trivial.

α *limit.* By Claim 2.3(1).

$\alpha = \beta + 1$, (F) *applies to* β. By 2.14(1).

$\alpha = \beta + 1$, (D) *applies to* β. By Claim 2.11.

$\alpha = \beta + 1$, (E) applied to β. By 2.16 note:

($*$) if $i \in B = \{i : i \text{ inaccessible}, j < i \Rightarrow |P_j| < i\}$ then: if for i clause (E) or clause (F) occur then Q_i is semiproper. [Why? If $\Vdash_{P_i}$ "Rss($\aleph_2$)" by 1.7(5), otherwise clause (F) applies, $\mathbf{t}_i = 0$ and we can use 2.11.]

But clause (D) does not apply to i non-measurable so

($**$) for i non measurable $\Vdash_{P_i}$ "$\underset{\sim}{Q}_i$ is semiproper".

Now suppose $p \in P_\kappa$, $\langle \underset{\sim}{A}_i : i < \kappa\rangle$ a P_κ-name and $p \Vdash$ "$\underset{\sim}{A}_i \subseteq S_1$ is stationary".

Let $Y = \{\mu < \kappa : \mu \text{ strongly inaccessible}, \bigwedge_{i<\mu} |P_i| < \mu$ and $\Vdash_{P_i}$ "Rss($\aleph_2$)" and $\mathbf{t}_\mu = 1\}$. Note that in V^{P_κ}, $Y \subseteq \{\delta < \kappa : \mathrm{cf}^{V[P_\kappa]}(\delta) = \aleph_1\}$ is stationary because if $\mu < \kappa$ is measurable, limit of supercompacts, $\bigwedge_{j<\mu} |P_j| < \mu$ and D is a normal filter on μ, concentrating on non-measurable so $X_\mu = \{i < \mu : i$ inaccessible, not measurable, $\bigwedge_{j<i} |P_j| < \mu\} \in D$. We use 1.10(1) (noting $\mathrm{Rss}^+(\mu)$ holds, by 1.6(6)) to get $\mu \in Y$. Now for each $\mu \in Y$ choose $p_\mu \in P_\kappa$ and a P_μ-name $\underset{\sim}{B}_\mu$ such that:

$$p \leq p_\mu,$$

$$p_\mu{\restriction}\mu \Vdash \text{"}\underset{\sim}{B}_\mu \subseteq S_1 \text{ is stationary, } \underset{\sim}{B}_\mu \in \mathfrak{B}[P_\mu]\text{"},$$

$$p_\mu \Vdash_{P_\kappa} \text{"for every nonzero } X \in \mathfrak{B}[P_\mu],$$

$$\text{if } X \leq \underset{\sim}{B}_\mu \text{ then } X \cap \underset{\sim}{A}_\mu \text{ is stationary".}$$

Why does such a p_μ exist? As $\mathfrak{B}[P_\mu] \lessdot \mathfrak{B}[P_\kappa]$ (and see 0.1(4)(b)). Remember that P_μ satisfies μ-c.c. so $\underset{\sim}{B}_\mu \in H(\chi_\mu)$ for some $\chi_\mu < \mu$ and without loss of generality $\underset{\sim}{B}$ is a P_{χ_μ}-name and $\mathbf{t}_{\chi_\mu} = 1$ (i.e. by increasing χ_μ; also, Y is stationary by a hypothesis).

By Fodor's lemma, for some stationary $Y_1 \subseteq Y$, there are p and $\underset{\sim}{B}$ such that for $\mu \in Y_1 : p_\mu \restriction \mu = p$, and $\underset{\sim}{B}_\mu = \underset{\sim}{B}$.
As each $\underset{\sim}{A}_\zeta$ is a P_λ-name for some $\lambda > \zeta, \lambda \in Y$, without loss of generality $[\mu_1 < \mu_2$ in $Y_1 \Rightarrow \underset{\sim}{A}_{\mu_1}$ is a P_{μ_2}-name$]$. Now, for $\mu \in Y_1$ let $\underset{\sim}{A}'_\mu$ be $\underset{\sim}{A}_\mu$ if $p_\mu \in \underset{\sim}{G}_{P_\kappa}$ and S otherwise.

Note that $Y_1 \in V$ and every countable subset of Y_1 is contained in a countable set from V [Why? Remembering S_3 is stationary, by S_3-properness.] Now we apply the previous Claim 4.2 to $\underset{\sim}{B}, \langle \underset{\sim}{A}'_\mu : \mu \in Y_1\rangle$. $\square_{4.3}$

4.4. Theorem. Suppose $\kappa = \sup\{\lambda < \kappa : \lambda$ a compact cardinal$\}$ and $\langle S_1, S_2, S_3\rangle$ is a partition of ω_1 to stationary sets. *Then* for some forcing notion $P \in V$:

(i) V^P is a model of: ZFC + $2^{\aleph_0} = \aleph_1 + 2^{\aleph_1} = \aleph_2$,

(ii) in V^P, the statement Ulam$(\mathcal{D}_{\omega_1} + S_1)$ holds, where, for a uniform filter D on λ, Ulam(D) means: there are many λ λ-complete filters extending D, such that every D-positive set belongs to at least one of them (A is D-positive if $A \subseteq \lambda$, and $(\lambda \setminus A) \notin D$).

Remark. So in V^P, Ulam's problem has a positive solution: there are $\aleph_1$ measures on $[0,1]_{\mathbb{R}}$, each countably additive, such that every $A \subseteq [0,1]_{\mathbb{R}}$ is measurable with respect to at least one of them.

Proof. Before we do the forcing, we work out some combinatorics, which will tell us what will suffice.

4.5A. Context and Notation.
(1) $\lambda = \lambda^{<\lambda}$ is a fixed regular uncountable cardinal.
(2) W denotes a fixed class of ordinals (in the actual case $W \subseteq \lambda^+$), $0 \in W$, for every i, $i+1 \in W$, and

$$\aleph_0 \leq \mathrm{cf}(i) < \lambda \Rightarrow i \notin W.$$

(3) B will denote a Boolean algebra.

(4) For a Boolean Algebra B, let $B^+ = B \setminus \{0\}$.

(5) $\Pr(a_1, a_2, B_1, B_2)$ means: B_1, B_2 are Boolean algebras, $B_1 \subseteq B_2$, $a_1 \in B_1^+$, $a_2 \in B_2^+$, and $(\forall x)[x \in B_1^+ \ \& \ x \leq a_1 \rightarrow x \cap a_2 \neq 0]$.

(6) If the identity of B_2 is clear (when dealing with one Boolean Algebra and its subalgebras) we just write $Pr(a_1, a_2, B_1)$.

4.5B. Observation.

(a) $\Pr(1, x, B_1, B_2)$ for $x \in B_2^+, |B_1| = 2$;

(b) if $B_a \subseteq B_b \subseteq B$, $x \in B_a^+$, $y \in B_b^+$, $z \in B$ and $\Pr(x, y, B_a)$, $\Pr(y, z, B_b)$ then $\Pr(x, z, B_a)$;

(c) if $\Pr(x, y, B_1, B_2), 0 < x' \leq x, x' \in B_1, y \leq y' \in B_2$ *then* $\Pr(x', y', B_1)$ and $\Pr(x', y \cap x, B_1)$.

4.5C. Notation and Definition.

(1) We call $\bar{B}$ 1-o.k. (for W) if $\bar{B} = \langle B_i : i < \alpha \rangle$ is an increasing continuous sequence, each B_i a Boolean Algebra of cardinality $\leq \lambda$, $[i, j \in \alpha \cap W$ and $i < j \Rightarrow B_i \lessdot B_j,]$ and $[i \in W \cap \alpha \Rightarrow B_i$ is λ-complete$]$.

(2) We call $w \subseteq W \cap \alpha$ closed (subset of $W \cap \alpha$) *if*

 (i) for every accumulation point $\delta < \alpha$ of the closure of w (that is $\delta = \sup(w \cap \delta) \ \& \ \delta < \alpha$) we have

  (a) $\delta \notin W \ \& \ \delta + 1 < \alpha \Rightarrow \delta + 1 \in w$

  (b) $\delta \in W \ \& \ \delta + 1 < \alpha \Rightarrow \delta \in w$,

 (ii) for every $\delta < \alpha$ we have: $\mathrm{Min}(w) < (\delta + 1) \in w \ \& \ \aleph_0 \leq \mathrm{cf}(\delta) < \lambda \Rightarrow \delta = \sup(\delta \cap w)$,

 (iii) if $\mathrm{Min}(w) < \beta \in W$, $\beta + 1 \in w$ then $\beta \in w$.

(3) Let $\mathrm{CSb}(\alpha) = \{w : w$ a closed subset of $W \cap \alpha$ of power $< \lambda\}$, $\mathrm{CSb}_u(\alpha) = \{w \in \mathrm{CSb}(\alpha) : w$ unbounded below $\alpha\}$. (Clearly $\mathrm{CSb}_u(\alpha) \neq \emptyset \Rightarrow \aleph_0 \leq \mathrm{cf}(\alpha) < \lambda$).

(4) For $w \in \mathrm{CSb}(\alpha)$ and $\bar{B} = \langle B_i : i < \beta \rangle$ which is 1-o.k. such that $\beta > \alpha$, let

 (i) $\mathrm{Seq}_w(\bar{B}) = \{\langle a_i : i \in w \rangle : a_i \in B_i^+, a_i$ is decreasing;

if $i \in w, i = \delta + 1$, δ limit of course, $\delta \notin W, i > \mathrm{Min}(w)$ *then* $a_i = \bigcap_{j \in w \cap i} a_j$;

if $i \in w$, $i > \mathrm{Min}(w), \mathrm{cf}(i) \geq \lambda$, *then* $a_i = a_j \in B_j$ for some $j \in i \cap w$;

and if $i < j$ are in $w (\subseteq W)$ then $\Pr(a_i, a_j, B_i, B_j)\}$.

(ii) Let $\mathrm{Seq}(\bar{B}) \stackrel{\mathrm{def}}{=} \bigcup\{\mathrm{Seq}_w(\bar{B})$: for some α $(\leq \ell g(\bar{B}))$, $\alpha = \ell g(\bar{B})$ or α is successor of a member of W, we have $w \in \mathrm{CSb}(\alpha)\}$.

Let $\mathrm{Seq}_u(\bar{B}) \stackrel{\mathrm{def}}{=} \bigcup\{\mathrm{Seq}_w(\bar{B}) : w \in \mathrm{CSb}_u(\ell g(\bar{B}))\}$.

It is naturally ordered by $\bar{a}^1 \leq \bar{a}^2$ if letting $\bar{a}^\ell = \langle a^\ell_i : i \in w_\ell\rangle$ then $w^1 \subseteq w^2$ and $[\zeta \in w^1 \Rightarrow a^1_\zeta \geq a^2_\zeta]$.

(iii) *When* $\alpha = \delta + 1 \leq \ell g(\bar{B}), w \in \mathrm{CSb}(\delta)$ let

$$Z_w(\bar{B}) \stackrel{\mathrm{def}}{=} \left\{\bigcap_{i \in w} a_i : \langle a_i : i \in w\rangle \in \mathrm{Seq}_w(\bar{B}\restriction(\delta+1))\right\},$$

$Z^\delta(\bar{B}) \stackrel{\mathrm{def}}{=} \bigcup\{Z_w(\bar{B}) : w \in \mathrm{CSb}_u(\delta)\}$.

If $\ell g(\bar{B}) = \delta + 2$, we may omit δ.

(5) We call $\bar{B}$ 2-o.k. if for every limit $\delta < \ell g(\bar{B})$, $0 \notin Z^\delta(\bar{B})$ and $\bar{B}$ is 1-o.k.

(6) We call $\bar{B}$ 3-o.k. if it is 2-o.k. and for limit $\delta < \ell g(\bar{B})$ of cofinality $< \lambda$ we have: $Z^\delta(\bar{B})$ is a dense subset of $B_{\delta+1}$.

(7) If $\bar{B}$ is not continuous, we identify it with the obvious correction for the purpose of our definitions.

(8) We call $\Upsilon \subseteq \mathrm{Seq}_u(\bar{B})$ dense if for every $\bar{a} \in \mathrm{Seq}_u(\bar{B})$ for some $\bar{a}' \in \Upsilon$ we have $\bar{a} \leq \bar{a}'$. We say Υ' refines Υ if $(\forall \bar{a} \in \Upsilon)(\exists \bar{a}' \in \Upsilon')$ $[\bar{a} \leq \bar{a}']$. We say $\Upsilon \subseteq \mathrm{Seq}_u(\bar{B})$ is open if $\bar{a} \leq \bar{a}'$ (in $\mathrm{Seq}_u(\bar{B})$), $\bar{a} \in \Upsilon$ implies $\bar{a}' \in \Upsilon$.

4.5D. Fact. Suppose $\bar{B}$ is 2-o.k., $\bar{B} = \langle B_i : i \leq \delta + 1\rangle, \aleph_0 \leq \mathrm{cf}(\delta) < \lambda$. Then:

(0) (i) $\mathrm{CSb}_u(\delta) \neq \emptyset$, moreover for every $\alpha \leq \delta + 1$ and $v \subseteq \alpha$ of cardinality $< \lambda$ we have:

$\aleph_0 \leq \mathrm{cf}(\alpha) < \lambda \Rightarrow$ there is $w \in \mathrm{CSb}_u(\alpha)$ such that $v \subseteq w$ and

$\mathrm{cf}(\alpha) \geq \lambda \Rightarrow$ there is $w \in \mathrm{CSb}(\alpha)$, such that $v \subseteq w$ and

$\alpha = i + 1 \,\&\, i \in W \Rightarrow$ there is $w \in \mathrm{CSb}(\alpha)$ such that $v \subseteq w$ and $i \in w$.

(ii) If $w \in \mathrm{CSb}_u(\delta), \alpha < \delta$, *then* $w \setminus \alpha \in \mathrm{CSb}_u(\delta)$; similarly for $\mathrm{CSb}(\alpha)$.

(iii) If $\alpha < \delta$ and $w \in \mathrm{CSb}_u(\alpha)$ and $\alpha = \varepsilon + 1$ and $\aleph_0 \leq \mathrm{cf}(\varepsilon) < \lambda$ *then* $w \cup \{\alpha\} \in \mathrm{CSb}(\delta)$.

(iv) If $w \in \mathrm{CSb}_u(\delta), \alpha \in w$ *then* there is $\beta \in w \setminus \alpha$ such that $\beta \notin \{\varepsilon + 1 : \aleph_0 \leq \mathrm{cf}(\varepsilon) < \lambda\}$; in fact $\beta = \min(w \setminus (\alpha + 1))$ is as required.

(v) If $w \in \mathrm{CSb}(\alpha)$ and $\beta < \alpha$, *then* $w \cap \beta \in \mathrm{CSb}(\beta)$.

(1) If $w \in \mathrm{CSb}(\delta+1)$ *then* $Z_w(\bar{B})$ includes $B_{\mathrm{Min}(w)}$, hence: $\aleph_0 \leq \mathrm{cf}(\delta) < \lambda \Rightarrow B_\delta \subseteq Z^\delta(\bar{B})$.

(2) (i) If $w_1,\ w_2 \in \mathrm{CSb}(\delta)$ and $\mathrm{Min}(w_\ell) < \mathrm{Min}(w_{3-\ell}) \Rightarrow \mathrm{Min}(w_{3-\ell}) \in w_\ell$ *then* $w_1 \cup w_2 \in \mathrm{CSb}(\delta)$.

(ii) Similarly for $\mathrm{CSb}_u(\delta)$.

(3) (i) If $w_1 \subseteq w_2$ are both in $\mathrm{CSb}(\alpha), \min(w_1) = \min(w_2)$, $\langle a_i : i \in w_1\rangle \in \mathrm{Seq}_{w_1}(\bar{B})$ *then* $\langle a_i : i \in w_2\rangle \in \mathrm{Seq}_{w_2}(\bar{B})$ provided that for $i \in w_2 \backslash w_1$ we define $a_i = a_{\max(i \cap w_1)}$ which is well defined.

(ii) If $\alpha < \sup(w)$, and $w \in \mathrm{CSb}(\delta)$, and $\langle a_i : i \in w\rangle \in \mathrm{Seq}_w(\bar{B})$ *then* $\langle a_i : i \in w \setminus \alpha\rangle \in \mathrm{Seq}_{w\setminus\alpha}(\bar{B})$ and $\langle a_i : i \in w \cap \alpha\rangle \in \mathrm{Seq}_{w\cap\alpha}(\bar{B})$.

(iii) If $w_1 \subseteq w_2$ are both in $CSb(\delta)$ and $\langle a_i : i \in w_2\rangle \in \mathrm{Seq}_{w_2}(\delta)$ and $[i \in w_2 \,\&\, \mathrm{cf}(i) \geq \lambda \,\&\, \mathrm{Min}(w_1) < i \Rightarrow (\exists j)(j \in w_1 \cap i \,\&\, a_j = a_i)]$ *then* $\langle a_i : i \in w_1\rangle \in \mathrm{Seq}_{w_1}(\delta)$.

(iv) If $\beta < \delta$, β is a successor of a limit ordinal, $w \in \mathrm{CSb}_u(\beta - 1)$, $\mathrm{Min}(w) < \beta \leq \varepsilon + 1$, $w_1 = w \cup \{\beta\}$ and $\langle a_i : i \in w\rangle \in Z_w(\bar{B})$ *then* we can find a_β such that $\langle a_i : i \in w_1\rangle \in Z_{w_1}(\bar{B})$.

(4) If $w_1 \subseteq w_2$ are both in $\mathrm{CSb}_u(\delta)$ and $\min(w_1) = \min(w_2)$ *then* $Z_{w_1}(\bar{B}) \subseteq Z_{w_2}(\bar{B})$.

(5) If $\langle a_i : i \in w_1\rangle, \langle b_j : j \in w_2\rangle$ are in $\mathrm{Seq}_{w_1}(\bar{B})$, $\mathrm{Seq}_{w_2}(\bar{B})$ respectively, and $(\forall i \in w_1)(\exists j \in w_2) a_i \leq b_j$ *then* $\bigcap_{i\in w_1} a_i \leq \bigcap_{j \in w_2} b_j$.

(6) If $\langle a_i : i \in w\rangle \in \mathrm{Seq}_w(\bar{B}), 0 < b < a_{\min(w)}$ and $b \in B_{\min(w)}$ *then* $\langle a_i \cap b : i \in w\rangle \in \mathrm{Seq}_w(\bar{B})$.

(7) If $\bar{B} = \langle B_i : i \leq \alpha\rangle$ is l-o.k. ($l = 1, 2, 3$), $\gamma_i \leq \alpha$ (for $i \leq i(*)$) is strictly increasing continuous and $[i \in W \Leftrightarrow \gamma_i \in W]$ and $[\aleph_0 \leq \mathrm{cf}(i) < \lambda \Rightarrow \gamma_{i+1} = \gamma_i + 1]$ *then* $\langle B_{\gamma_i} : i \leq i(*)\rangle$ is l-o.k.

(8) Assume $\bar{B} = \langle B_i : i \leq \alpha\rangle$ is 1-o.k.

(i) if $\beta < \gamma < \alpha$, $\beta \in W$, $\gamma \in W$, $b \in B_\gamma$ *then* for at most one a we have:

(*) $a \in B_\beta$ and $B_\gamma \models$ "$a \geq b$" and $\Pr(a, b, B_\beta, B_\gamma)$,

(ii) if $\bar{a}^\ell \in \mathrm{Seq}_{w_\ell}(\bar{B})$ for $\ell = 1, 2$, *then* $\{\beta : \beta \in w_1 \cap w_2$ and $a^1_\beta = a^2_\beta\}$ is an initial segment of $w_1 \cap w_2$.

(9) Assume $\bar{B} = \langle B_i : i \leq \alpha \rangle$ is 3-o.k., $[\delta < \alpha \,\&\, \mathrm{cf}(\delta) \geq \lambda \Rightarrow \delta \in W]$.

(i) If $v \in \mathrm{CSb}(\beta)$, $\beta \leq \alpha + 1$ is a successor of a member of W, and $\gamma \in v$ and $d \in B_\gamma$ *then* the set $\Gamma = \Gamma_{\alpha,v,\gamma,d} = \{\bar{a} \in \mathrm{Seq}\,(\bar{B}) : v \subseteq \mathrm{Dom}(\bar{a})$ and $[a_\gamma \cap d = 0$ *or* $a_\gamma \leq d]\}$ is a dense and open subset of $\mathrm{Seq}\,(\bar{B})$.

(ii) If $\aleph_0 \leq \mathrm{cf}(\alpha) < \lambda$, $v \in \mathrm{CSb}_u(\alpha)$, $\gamma \in v$ and $d \in B_\gamma$ *then* the set $\Gamma = \Gamma_\alpha = \{\bar{a} \in \mathrm{Seq}_u(\bar{B}) : v \subseteq \mathrm{Dom}(\bar{a})$ and $[a_\gamma \cap d = 0$ or $a_\gamma \leq d]\}$ is a dense open subset of $\mathrm{Seq}\,(\bar{B})$.

(iii) If $\beta \leq \alpha$, $d \in B_\beta \setminus \{0\}$ and $v \subseteq \beta + 1$, $|v| < \lambda$ *then* there is w satisfying $v \subseteq w \in \mathrm{CSb}(\beta + 1)$, $\beta \in w$ and $\bar{a} \in \mathrm{Seq}_w(\bar{B})$ such that $a_\beta \leq d$.

Proof. Easy, e.g.,

0)(i) We prove it by induction on α. For α non-limit the result is trivial so assume α is a limit. So for every $j < \alpha$ there is w_j such that: $v \cap j \subseteq w_j$ and $[\aleph_0 \leq \mathrm{cf}(j) < \lambda \Rightarrow w_j \in \mathrm{CSb}_u(j)]$ and $[\mathrm{cf}(j) \notin [\aleph_0, \lambda) \Rightarrow [\mathrm{cf}(j) \geq \lambda \vee (\exists i)(j = i + 1 \,\&\, i \in W) \Rightarrow w_j \in \mathrm{CSb}(j)]]$. Let $\langle j_\varepsilon : \varepsilon < \mathrm{cf}(\alpha) \rangle$ be an increasing continuous sequence of ordinals $< \alpha$ with limit α. If $\mathrm{cf}(\alpha) = \aleph_0$ *then* w.l.o.g. $j_n + 2 \in v$ for $n < \omega$ and then $w \stackrel{\mathrm{def}}{=} \bigcup \{w_{j_{n+1}+3} \setminus (j_n + 3) : n < \omega\} \cup w_{j_0+3}$ is as required (remember $i + 1 \in W$ for any ordinal i by 4.5A(2)). If $\mathrm{cf}(\alpha) \geq \lambda$ then for some $j < \alpha$, we have $v \subseteq j$ and we can use the induction hypothesis. If $\mathrm{cf}(\alpha) > \aleph_0$ but still it is $< \lambda$, without loss of generality each j_ε is a limit ordinal with cofinality $< \mathrm{cf}(\alpha) < \lambda$. Let $w = \{j_\varepsilon + 1 : \varepsilon < \mathrm{cf}(\alpha)\} \cup \bigcup_{\varepsilon < \mathrm{cf}(\alpha)} (w_{j_{\varepsilon+1}+3} \setminus (j_\varepsilon + 3)) \cup w_{j_0+3}$ and note that it belongs to $\mathrm{CSb}_u(\alpha)$ and includes v, as required.

(0)(iv) See the last phrase of 4.5C(2).

(1) For the first phrase note that for $a \in B_{\min(w)}$, $\bar{b}_a = \langle a : i \in w \rangle \in \mathrm{Seq}_w(\bar{B})$ (see Definition 4.5C(4)(i), (iii)).

The second phrase follows by the definition of $Z^\delta(\bar{B})$ and 4.5D(0)(i).

(3)(i) Why is $\max(i \cap w_1)$ well defined?

First note: $i \cap w_1 \neq \emptyset$ as $i \notin w_1$ implies $i \neq \min(w_1)$, but $\min(w_1) = \min(w_2)$

so $i > \min(w_1)$ hence $\min(w_1) \in i \cap w_1$.

Second note: if $i \cap w_1$ has no last element, let $\beta = \sup(i \cap w_1)$, so $\beta \leq i$ and $\aleph_0 \leq \mathrm{cf}(\beta) \leq |w_1| < \lambda$, hence $\beta \notin W$, so $\beta \notin w_2$ and $\beta < i$. Also $\beta+1 \in w_1$ (as w_1 is closed and $\beta < i < \alpha$ so $\beta + 1 < \delta$), so $\beta + 1$ cannot be in $i \cap w_1$, hence $i = \beta + 1 \in w_1$, contradicting the assumption on i (i.e. $i \in w_2 \setminus w_1$).

(3)(iv) Note that, as $w \in \mathrm{CSb}_u(\beta - 1)$, necessarily $\mathrm{cf}(\beta - 1) < \lambda$. Also $w_1 \in \mathrm{CSb}(\delta + 2)$, so $Z_w(\bar{B})$ is well defined. Also $a_\beta \overset{\mathrm{def}}{=} \bigcap_{i \in w} a_i$ is well defined as B_β is λ-complete, and $|w| < \lambda$ as $w \in\subseteq \mathrm{CSb}_u(\beta) \subseteq \mathrm{CSb}(\beta)$ (see Definition 4.5C(3)). As $\bar{B}$ is 2-o.k. (see Definition 4.5C(5)), $0_{B_{\delta+1}} \notin Z^\delta(\bar{B})$, but clearly $a_\beta \in Z_w(\bar{B}) \subseteq Z^\beta(\bar{B})$ hence $a_\beta \neq 0_{B_{\delta+1}}$. The order requirements for $\langle a_i : i \in w_1 \rangle \in \mathrm{Seq}_{w_1}(\bar{B})$ are easy too.

(4) Use (3)(i).

(6) Let for $i \in w$, $c_i \overset{\mathrm{def}}{=} a_i \cap b$ and $\beta = \min(w)$. So

(i) $c_i = a_i \cap b \in B_i$ [as $a_i \in B_i, b \in B_\beta \subseteq B_i$];

(ii) for $i < j$ from w, $c_j \leq c_i$ [as $a_j \leq a_i$, clearly $a_j \cap b \leq a_i \cap b$];

(iii) for $i < j$ from w, $\Pr(c_i, c_j, B_i)$.

[Why? Let $0 < d \leq c_i$, $d \in B_i$ then $0 < d \leq a_i$, $d \in B$, hence (by $\Pr(a_i, a_j, B_i)$) $d \cap a_j \neq 0$ and $d \leq c_i = a_j \cap b \leq b$ so $d \cap b = d$, hence

$$d \cap c_j = d \cap (a_j \cap b) = (d \cap b) \cap a_j = d \cap a_j$$

so $d \cap c_j \neq 0$ as required.]

The other conditions are easy too.

So $\langle c_j : j \in w \rangle \in \mathrm{Seq}_w(\bar{B})$.

(9) We prove this by induction on α ((i), (ii) and (iii) together). In parts (i) and (ii), Γ being open is immediate, so let us prove density. So assume $\bar{c} = \langle c_i : i \in v_0 \rangle$ belongs to $\mathrm{Seq}(\bar{B})$ (for 9(i)) or $\mathrm{Seq}_u(\bar{B})$ (for 9(ii)), $v \subseteq \alpha$, $\gamma \in v$, $d \in B_\gamma$ as there and we shall find $\bar{b}$, $\bar{c} \leq \bar{b} \in \Gamma$ (see end of 4.5C(4)(ii)). In the cases below for 9(iii) only the assumption on α is relevant.

Case 1: $\alpha = 0$

Trivial.

Case 2: $\alpha = \varepsilon + 1$, $\varepsilon \in W$.

For 9(iii) note that by the induction hypothesis we have to prove only the case $\beta = \alpha = \varepsilon + 1$ and $d \in B_\beta = B_\alpha$ is given. Let $d_1 \in B^+_\beta$ be such that $\mathrm{Pr}_1(d_1, d, B_\varepsilon, B_\alpha)$. By the induction hypothesis we can find $w_1 \in \mathrm{CSb}(\varepsilon + 1)$, such that $\varepsilon \in w_1$, and $\bar{a} \in \mathrm{Seq}_{w_1}(\bar{B})$ such that $a_\varepsilon \le d_1$.
Let $w \stackrel{\text{def}}{=} w_1 \cup \{\alpha\}$, $a_\alpha \stackrel{\text{def}}{=} a_\beta \cap d \in B^+$ (not zero as $a_\beta \le d_1$ and the choice of d_1). So $\langle a_i : i \in w\rangle \in \mathrm{Seq}_w(\bar{B})$ is as required.

Now as α is a successor only 9(i) is left, by the induction hypothesis $\beta = \alpha + 1$ and by the assumptions of 9(i), β is a successor of a member of W so $\alpha \in W$ hence v_0 has a last element. Let d_0 be: $c_{\max(v_0)} \cap d$ if not zero and $c_{\max(v_0)}$ otherwise and as we have proved 9(iii), there is $\langle a_i : i \in w\rangle \in \mathrm{Seq}_w(\bar{B})$ satisfying $w \in \mathrm{CSb}(\alpha + 1)$ such that: $v_0 \cup v \cup \{\varepsilon, \alpha\} \subseteq w$, and $a_\alpha \le d_0$. So $a_\alpha \le d$ or $a_\alpha \cap d = 0$; by 4.5D(8)(ii) we are done.

Case 3: $\alpha = \varepsilon + 1$, $\varepsilon \notin W$ (so only 9(i)+(iii) apply and ε is a limit ordinal) (as $\beta \notin W$, $\mathrm{cf}(\varepsilon) < \lambda$).

For 9(i) as in case 2 it follows from 9(iii), so let us prove 9(iii), by the induction hypothesis w.l.o.g. $\beta = \alpha$. As $\bar{B}$ is 3-o.k. by Definition 4.5C(6) there are $w_0 \in \mathrm{CSb}_u(\varepsilon)$ and $\langle b^0_i : i \in w_0\rangle \in \mathrm{Seq}_{w_0}(\bar{B})$ such that $\bigcap_{i \in w_0} b^0_i$ is not zero and is $d_0 \le d$.

By the induction hypothesis we can apply 4.5D(9)(ii) to ε, $\mathrm{CSb}_u(\varepsilon)$, $\langle b^0_i : i \in w_0\rangle$ and so we can find $\langle b_i : i \in w_1\rangle$ such that $\langle b^0_i : i \in w_0\rangle \le \langle b_i : i \in w_1\rangle \in \mathrm{Seq}(\bar{B}{\restriction}\alpha)$ and $v \subseteq w_1$. As $\bar{B}$ is 2-o.k., $b_\alpha \stackrel{\text{def}}{=} \bigcap_{i \in w_1} b_i \in B_\alpha$ is not zero. Let $w = w_1 \cup \{\alpha\}$, so $w \in \mathrm{CSb}(\alpha + 1)$ and $\langle b_i : i \in w\rangle \in \mathrm{Seq}_u(\bar{B})$ is as required.

Case 4: α is a limit ordinal, $\mathrm{cf}(\alpha) \ge \lambda$ (so $\alpha \in W$ by an assumption of 4.5D(9)). As $B_\alpha = \bigcup_{\beta<\alpha} B_\beta$ and the third requirement in the definition of $\bar{a} \in \mathrm{Seq}$ (see 4.5C(4)(i)) it is easy.

Case 5: α is a limit ordinal, $\mathrm{cf}(\alpha) < \lambda$.

So 9(i), 9(iii) does not apply. First as for 9(ii) we can assume $v \setminus \min(v_0) = v_0$.

[Why? By 4.5D(0)(i) w.l.o.g. $0 \in v$ & $v_0 \subseteq v$. But $\sup(v_0 \cup v) \geq \sup(v_0) = \alpha$, so $v = v_0 \cup v \in \mathrm{CSb}_u(\alpha)$, and lastly apply 4.5D(3)(i) to replace v_0 by $v \setminus \min(v_0)$.] Second we are given $\gamma \in v$, $d \in B_\gamma$ (so, as we can increase γ w.l.o.g. $\gamma \in v_0$). Now $v_0 \cap (\gamma+1) \in \mathrm{CSb}(\gamma+1)$ and so $\langle c_i : i \in v_0 \cap (\beta+1)\rangle \in \mathrm{Seq}\,(\bar{B})$, and by the induction hypothesis (on 9(i)) we can find $\langle b_i^0 : i \in w_0\rangle \in \mathrm{Seq}\,(B\restriction(\beta+1))$ such that $\langle c_i : i \in v_0 \cap (\beta+1)\rangle \leq \langle b_i^0 : i \in w_0\rangle$ and $b_\beta^0 \leq d \vee b_\beta^0 \cap d = 0$. Define $w = w_0 \cup v_0$,

$$b_i = \begin{cases} b_i^0 & \textit{if } i \leq \beta \text{ (so } i \in w_0); \\ b_\beta^0 \cap c_i & \textit{if } i > \beta \text{ (so } i \in v_0). \end{cases}$$

Now $\langle b_i : i \in w\rangle$ is as required. $\square_{4.5D}$

4.5E. Claim. If $\bar{B} = \langle B_i : i \leq \lambda^+\rangle$ is 3-o.k. and $[i < \lambda^+ \;\&\; \mathrm{cf}(i) \geq \lambda \Rightarrow i \in W]$ *then* $B_{\lambda^+}^+$ is the union of λ many λ-complete filters.

Proof. Note that by 4.5D(9) we have:

(∗) for every $\alpha \in W$ and $x \in B_\alpha^+$ for some $w \in \mathrm{CSb}(\alpha+1)$ and $\langle a_i : i \in w\rangle \in \mathrm{Seq}_w(\langle B_i : i < \alpha+1\rangle)$ we have $0 < \bigcap_{i\in w} a_i \leq x$, $0 \in w$, and w is closed and has a last element α.

Now remember that $\mathrm{Seq}\,(\bar{B}) = \bigcup\{\mathrm{Seq}\,(\langle B_i : i \leq \alpha\rangle) : \alpha < \lambda^+$ is a successor of a member of $W\}$.

It is well known that there is $H : \{w \subseteq \lambda^+ : |w| < \lambda\} \to \lambda$ such that: $H(w) = H(u), \alpha \in w \cap u$ implies $\alpha \cap w = \alpha \cap u$; also $H(w) = H(u)$ implies that w, u have the same order type (let $f_\alpha : \alpha \to \lambda$ be one to one, $H^0(w) = \{\langle \mathrm{otp}(w \cap \alpha), \mathrm{otp}(w \cap \beta), f_\beta(\alpha)\rangle : \alpha < \beta \text{ in } w\}$. Now H^0 is as required except that $\mathrm{Rang}(H^0) \not\subseteq \lambda$, but $|\mathrm{Rang}(H^0)| = \lambda$, so we can correct this).

Let F_i be a one-to-one function from B_{i+1} into λ. We say $\langle a_i^1 : i \in w_1\rangle, \langle a_i^2 : i \in w_2\rangle \in \mathrm{Seq}\,(\bar{B})$ (hence w_1, w_2 have last element) are *equivalent if*:

(a) $H(w_1) = H(w_2)$ and

(b) *if* $\alpha_1 \in w_1$ and $\alpha_2 \in w_2$, and $w_1 \cap \alpha_1, w_2 \cap \alpha_2$ have the same order type and $\alpha_1 = \gamma_1 + 1$, $\alpha_2 = \gamma_2 + 1$, *then*

$$F_{\gamma_1}(a_{\alpha_1}^1) = F_{\gamma_2}(a_{\alpha_2}^2).$$

Now the number of equivalence classes is $\leq \lambda^{<\lambda} = \lambda$. So it is enough to show that if $\langle a_i^\zeta : i \in w_\zeta \rangle \in \mathrm{Seq}(\bar{B})$ are equivalent for $\zeta < \zeta(*) < \lambda$, $0 \in w_\zeta$, $\max(w_\zeta) \in w_\zeta$, then $\bigcap_{\zeta<\zeta(*)} a^\zeta{}_{\max(w_\zeta)} \neq 0$ (see $(*)$). Note that if $\alpha \in w_{\zeta_1} \cap w_{\zeta_2}$, then $a_\alpha^{\zeta_1} = a_\alpha^{\zeta_2}$.

Toward this end we prove by induction on $\alpha \in W$:

$(*)$ (1) $x_\alpha \stackrel{\text{def}}{=} \bigcap_{\zeta<\zeta(*)} a^\zeta{}_{\max(w_\zeta\cap(\alpha+1))}$ is not zero (and belongs to B_α);

(2) if $\gamma < \alpha$ (and $\gamma \in W$) then $\Pr(x_\gamma, x_\alpha, B_\beta)$;

(3) if $\gamma \leq \alpha$ is a limit ordinal then:

(a) $\mathrm{cf}(\gamma) < \lambda \Rightarrow x_{\gamma+1} = \cap\{x_\varepsilon : \varepsilon \in \gamma \cap W\}$,

(b) $\mathrm{cf}(\gamma) \geq \lambda \Rightarrow x_\gamma = x_\varepsilon$ for every large enough $\varepsilon < \gamma$.

Clearly x_α is decreasing (as a_α^ζ is decreasing in α for each ζ) and well defined as $\max(w_\zeta \cap (\alpha+1))$ belongs to w_ζ when $\alpha \in W$ (remembering $0 \in w_\zeta$).

Case 1. $\alpha = 0$

Then $\max(w_\zeta \cap (\alpha+1)) = 0$ and $a_0^\zeta = a_0^0 \in B_0^+$ for every $\zeta < \zeta(*)$. So $(*)(1)$ holds and $(*)(2)$, (3) do not apply.

Case 2. $\alpha = \beta+1, \beta \in W$

Note that if ($\zeta < \zeta(*)$ and) $\alpha = \beta + 1 \notin w_\zeta$ then $a^\zeta{}_{\max(w_\zeta\cap(\alpha+1))} = a^\zeta{}_{\max(w_\zeta\cap(\beta+1))}$.

So if $\alpha \notin w_\zeta$ for every $\zeta < \zeta(*)$ then $x_\alpha = x_\beta$, so $(*)(1)$ holds. As for $(*)(2)$: for $\gamma < \beta$ use the induction hypothesis; for $\gamma = \beta$ this is easy. Similarly for $(*)(3)$.

If for some $\zeta < \zeta(*)$ we have $\alpha \in w_\zeta$, let $v = \{\zeta < \zeta(*) : \alpha \in w_\zeta\}$. So $x_\alpha = \bigcap_{\zeta\notin v} a^\zeta{}_{\max(w_\zeta\cap(\beta+1))} \cap \bigcap_{\zeta\in v} a_\alpha^\zeta$. By the definition of the equivalence relation and the F_i's, for some a we have $[\zeta \in v \Rightarrow a_\alpha^\zeta = a \leq a^\zeta{}_{\max(w_\zeta\cap(\beta+1))}]$ and $[\zeta, \xi \in v \Rightarrow w_\zeta \cap (\alpha+1) = w_\xi \cap (\alpha+1)]$. Clearly

$$\begin{aligned} x_\alpha &= \bigcap_{\zeta\notin v} a^\zeta{}_{\max(w_\zeta\cap(\beta+1))} \cap \bigcap_{\zeta\in v} a_\alpha^\zeta \\ &= \bigcap_{\zeta<\zeta(*)} a^\zeta{}_{\max(w_\zeta\cap(\beta+1))} \cap \bigcap_{\zeta\in v} a_\alpha^\zeta \\ &= x_\beta \cap a. \end{aligned}$$

Now as $\beta \in W$, B_β is λ-complete, hence $x_\beta \in B_\beta$. Now $a \in B_\alpha$ and let $\zeta(0) = \min(v)$, $\gamma(0) = \max(w_{\zeta(0)} \cap (\beta+1))$, the maximum exists as said above. Clearly $\gamma(0) = \beta$ (see 4.5C(2)(iii)), $a \leq a^{\zeta(0)}_{\gamma(0)}$ and $\Pr(a^{\zeta(0)}_{\gamma(0)}, a, B_\beta)$ by the last clause in the definition of $\langle a^{\zeta(0)}_i : i \in w_{\zeta(0)} \rangle \in \operatorname{Seq}_{w_{\zeta(0)}}(\bar{B})$ (see 4.5D(4)(i)). As $x_\beta \in B_\beta$, and easily $a^{\zeta(0)}_{\gamma(0)} \geq x_\beta > 0$, clearly $x_\beta \cap a \neq 0$. So $(*)(1)$ holds. As for $(*)(2)$, by 4.5B(b) as there is a maximal $\gamma \in w \cap \alpha$, i.e. $\beta = \gamma(0)$ (see above) it is enough to prove $(*)(2)$ for $\gamma = \beta = \gamma(0)$. So let $d \in B_\beta$, $0 < d \leq x_\beta$. Then $d \leq a^{\zeta(0)}_{\gamma(0)}$, hence by $\Pr(a^{\zeta(0)}_{\gamma(0)}, a, B_\beta), a \cap d \neq 0$, but $a \cap d = d \cap x_\beta \cap a = d \cap x_\alpha$, so we are done. Lastly $(*)(3)$ holds by the induction hypothesis.

Case 3. $\alpha = \beta + 1, \beta \notin W$

By an assumption of 4.5E, $\aleph_0 \leq \operatorname{cf}(\beta) < \lambda$ so by 4.5D(0)(i) there is $w \in \operatorname{CSb}_u(\beta)$ such that $\zeta < \zeta(*) \Rightarrow w_\zeta \subseteq w$ and $i \in w \,\&\, \operatorname{cf}(i) = \lambda \Rightarrow (\exists j)(\sup(\bigcup_\zeta w_\zeta \cap i) < j < i \,\&\, j \in w)$. Note that

$$a^\zeta{}_{\max(w_\zeta \cap (\alpha+1))} = \bigcap_{\gamma<\beta} a^\zeta{}_{\max(w_\zeta \cap (\gamma+1))}.$$

[Why? If $\alpha \notin w_\zeta$, as $\langle a^\zeta{}_{\max(w_\zeta \cap (\gamma+1))} : \gamma < \beta \rangle$ is nonincreasing and eventually constant (because $\langle \max(w_\zeta \cap (\gamma + 1)) : \gamma < \beta \rangle$ is eventually constant), it is equal to

$$a^\zeta{}_{\max(w_\zeta \cap (\alpha+1))} = \bigcap_{\gamma<\beta} a^\zeta{}_{\max(w_\zeta \cap (\gamma+1))}.$$

If $\alpha \in w_\zeta$, as $\langle a^\zeta_\gamma : \gamma \in w_\zeta \rangle \in \operatorname{Seq}(B)$ (see in Definition 4.5C(4)(i) the second clause in the definition of $\bar{a}^\zeta \in \operatorname{Seq}(\bar{B})$).] Now:

$$x_\alpha = \bigcap_{\zeta<\zeta(*)} a^\zeta{}_{\max(w_\zeta \cap (\alpha+1))} = \bigcap_{\zeta<\zeta(*)} \bigcap_{\gamma<\beta} a^\zeta{}_{\max(w_\zeta \cap (\gamma+1))}$$
$$= \bigcap_{\gamma<\beta} \Big(\bigcap_{\zeta<\zeta(*)} a^\zeta{}_{\max(w_\zeta \cap (\gamma+1))} \Big) = \bigcap_{\gamma<\beta} x_\gamma.$$

So $(*)(3)$ holds (as $\gamma = \beta$ is the only new case). Also it can be checked that $\langle x_\varepsilon : \varepsilon \in w \rangle \in \operatorname{Seq}_w(\bar{B})$ (in Definition 4.5C(4)(i), the first clause by the definition of x_α and $(*)(1)$, the second clause (x_ε decreasing) is shown above, the third clause (continuity) by $(*)(3)$, the fourth clause by the choice of w and the definition of x_ε, the fifth clause by $(*)(2)$). As $\bar{B}$ is 2-o.k. (see 4.5C (5)) (as $(*)(2)$ holds below β) we get that $x_\alpha = \bigcap\{x_\varepsilon : \varepsilon \in u \cap \beta\} \neq 0$. Similarly, using 4.5D(6), we can check $(*)(2)$.

Case 4. α limit

As $\alpha \in W$, necessarily $\mathrm{cf}(\alpha) = \lambda$. But then, by the definition of $\mathrm{Seq}_{\underset{\sim}{w}}(\bar{B})$, if $\alpha \in w_\zeta$ though necessarily $\max(w_\zeta \cap (\alpha+1)) \neq \max(w_\zeta \cap (\gamma+1))$ for $\gamma < \alpha$, still for $\gamma < \alpha$ large enough $a^\zeta_\alpha = {a^\zeta}_{\max(w_\zeta \cap (\gamma+1))}$, hence ${a^\zeta}_{\max(w_\zeta \cap (\alpha+1))} = \bigcap_{\gamma<\beta} {a^\zeta}_{\max(w_\zeta \cap (\gamma+1))}$ for every large enough $\beta < \alpha$. If $\alpha \notin w_\zeta$ this holds on simpler grounds. But $\zeta(*) < \lambda = \mathrm{cf}(\alpha)$. So $x_\alpha = x_\gamma$ for every large enough $\gamma < \alpha$, and we can finish easily. $\square_{4.5E}$

4.5F Remark. The proof is written such that it will be easy to change it for $\bar{B} = \langle B_i : i < \gamma\rangle$, $\gamma < (2^\lambda)^+$, so $|B_i| = |i| + \lambda$, B_{i+1} is generated by $B_i \cup B'_i$, $|B'_i| = \lambda$, B'_i is λ-complete and in the definition of $\mathrm{Seq}_{\underset{\sim}{w}}(\bar{B})$ add: if $i = \beta+1, \beta \in w$ then $(\exists x \in B'_\beta)[a_i = a_\beta \cap x]$.

Just use $H : \{a : a \subseteq 2^\lambda, |a| < \lambda\} \to \lambda$ such that $H(a) = H(b)$ & $\alpha \in a \cap b \Rightarrow \mathrm{otp}\,(a \cap \alpha) = \mathrm{otp}(b \cap \alpha)$ which exists by Engelking and Karlowic [EK]. But it is not clear whether there is interest in this.

4.6 Definition. 1) We say $\bar{Q} = \langle P_i, \underset{\sim}{Q}_j, \mathbf{t}_j, \underset{\sim}{\mathcal{A}}_i : i \leq \alpha, j < \alpha\rangle$ is an $\bar{S}$-o.k. sequence for $\underset{\sim}{W}$ (where $\bar{S} = \langle S_1, S_2, S_3\rangle$, a partition of ω_1) if:

(A) $\bar{Q}$ is a S_1-suitable iteration (forgetting the $\underset{\sim}{\mathcal{A}}_i$'s).

(B) Each Q_i is S_3-complete.

(C) $\underset{\sim}{\mathcal{A}}_i$ is a P_i-name of a subalgebra (or just subset) of $\mathfrak{B}^{P_i}$.

(D) $\underset{\sim}{\mathcal{A}}_i$ is increasing continuous.

(E) $\mathbf{t}_j \in \{0,1\}$ and: if $\mathbf{t}_i = 1$, then $\Vdash_{P_i}$ "$\underset{\sim}{\mathcal{A}}_i \lessdot \mathfrak{B}^{P_i} \restriction S_1$".

(F) $\mathbf{t}_i = 1$ for every successor ordinal i.

(G) $\Vdash_{P_i}$ "$\langle \underset{\sim}{\mathcal{A}}_j : j \leq i, j \in \underset{\sim}{W}\rangle$ is 3-o.k. for $\underset{\sim}{W}$" where on $\underset{\sim}{W}$ see clause (H) below and λ from 4.5A(1) is chosen as $\aleph_1$ (see below and 4.5C(1), (5), (6)).

(H) If i is successor or zero then $i \in \underset{\sim}{W}$. If, in $V^{P_{i+1}}$, i is a limit of cofinality $\aleph_0$ then $i \notin \underset{\sim}{W}$. Also "$i \in \underset{\sim}{W}$" is a P_{i+1}-name and $\underset{\sim}{W} \subseteq \alpha$.

(I) $\Vdash_{P_{i+1}}$ "$\mathrm{Rss}(\aleph_2)$".

(J) If i is neither a limit nor a successor of a limit ordinal, then $\mathcal{A}_i = \mathfrak{B}^{P_i} \restriction S_1$.

2) If $\underset{\sim}{W}$ is not given we mean $\{i < \alpha : \text{if } i \text{ is limit then (in } V^{P_{i+1}}) \ \mathrm{cf}(\alpha) \geq \aleph_1\}$.

4.6A Remark. Note that $\underset{\sim}{W}$ determines $\langle \mathbf{t}_\alpha : \alpha < \kappa \rangle$ in Definition 4.6, so we could in 4.7 below forget it.

4.7 Proof of 4.4. Let $h : \kappa \to H(\kappa)$. We define $\bar{Q}^\alpha = \langle P_i, \underset{\sim}{Q}_j, \mathbf{t}_j, \underset{\sim}{\mathcal{A}}_i : i \leq \alpha, j < \alpha \rangle$ by induction on $\alpha \leq \kappa$ such that in stage α, the objects $\underset{\sim}{Q}_j (j < \alpha), P_j (j \leq \alpha)$, $\mathbf{t}_j$ $(j + 1 \leq \alpha + 1)$ and $\underset{\sim}{\mathcal{A}}_j (j \leq \alpha)$ (and the truth value of "$j \in \underset{\sim}{W}$" is as in 4.6(2)) have already been defined and for successor i, $\mathfrak{B}^{P_i} \lessdot \underset{\sim}{\mathcal{A}}_{i+1}$ and:

(A) $\bar{Q}^\alpha$ is $\bar{S}$-o.k. (and increases with α).

(B) $\bar{Q}^\alpha \in H(\kappa)$ for $\alpha < \kappa$.

(C) If α is non-limit, let $\kappa_{\alpha+1}$ be the first compact cardinal $> |P_\alpha|$, and $\underset{\sim}{Q}_\alpha = \mathrm{SSeal}(\mathfrak{B}^{P_\alpha}, S_1, \kappa_{\alpha+1})$ if α is successor and $\mathrm{Levy}(\aleph_1, < \kappa_{\alpha+1})$ if α is zero and $\mathcal{A}_{\alpha+1} = \mathfrak{B}^{P_{\alpha+1}} \restriction S_1$ (and $\underset{\sim}{\mathcal{A}}_0$ the trivial algebra). Lastly of course $\mathbf{t}_{\alpha+1} = 1$ and "$\alpha \in \underset{\sim}{W}$" is true also $\mathbf{t}_0 = 0$ and "$0 \in \underset{\sim}{W}$".

(D) *If* α is a limit ordinal, $h(\alpha) = \langle \mathbf{t}, \underset{\sim}{Q}, \underset{\sim}{\mathcal{A}} \rangle$, $\underset{\sim}{Q}$ a P_α-name of a forcing notion, $\underset{\sim}{\mathcal{A}}$ a $P_\alpha * \underset{\sim}{Q}$-name, and for some $\underset{\sim}{R} \in H(\kappa)$ we have $\Vdash_{P_i}$ "$\underset{\sim}{Q} \lessdot \underset{\sim}{R}$" and by the following choices for $\bar{Q}^{\alpha+1}$ we get a $\bar{S}$-o.k., *then* so we choose $\bar{Q}^{\alpha+1}$; where the choices are: $\mathbf{t}_\alpha = \mathbf{t}, \underset{\sim}{Q}_\alpha = \underset{\sim}{R}$, and $\mathcal{A}_\alpha = \mathfrak{B}^{\bar{Q} \restriction \alpha} \restriction S_1$, $P_{\alpha+1} = P_\alpha * \underset{\sim}{R}$ and $\mathcal{A}_{\alpha+1} = \mathfrak{B}[P_{\alpha+1}]$. If possible we choose $\underset{\sim}{R} = \underset{\sim}{Q}$.

(E) If clauses (C), (D) do not produce a definition of $\bar{Q}^{\alpha+1}$, let $\kappa_{\alpha+1}$ be the first compact cardinal $> |P_\alpha|$, and then:

first case if in $V^{P_\alpha}, \mathrm{cf}(\alpha) > \aleph_0$ *then*

$$\begin{aligned}
\underset{\sim}{\mathcal{A}}_\alpha &\stackrel{\text{def}}{=} \mathfrak{B}^{\bar{Q}\restriction\alpha}\restriction S_1 \text{ i.e. } \underset{\sim}{\mathcal{A}}_\alpha \stackrel{\text{def}}{=} \bigcup_{j<\alpha} \mathfrak{B}^{P_{j+2}}\restriction S_1 = \bigcup_{j<\alpha} \underset{\sim}{\mathcal{A}}_j, \\
\underset{\sim}{Q}_\alpha &= \mathrm{SSeal}(\langle \underset{\sim}{\mathcal{A}}_j : j \leq \alpha\rangle, S_1, \kappa_{\alpha+1}) = \\
&= \mathrm{SSeal}(\langle \mathfrak{B}^{P_{j+2}} : j < \alpha\rangle \hat{}\, \langle \mathcal{A}^{P_\alpha}\rangle, S_1, \kappa_{\alpha+1}) \\
\underset{\sim}{\mathcal{A}}_{\alpha+1}, &= \underset{\sim}{\mathfrak{B}}^{P_{\alpha+1}}, \\
\mathbf{t}_\alpha &= \mathbf{t}_{\alpha+1} = 1
\end{aligned}$$

:

second case if in $V^{P_\alpha}, \mathrm{cf}(\alpha) = \aleph_0$ (i.e. $\alpha \notin W$) *then*

$$\underset{\sim}{\mathcal{A}}_\alpha = \mathfrak{B}^{\bar{Q}\restriction\alpha}\restriction S_1,$$

(in V^{P_α}) let $\underset{\sim}{\mathcal{A}}'_{\alpha+1}$ be $Z^\alpha = Z^\alpha(\langle \underset{\sim}{\mathcal{A}}_j : j \leq \alpha\rangle \hat{}\, \langle \mathfrak{B}^{P_\alpha}\restriction S_1\rangle)$ (a subset of $\mathfrak{B}^{P_\alpha}\restriction S_1$, see Definition 2.4(2)) and $\underset{\sim}{\mathcal{A}}_{\alpha+1}$ be the subalgebra of $\mathfrak{B}^{P_{\alpha+1}}\restriction S_1$ which $\underset{\sim}{\mathcal{A}}'_{\alpha+1}$ generates. We let

$$\underset{\sim}{Q}_\alpha = \mathrm{SSeal}(\langle \mathfrak{B}^{P_{j+2}} : j < \alpha\rangle \hat{}\, \langle \underset{\sim}{\mathcal{A}}_{\alpha+1}\rangle, S_1, \kappa_{\alpha+1}),$$

$$\mathbf{t}_\alpha = 0, \quad \mathbf{t}_{\alpha+1} = 1.$$

If we succeed to carry the induction, then letting $G \subseteq P_\kappa$ be generic over V we know:

(a) $\aleph_1^{V[G]} = \aleph_1^V$ and $\langle S_1, S_2, S_3\rangle$ is a partition of ω_1 to stationary subsets (as P_κ is semiproper by clause (A) of Definition 4.6).

(b) $\aleph_2^{V[G]} = \kappa$ (similarly, noting that P_κ satisfies the κ-c.c.).

(c) Every countable set of ordinals from V^{P_κ} is included in one from V (see (e) below).

(d) $\langle \underset{\sim}{\mathcal{A}}_i[G] : i < \kappa\rangle$ is 3-o.k. (by clause (G) of Definition 4.6).

(e) P_κ is S_3-complete (see clause (B) of Definition 4.6) hence, as S_3 is sationary, P_κ adds no reals so $V[G_\kappa] \models$ "$2^{\aleph_0} = \aleph_1$, so $\lambda = \lambda^{<\lambda}$".

(f) $\mathfrak{B}^{V[G]} \restriction S_1$ is Ulam i.e., omitting 0, it is the union of $\lambda = \aleph_1$ many λ-complete filters. [Why? By 4.5(E) and (d) above (as $W = \{\alpha < \kappa : \alpha$ zero, successor or has cofinality $\geq \aleph_1$ (in V, equivalently in $V[G_\kappa]$)$\}$).]

To carry the induction it is enough to show that when clause (E) in the construction is applied, we get an $\bar{S}$-o.k. iteration; this is dealt by 4.9 below $+2.13 + 2.16$ for $\langle \mathfrak{B}^{\bar{Q}} \restriction S \rangle$ for the first case, and by 4.10 below $+2.13 + 2.16$ for W defined by 4.10 for the second case. $\square_{4.7}$

4.8 Remarks. 1) We could have allowed in clause (D) during the proof 4.7 (of 4.4) to decide if $i \in \underset{\sim}{W}$, i.e. decide $\underset{\sim}{W}_i = \underset{\sim}{W} \cap i$, i.e. demand $h(\alpha) = \langle \mathbf{t}, \underset{\sim}{\mathbf{s}}, \underset{\sim}{Q}, \underset{\sim}{A} \rangle$, and try to define $\bar{Q}^{\alpha+1}$ as there with the following addition: the truth value of "$\alpha \in \underset{\sim}{W}$" is $\underset{\sim}{\mathbf{s}}$, a $P_\alpha * \underset{\sim}{Q}$-name, and at the end shoot a suitable club of κ through the "good" places.

2) We could have gotten a forcing axiom, as before.

3) In fact we can weaken the large cardinals demand to "$\kappa = \sup\{\lambda < \kappa : \lambda$ strongly inaccessible and $\mathrm{Rss}^+(\lambda)$ or at least $\bigwedge_{\mu<\kappa} \mathrm{Rss}^+(\lambda, \mu)\}$".

4.9 Claim. Suppose $S \subseteq \omega_1$ is stationary, $\bar{Q} = \langle P_i, \underset{\sim}{Q}_j : i \leq \alpha,\ j < \alpha \rangle$ is a semiproper iteration, α a limit ordinal, and, for simplicity, $\Vdash_{P_{i+1}}$ "$\mathrm{Rss}(\aleph_2[V^{P_{i+1}}])$" for $i < \alpha$. Let $\underset{\sim}{\Upsilon}$ be a P_α-name of a dense subset of $\mathfrak{B}^{\bar{Q}} \restriction S = \bigcup_{i<\alpha} \mathfrak{B}^{P_i} \restriction S$ for $i \in W^*$.

Then

$\otimes$ *if* λ is regular and large enough, $N \prec (H(\lambda), \in, <^*_\lambda)$ is countable, and $\bar{Q}$, λ, p, $\underset{\sim}{\Upsilon}$, belong to N, $p \in P_\alpha \cap N$, $\beta \in \alpha \cap N$ a successor ordinal and $q \in P_\beta$ is (N, P_β)-semi-generic, $p \restriction \beta \leq q$ and $N \cap \omega_1 \in S$, *then* there is a countable N', $N \leq_\beta N' \prec (H(\lambda), \in, <^*_\lambda), N \cap \omega_1 = N' \cap \omega_1$, successor $\gamma \in [\beta, \alpha)$, P_γ-name $\underset{\sim}{A} \in N'$ and q', p' satisfying $p \leq p' \in P_\alpha \cap N'$, $q' \in P_\gamma$, $p' \restriction \gamma \leq q'$, $q' \restriction \beta = q$ and $p' \restriction \beta = p \restriction \beta$ such that $q' \Vdash_{P_\alpha}$ "$N \cap \omega_1 \in \underset{\sim}{A}$" and $p' \Vdash$ "$\underset{\sim}{A} \in \underset{\sim}{\Upsilon}$".

4.9A Remark. 1) Note that $\mathfrak{B}^{P_{i+1}} \restriction S \lessdot \mathfrak{B}^{\bar{Q}} \restriction S$ for $i < \alpha$.

2) The claim gives more chains than used in 4.7.

3) This is naturally used together with 2.13.

Proof. Let us fix p, $\underset{\sim}{\Upsilon}$, β and work in $V[G_\beta]$ where $G_\beta \subseteq P_\beta$ generic over V and $q \in G_\beta$. Let λ be large enough and

$$\begin{aligned}\mathbf{W} \stackrel{\text{def}}{=} \{M \prec (H(\lambda), \in, <^*_\lambda) : {} & M \text{ is countable, } M \cap \omega_1 \in S, \text{ but} \\ & \text{there are no successor } \gamma \in M \cap [\beta, \alpha), r \in P_\gamma/G_\beta \text{ and} \\ & \underset{\sim}{A} \in M \text{ (a } P_\gamma\text{-name) and } p' \in P_\alpha/G_\beta \cap N \text{ such that:} \\ & r \text{ is } (M, P_\gamma/G_\beta)\text{-semi-generic, } p \leq p', p' \restriction \gamma \leq r \text{ and} \\ & r \Vdash_{P_\alpha/G_\beta} \text{“} M \cap \omega_1 \in \underset{\sim}{A}\text{”}, p' \Vdash \text{“}\underset{\sim}{A} \in \underset{\sim}{\Upsilon}\text{”}\}.\end{aligned}$$

If $\mathbf{W} = \emptyset \bmod \mathcal{D}_{<\aleph_1}(H(\lambda))$, we can easily get the desired result (as in the proof of 1.11)).

So (in $V[G_\beta]$) the set $\mathbf{W}$ is a stationary subset of $\mathcal{S}_{<\aleph_1}(H(\lambda))$, hence semi-stationary. As $V[G_\beta] \models$ “Rss$(\aleph_2)$” (remember β is a successor ordinal and clause (I) of Definition 4.6) there is $u \subseteq H(\lambda)$ such that $\omega_1 \subseteq u$ and $|u| < \aleph_2$ (in $V[G_\beta]$) and $W \cap \mathcal{S}_{<\aleph_1}(u)$ is semi-stationary. Now without loss of generality $(u, \in, <^*_\lambda \restriction u) \prec (H(\lambda), \in, <^*_\lambda)$. Let $u = \bigcup_{\zeta<\omega_1} u_\zeta$, u_ζ is countable, increasing and continuous. So

$$B_1 \stackrel{\text{def}}{=} \{\zeta \in S : (\exists M \in \mathbf{W})(\omega_1 \cap u_\zeta \subseteq M \subseteq u_\zeta)\}$$

is a stationary subset of $S \subseteq \omega_1$ (see 1.2(4)), it belongs to $\mathfrak{B}[P_\beta]$, and we shall prove:

(∗) $\quad p \Vdash_{P_\alpha/G_\beta}$ “ for every $X \in \underset{\sim}{\Upsilon}$ the set $X \cap \underset{\sim}{A} \cap B_1$ is not stationary”.

[Why (∗)? If not then for some p' and P_α-name $\underset{\sim}{A}$, $p \leq p' \in P_\alpha/G_\beta$ and $p' \Vdash_{P_\alpha/G_\beta}$ “$\underset{\sim}{A} \in \underset{\sim}{\Upsilon}$ and $\underset{\sim}{A} \cap B_1$ is stationary”. As $\underset{\sim}{\Upsilon} \subseteq \mathfrak{B}^{\bar{Q}}$, for some $\gamma < \alpha$, $\underset{\sim}{A}[G_{P_\alpha}]$ is in $\mathfrak{B}^{P_\gamma}$, so (possibly increasing p) without loss of generality for some successor $\gamma \in [\beta, \alpha)$, $\underset{\sim}{A}$ is a P_γ-name of a member of $\mathfrak{B}^{P_\gamma}$. For $\zeta \in B_1$, let the model M_ζ be any member of $\mathbf{W}$ which satisfies $\omega_1 \cap u_\zeta \subseteq M_\zeta \subseteq u_\zeta$ (see the definition of B_1). For $\xi < \omega_1$, let N'_ξ be the Skolem Hull (in $(H(\lambda), \in, <^*_\lambda)$) of $\{\zeta : \zeta < \xi\} \cup \{p, p', \underset{\sim}{A}, W, \langle u_\zeta, M_\zeta : \zeta \in B_1\rangle\}$, and

$$\underset{\sim}{C} = \{\xi < \omega_1 : N'_\xi[G_{P_\alpha}] \cap \omega_1 = \xi \text{ and } N'_\xi[G_{P_\alpha}] \cap u = u_\xi\}.$$

As $\langle N'_\xi[G_{P_\alpha}] : \xi < \omega_1\rangle$ is increasing continuous, $\underset{\sim}{C}$ is a P_α/G_β-name of a club of ω_1. Clearly $\underset{\sim}{C} \cap \underset{\sim}{A}$ is necessarily disjoint to B_1 by the definition of W: if $\zeta < \omega_1, q \in P_\alpha/G_\beta$, and $q \Vdash_{P_\alpha/G_\beta}$ "$\zeta \in \underset{\sim}{C} \cap \underset{\sim}{A} \cap \underset{\sim}{B}_1$", then $N_\zeta \in W$ is defined, q_α is $(N_\zeta, P_\alpha/G_\beta)$-semi-generic, and $q_\alpha \Vdash_{P_\alpha/G_\beta}$ "$N_\zeta \cap \omega_1 \in \underset{\sim}{A}$", contradicting "$N_\zeta \in W$" so $(*)$ holds.]

But $(*)$ contradicts $p \Vdash_{P_\alpha/G_\beta}$ "$\underset{\sim}{\Upsilon} \subseteq \mathfrak{B}^{\bar{Q}}$ is dense " as $B_1 \in \mathfrak{B}^{P_\beta} \subseteq \mathfrak{B}^{\bar{Q}}$.

$\square_{4.9}$

4.10 Claim. Suppose $S \subseteq \omega_1$ is stationary, $\bar{Q} = \langle P_i, \underset{\sim}{Q}_j, \mathbf{t}_j, \underset{\sim}{\mathcal{A}}_i : i \leq j, j < \alpha\rangle$ 3-o.k. sequence for W, S-suitable iteration, α limit ordinal and (for simplicity) $\mathrm{cf}(\alpha) = \aleph_0$ and let $\underset{\sim}{\mathcal{A}}_\alpha = \mathfrak{B}^{\bar{Q}}$ (a P_α-name). Let $\bar{\mathcal{A}} = \langle \underset{\sim}{\mathcal{A}}_i : i \leq \alpha \text{ and } i \in \underset{\sim}{W}\rangle$. *Then:*

- $\otimes$ *if* λ is regular large enough, $N \prec (H(\lambda), \in, <^*_\lambda)$ countable and $\bar{Q}, \lambda, p$ belong to N, $p \in P_\alpha \cap N$, $\beta \in \alpha \cap N$ a successor ordinal and $q \in P_\beta$ is (N, P_β)-semi generic, $p{\restriction}\beta \leq q$ and $N \cap \omega_1 \in S$ *then* there is an (N, P_α)-semi-generic $q' \in P_\alpha, q'{\restriction}\beta \geq q$ and $q' \Vdash_{P_\alpha}$ " for every dense open $\Upsilon \subseteq \mathrm{Seq}_u(\bar{\mathcal{A}})$ (see 4.5A(4)(ii), (9)) which belongs to $N[G_\alpha]$, for some $\langle A_i : i \in w\rangle \in N[G_\alpha] \cap \Upsilon$ we have $N \cap \omega_1 \in \bigcap_{i \in w} A_i$".

4.10A Remark. 1) If "$\mathrm{cf}(\alpha) \neq \aleph_0$" we can still assume p forces $\mathrm{cf}(\alpha) = \aleph_0$ or p forces $\mathrm{cf}(\alpha) = \aleph_1$ or "α is inaccessible, $\bigwedge_{i<\alpha} |P_i| < \alpha$" and in the first case prove 4.10 with minimal changes.
2) Note that $Z^\alpha[\langle \underset{\sim}{\mathcal{A}}_i : i \leq \alpha\rangle]$ is a subset of $\mathfrak{B}^{P_\alpha}$ extending $\mathfrak{B}^{\bar{Q}}$, 0 is not in it, *but* there is no reason for it to be closed under differences.

Proof. Standard, by now. Let $\langle \beta_\ell : \ell < \omega\rangle \in N$ be an increasing sequence of successor ordinals with $\beta_0 = \beta$, $\bigcup_{\ell<\omega} \beta_\ell = \alpha$. Let $\bar{\underset{\sim}{\Upsilon}} = \langle \underset{\sim}{\Upsilon}_n : n < \omega\rangle$ list the sets $\underset{\sim}{\Upsilon} \in N$ which are P_α-names (forced to be) pre-dense subsets of $\mathrm{Seq}_u(\bar{\mathcal{A}})$. We choose by induction $p_n, q_n, N_n, \bar{a}^n, G_{\beta_n}$ such that:

(a) $G_{\beta_n} \subseteq P_{\beta_n}$ generic over V, $G_{\beta_n} \subseteq G_{\beta_{n+1}}$,

(b) $N_0 = N, p_0 = p, q_0 = q$,

(c) $N_n \leq_{\omega_2} N_{n+1}, N_n \prec (H(\lambda), \in, <^*_\lambda)$ and $N_n \in V[G_{\beta_n}]$,

(d) $p_n \leq p_{n+1}, p_n \in N \cap P_\alpha / G_{\beta_n}$,

(e) $q_n \in G_{\beta_n} \subseteq P_{\beta_n}$ is (N_n, P_{β_n})-semi-generic,

(f) $p_n \restriction \beta_n \leq q_n$ (in P_{β_n}),

(g) $\bar{a}^n = \langle \underset{\sim}{a}^n_\zeta : \zeta \in w_n \rangle \in \mathrm{Seq}\,(\langle \mathcal{A}_i : i < \alpha \rangle) \cap \Upsilon_n$,

(h) $w_n \subseteq w_{n+1}$ and $a^{n+1}_\zeta \leq a^n_\zeta$ for $\zeta \in w_n$,

(i) $\{\beta_\ell : \ell < n\} \subseteq w_n$,

(j) $N_n \cap \omega_1 \in \underset{\sim}{a}^n_{\beta_n}$ that is $q_n \cup p_n \restriction [\beta_n, \alpha)$ forces this.

The induction step is by 4.1 (and 4.5D(9)). As we are using RCS iteration, this suffices (i.e. we can make the G_{β_n} disappear).
The details are left to the reader. This induction suffices as we can use RCS iteration, so we can find q' as required. $\square_{4.10}$