Epilogue

In contrast with those of a work of fiction, the lives of the characters of this book extend well beyond its covers. For the reader who has a yen to follow them a bit further, we discuss here briefly some of the more recent developments and offer some suggestions for further reading. In many cases the material is still very much in a state of development and we are attempting not to give a thorough and orderly presentation but rather to convey some of the flavor of current research. Some of the notational conventions of the preceding chapters do not apply here.

We begin with Inductive Definability, which has been either on stage or just in the wings in almost every section of the book. One of the most elementary questions which was not answered in the text is that of the closure ordinals and sets for most classes of non-monotone inductive operators over ω. To state these results concisely, we introduce some notation. For any class X of relations, let

$$|X| = \sup^+\{|\Gamma| : \Gamma \text{ is an inductive operator and } \Gamma \in X\};$$

$$X\text{-Ind} = \{R : R \text{ is reducible to } \bar{\Gamma} \text{ for some inductive operator } \Gamma \in X\};$$

$$X\text{-Hyp} = \{R : \text{both } R \text{ and } {\sim}R \text{ belong to } X\text{-Ind}\}.$$

Replacing "inductive" by "monotone" yields the definitions of $|X\text{-mon}|$, X-mon-Ind, and X-mon-Hyp.

For comparison we state first some of our earlier results in this notation:

(1) $|\Sigma^0_1| = \omega;$ (III.3.3)

$\Sigma^0_1\text{-mon-Ind} = \Sigma^0_1;$ (III.3.5)

$\Delta^0_{(\omega)} \subsetneqq \Sigma^0_1\text{-Ind} \subsetneqq \Delta^1_1;$ (III.3.6, 7)

(2) $|\Pi^0_1| = |\Pi^0_1\text{-mon}| = |\Pi^1_1\text{-mon}| = \omega_1;$ (IV.2.15, 16, 21)

$\Pi^0_1\text{-Ind} = \Pi^0_1\text{-mon-Ind} = \Pi^1_1\text{-mon-Ind} = \Pi^1_1;$ (III.3.1, 2, IV.2.17)

(3) $|\Sigma^1_1\text{-mon}| = \omega_1[\mathsf{E}^{\#}_1];$

$\Sigma^1_1\text{-mon-Ind} = \{R : R \text{ is semi-recursive in } \mathsf{E}^{\#}_1\};$ (VI.6.14)

(4) for all $r \geq 2$,

$|\Delta_r^1| = \delta_r^1$; (IV.7.10, III.3.34)

Δ_r^1-Ind $= \Delta_r^1$; (III.3.10)

Σ_r^1-mon-Ind $= \Sigma_r^1$; (III.3.14)

Π_r^1-mon-Ind $= \Pi_r^1$. (III.3.1)

Aanderaa [1974] proves the following:

(5) $|\Pi_1^1| < |\Sigma_1^1|$ and $|\Sigma_2^1| < |\Pi_2^1|$;

if $V = L$, then for all $r \geq 2$,

$$|\Delta_r^1| = |\Sigma_r^1\text{-mon}| < |\Sigma_r^1| < |\Pi_r^1|;$$

if PD, then as above for all even $r \geq 2$, while for all odd $r \geq 2$,

$$|\Delta_r^1| = |\Pi_r^1\text{-mon}| < |\Pi_r^1| < |\Sigma_r^1|.$$

The proof is based on the pre-wellordering property.

Recall from Exercise VIII.7.27 the notion κ is $\forall_r^0$-reflecting. In view of (i) of that exercise and Theorem VIII.3.4(i), the following result of Aczel–Richter [1974] is a generalization of (2):

(6) for all $r \geq 1$,

$|\Pi_r^0| =$ least $\kappa . L_\kappa$ is $\forall_{r+1}^0$-reflecting;

Π_r^0-Ind $= \{R : R$ is $|\Pi_r^0|$-semi-recursive in parameters$\}$.

Similarly, if we add to $\mathscr{L}_{\mathrm{ZF}}$ second-order variables and define the classes of $\exists_r^1$ and $\forall_r^1$ formulas analogously, we have:

(7) $|\Sigma_1^1| = |\Sigma_1^1\text{-mon}| =$ least $\kappa . L_\kappa$ is $\exists_1^1$-reflecting;

Σ_1^1-Ind $= \Sigma_1^1$-mon-Ind $= \{R : R$ is $|\Sigma_1^1|$-semi-recursive in parameters$\}$;

(8) $|\Pi_1^1| =$ least $\kappa . L_\kappa$ is $\forall_1^1$-reflecting;

Π_1^1-Ind $= \{R : R$ is $|\Pi_1^1|$-semi-recursive in parameters$\}$.

Richter [1975] shows that

(9) $|\Pi_2^1| =$ least $\kappa . L_\kappa$ is $\forall_2^1$-reflecting;

Π_2^1-Hyp $= \{R : R$ is $|\Pi_2^1|$-recursive in parameters$\}$;

Π_2^1-Ind $\supsetneq \{R : R$ is $|\Pi_2^1|$-semi-recursive in parameters$\}$.

The direct analogues of (7)–(9) for Σ_2^1 or higher levels of the analytical hierarchy are not true, but Aczel–Richter [1974] have modified versions. They, and independently Cenzer [1974], obtained the interesting characterization

(10) $\quad |\Pi_1^1| = \text{least } \kappa \,.\, \kappa \text{ is } \kappa^+ \text{ stable},$

where κ^+ denotes the least admissible ordinal greater than κ. Cenzer [1974] has similar characterizations for $|\Sigma_2^1|$ and for higher levels under the hypothesis $V = L$. Many of these facts are also proved in a more general form in Moschovakis [1974b] (see (31)–(33) below).

The results of the text concerning inductive operators over ${}^\omega\omega$, other than decomposable ones, are even more limited. We have only

(11) for all $r > 0$,

$\Pi_r^2\text{-mon-Ind} = \Pi_r^2;$ (VI.7.10)

$\Sigma_r^2\text{-mon-Ind} = \Sigma_r^2;$ (VI.7.15)

$\Delta_r^2\text{-Ind} = \Delta_r^2.$ (VI.7.13)

Cenzer [1974b] announces (among others) the following results for operators over ${}^\omega\omega$:

(12) $\quad |\Delta_1^0| = |\Pi_1^0\text{-mon}| = |\Pi_1^1\text{-mon}| = |\Sigma_1^1\text{-mon}| = \aleph_1;$

$\Pi_1^0\text{-mon-Ind} = \Pi_1^1\text{-mon-Ind} = \Pi_1^1;$

$\Delta_1^1\text{-Ind} \subseteq \Sigma_1^1\text{-mon-Ind} = \Sigma_2^1;$

(13) $\quad |\Pi_1^1| = \boldsymbol{\omega}_1[\mathbb{E}]$

$\Pi_1^1\text{-Ind} = \{\mathsf{R} : \mathsf{R} \text{ is semi-recursive in } \mathbb{E} \text{ (semi-hyperanalytical)}\};$

(14) $\quad |\Sigma_2^1| = |\Sigma_2^1\text{-mon}| = \boldsymbol{\omega}_1[\mathbb{E}^{\#}];$

$\Sigma_2^1\text{-Ind} = \Sigma_2^1\text{-mon-Ind} =$

$\{\mathsf{R} : \mathsf{R} \text{ is semi-recursive in } \mathbb{E}^{\#} \text{ (semi-hyperprojective)}\};$

$|\Pi_2^1| > \boldsymbol{\omega}_1[\mathbb{E}^{\#}].$

The characterizations of Π_1^1-Ind and Σ_2^1-(mon)-Ind are essentially contained in Grilliot [1971a] read in conjunction with Hinman–Moschovakis [1971].

We turn next to an important unifying concept which is currently playing an important role in definability theory, that of a Spector class of relations over ω or over an arbitrary set M. A class X of relations over ω is called a *Spector class* iff it contains the recursive relations, is closed under $\cap$, $\cup$, $\exists^0$, $\forall^0$, and relational composition with recursive functions, is ω-indexable, and has the pre-wellordering property. This book is full of examples of Spector classes: Π_1^1, Σ_2^1, and the classes of relations semi-recursive in E, I for any type-2 functional I, in $\mathbb{E}$, $\mathbb{I}$ for any type-3 functional $\mathbb{I}$, or κ-semi-recursive for an admissible κ. The similarities in structure among these classes are largely explained by the fact that many of their properties are properties of any Spector class. Moschovakis [1974, 9A–C] gives a good exposition of the theory of Spector classes.

Call a type-2 functional I *normal* iff E is recursive in I. The 1-*envelope* and 1-*section* of I are the classes:

$$1\text{-env}(\mathsf{I}) = \{R : R \text{ is semi-recursive in } \mathsf{I}\};$$

$$1\text{-sc}(\mathsf{I}) = \{R : R \text{ is recursive in } \mathsf{I}\}.$$

For any normal I, 1-env(I) is a Spector class. The next two results characterize 1-env(I) and the collection of all 1-envelopes of normal type-2 functionals among all Spector classes. A class X of relations is called *closed under* I-*application* iff for any partial function $F \in X$ (that is, $\mathrm{Gr}(F) \in X$), if $G(m) \simeq \mathsf{I}(\lambda p\,.\,F(p, m))$, then also $G \in X$. Moschovakis [1974a] shows:

(15) for any normal type-2 functional I, 1-env(I) is the smallest Spector class closed under I-application.

For any Spector class X, let $\Delta(X)$ denote the symmetric class $X \cap cX$ and let $\omega_1[X]$ denote the least ordinal not the order-type of a $\Delta(X)$ well-ordering of ω. For two Spector classes $X \subseteq Y$, set

$$X < Y \leftrightarrow X \subseteq \Delta(Y) \leftrightarrow \omega_1[X] < \omega_1[Y]$$

(the equivalence is a generalization of Theorem IV.2.14). A Spector class X is called *Mahlo* iff for any I such that X is closed under I-application, there exists a Spector class $Y < X$ also closed under I-application. Harrington–Kechris [1975] and independently Simpson show:

(16) A Spector class X is the 1-envelope of some normal type-2 functional iff X is not Mahlo.

These results have generalizations to higher types. For example, a type-3 functional $\mathbb{I}$ is called *normal* iff $\mathbb{E}$ is recursive in $\mathbb{I}$ and 2-env($\mathbb{I}$) and 2-sc($\mathbb{I}$) are the analogous classes of type-2 relations R. The criterion for X to be a Spector class over ${}^\omega\omega$ includes the conditions that X be closed under $\exists^1$ and $\forall^1$, so Corollary VII.2.11 implies that in general 1-env($\mathbb{I}$) is *not* a Spector class. It is, however, a *semi-Spector class*, defined by replacing closure under $\exists^1$ by the weaker closure under *deterministic* $\exists^1$: if P, $\mathsf{Q} \in X$, $\mathsf{P} \cap \mathsf{Q} = \varnothing$ and

$$\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \forall \beta\, [\mathsf{P}(\mathbf{m}, \boldsymbol{\alpha}, \beta) \vee \mathsf{Q}(\mathbf{m}, \boldsymbol{\alpha}, \beta)] \wedge \exists \beta\, \mathsf{P}(\mathbf{m}, \boldsymbol{\alpha}, \beta),$$

then also $\mathsf{R} \in X$. Moschovakis [1974a] proves

(17) for any normal type-3 functional $\mathbb{I}$, 2-env($\mathbb{I}$) is the smallest semi-Spector class over ${}^{\omega}\omega$ closed under $\mathbb{I}$-application,

and Kechris [1973a] shows

(18) a semi-Spector class X is the 2-envelope of some normal type-3 functional iff X is not Mahlo.

The extensions to types 4 and higher are now straightforward.

Sacks [1974] and [1978] has established parallel characterizations of the k-section of a normal type-n functional. For $k = 1$ and $n = 2$ this goes as follows. Let Code be the smallest class of subsets of ω such that $\{0\} \in$ Code and if for all n, $A_n \in$ Code, then also $\{\langle m, n\rangle : m \in A_n\} \in$ Code. To each $A \in$ Code, assign a set by:

$$\text{set}(\{0\}) = \varnothing, \quad \text{and for} \quad A \neq \{0\},$$

$$\text{set}(A) = \{\text{set}(\{m : \langle m, n\rangle \in A\}) : n \in \omega\}.$$

The hereditarily countable sets are exactly those of the form set(A) for some $A \in$ Code. Let As_1 (abstract 1-section) denote the theory in the language $\mathscr{L}_{\mathbf{ZF}}$ generated by the following axioms: Pair, Union, Δ_0-Separation (Definition VIII.7.3) and

(*Local Countability*) $\forall x.\ x$ is countable;

(Δ_0-DC) $\forall x\, \exists y\, \mathfrak{A} \rightarrow \exists f\, [f$ is a function with domain ω and $\forall n\, \mathfrak{A}(f(n)/x, f(n+1)/y)]$

for all Δ_0 formulas $\mathfrak{A}$.

Sacks then proves by a forcing argument:

(19) a countable class X of relations over ω closed under "recursive in" is the 1-section of some normal type-2 functional iff the structure $(\{\text{set}(A) : A \in X\}, \in)$ is a model of As_1.

In particular, this implies that all of the following classes of relations are 1-sections of normal type-2 functionals: Δ^1_r $(r \geqslant 1)$, $\{R : R \in \mathrm{L}_\kappa\}$ for any countable admissible κ, and the 1-section of any normal functional of type $\geqslant 2$. The extension of this last fact to higher types yields the following result known as the *Plus-One Theorem*:

(20) for any $k > 0$, the k-section of any normal functional of type $\geqslant (k+1)$ is also the k-section of some normal type-$(k+1)$ functional.

Moschovakis [1974a] shows that there is no corresponding result for envelopes — indeed:

(21) for any normal type-3 functional $\mathbb{I}$, 1-env($\mathbb{I}$) is not the 1-envelope of any normal type-2 functional.

The same technique shows that the classes Σ_r^1 $(r \geqslant 1)$ and Π_r^1 $(r \geqslant 2)$ are not 1-envelopes of normal type-2 functionals. However, Harrington [1973] proves the following *Plus-Two Theorem*:

(22) for any $k > 0$, the k-envelope of any normal functional of type $\geqslant (k+2)$ is also the k-envelope of some normal type-$(k+2)$ functional.

Before we leave the subject of recursion in higher types, we should mention the elegant Kechris–Moschovakis [1977]. Here is given an exposition of the basic notions and facts of this theory in the context of the general theory of inductive definability. Many of the difficult results are derived from simpler, more general results about inductive definitions.

One of the most surprising developments in recent years is that a large part of the theory developed for the type-structure over ω can be extended to the type-structure over an arbitrary set. The two basic tools for the development of this theory are inductive definability and admissible sets (with urelements). Indeed, many parts of the theory can be viewed from either point of view. The basic references here are Moschovakis [1974] and Barwise [1975]. The theory is quite complex by now and we can give here only a few examples of its scope.

Let $\mathfrak{M} = (M, R_0, \ldots, R_{n-1})$ be a relational structure for some first-order language $\mathscr{L}$. Add to $\mathscr{L}$ names for the elements of M and second-order variables $U, V, W, \ldots$ ranging over relations. A formula $\mathfrak{A}$ is called ***first-order positive*** iff all occurrences of second-order variables are free and positive. If $\mathfrak{A}$ is first-order positive with $x_0, \ldots, x_{k-1}$, U free, then $\mathfrak{A}$ defines a monotone operator $\Gamma_{\mathfrak{A}}$ over ${}^kM$: for $R \subseteq {}^kM$,

$$\Gamma_{\mathfrak{A}}(R) = \{\mathbf{a} : \mathfrak{M} \models \mathfrak{A}[\mathbf{a}, R]\}.$$

(Monotonicity follows from the fact that U occurs only positively.) A relation

$S \subseteq {}^kM$ is $\mathfrak{M}$-*positive-Inductive* ($\mathfrak{M}$-pos-Ind) iff for some operator $\Gamma_{\mathfrak{A}}$ over ${}^{k+l}M$ and some $\mathbf{b} \in {}^lM$, for all $\mathbf{a} \in {}^kM$,

$$S(\mathbf{a}) \leftrightarrow (\mathbf{a}, \mathbf{b}) \in \bar{\Gamma}_{\mathfrak{A}}.$$

S is $\mathfrak{M}$-*positive-Hyperelementary* ($\mathfrak{M}$-pos-Hyp) iff both R and ${}^kM \sim R$ are $\mathfrak{M}$-pos-Ind. Much of the general theory is motivated by the examples of the two structures:

$$\mathfrak{N}_0 = (\omega, R_+, R_\cdot) = \text{standard (relational) structure for arithmetic;}$$

$$\mathfrak{N}_1 = (\omega \cup {}^\omega\omega, \omega, R_+, R_\cdot, \mathrm{Apl}) = \text{standard (relational) structure for analysis,}$$

where R_+ and $R_\cdot$ are the graphs of the functions $+$ and $\cdot$ and $\mathrm{Apl}(m, n, \alpha) \leftrightarrow \alpha(m) = n$. For these we have

$$\text{(23)} \qquad \mathfrak{N}_0\text{-pos-Ind} = \Pi^1_1 = 1\text{-env}(\mathsf{E});$$

$$\mathfrak{N}_1\text{-pos-Ind} = \bigcup\{2\text{-env}(\mathsf{E}^{\#}, \beta) : \beta \in {}^\omega\omega\}.$$

This second equality should be contrasted with (14); here arbitrary parameters from ${}^\omega\omega$ may be introduced in the reduction.

For any structure $\mathfrak{M}$ we call a class X of relations over M an $\mathfrak{M}$-*Spector-class* iff it contains the $\mathfrak{M}$-elementary (first-order definable over $\mathfrak{M}$) relations, is closed under $\cap$, $\cup$, $\exists^M$, $\forall^M$, and relational composition with $\mathfrak{M}$-elementary functions, is M-indexable (contains a universal relation) and has the pre-wellordering property. Moschovakis [1974] shows that for an arbitrary $\mathfrak{M}$, the class $\mathfrak{M}$-pos-Ind has all of these properties except perhaps M-indexability. Note that in the absence of a universal relation the pre-wellordering property may be established in the form of Exercise V.1.20(ii); the proof is along the lines of Exercise III.3.33. To secure indexability he restricts to the class of acceptable structures, which have the requisite coding ability. $\mathfrak{M}$ is *acceptable* iff there exist $N \subseteq M$, $\preccurlyeq\ \subseteq N \times N$, and a one–one (coding) function $\langle\cdot\rangle$ from the set of all finite sequences from M into M such that $\preccurlyeq$ orders N in type ω and N, $\preccurlyeq$, and the following relations and functions are all $\mathfrak{M}$-elementary:

$$\mathrm{Seq}(a) \leftrightarrow a \text{ belongs to the range of } \langle\cdot\rangle;$$

$$\begin{aligned} \mathrm{lh}(a) &= \dot{0}, \quad \text{if} \quad \neg\mathrm{Seq}(a);\\ &= \dot{n}, \quad \text{if} \quad a = \langle a_1, \ldots, a_n\rangle;\\ \mathrm{q}(a, \dot{m}) &= a_m, \quad \text{if} \quad a = \langle a_1, \ldots, a_n\rangle \quad \text{and} \quad 1 \leqslant m \leqslant n;\\ &= \dot{0}, \quad \text{otherwise;} \end{aligned}$$

where $\dot{n}$ denotes the n-th element of N in the ordering $\preccurlyeq$.

For an acceptable structure $\mathfrak{M}$ the proof that $\mathfrak{M}$-pos-Ind is M-indexable proceeds by showing that this is exactly the class of relations definable over $\mathfrak{M}$ by a *game formula*:

$$\forall b_0 \exists b_1 \forall b_2 \exists b_3 \cdots \bigvee_{n \in \omega} \mathfrak{A}(\langle b_0, \ldots, b_{2n-1} \rangle, \mathbf{a})$$

with $\mathfrak{A}$ elementary. This fact is a generalization of Exercise III.3.22. Then Moschovakis [1974] shows:

(24) for any acceptable structure $\mathfrak{M}$, $\mathfrak{M}$-pos-Ind is the smallest $\mathfrak{M}$-Spector class.

One of the most interesting results of the theory is a very broad generalization of (23). A relation $S \subseteq {}^kM$ is called $\mathfrak{M} - \Pi_1^1$ iff it is definable over $\mathfrak{M}$ by a second-order formula with parameters in which the second-order variables are quantified only universally. Then Moschovakis [1974, 8A.1] proves

(25) for any countable acceptable structure $\mathfrak{M}$,

$$\mathfrak{M}\text{-pos-Ind} = \mathfrak{M} - \Pi_1^1.$$

Note that the assumption of countability is really necessary here since $\mathfrak{N}_1$-pos-Ind is a proper subset of $\mathbf{\Pi}_1^2 = \mathfrak{N}_1 - \Pi_1^1$ (cf. discussion following VII.2.11). Barwise [1975, VI.5.2] shows that the hypothesis of acceptability in (25) (and elsewhere) may be replaced by the weaker requirement that M admit an M-pos-Ind pairing function.

We consider now the approach via admissible sets. For the moment we restrict attention to transitive $\in$-structures — that is, structures of the form $\mathfrak{M} = (M, \in, R_0, \ldots, R_{n-1})$ where M is a transitive set and $\in$ is the restriction of the membership relation to M. Below we mention how this restriction may be removed. For such $\mathfrak{M}$, set

$$\mathrm{HYP}(\mathfrak{M}) = \bigcap \{N : N \text{ is an admissible set and } M, R_0, \ldots, R_{n-1} \in N\}.$$

It turns out that $\mathrm{HYP}(\mathfrak{M})$ is itself admissible and is thus the smallest admissible set containing $M, R_0, \ldots, R_{n-1}$. Then Barwise [1975, IV.3.1, 3 and VI.5.1] shows

(26) for any countable transitive $\in$-structure $\mathfrak{M}$ and any $S \subseteq {}^kM$, S is $\mathfrak{M} - \Pi_1^1$ iff S is Σ definable over $(\mathrm{HYP}(\mathfrak{M}), \in)$;

(27) for any transitive $\in$-structure $\mathfrak{M}$ which admits an $\mathfrak{M}$-pos-Ind pairing function and any $S \subseteq {}^kM$, S is $\mathfrak{M}$-pos-Ind iff S is Σ definable over $(\mathrm{HYP}(\mathfrak{M}), \in)$.

Note that in (27) since $M \in \mathrm{HYP}(\mathfrak{M})$, it follows by Δ-Separation that S is $\mathfrak{M}$-pos-Hyp iff $S \in \mathrm{HYP}(\mathfrak{M})$. In view of Barwise [1975, VI.5.8], (27) provides an alternative method for proving that $\mathfrak{M}$-pos-Ind is a Spector class.

Applying (26) to the case $\mathfrak{M} = (\mathrm{L}_\kappa, \in)$, we obtain the following pretty generalization of Theorem VIII.3.4:

(28) for any countable admissible ordinal κ, and any $R \subseteq {}^k\kappa$,
(i) R is κ^+-semi-recursive iff R is $\kappa - \Pi^1_1$;
(ii) R is κ^+-recursive iff R is $\kappa - \Delta^1_1$;

where κ^+ is the least admissible ordinal greater than κ and R is $\kappa - \Pi^1_1$ iff for some κ-recursive relation $\mathsf{P} \subseteq {}^{k,1}\kappa$ (cf. VIII.2.26),

$$R(\boldsymbol{\mu}) \leftrightarrow (\forall f : \kappa \to \kappa)(\exists \pi < \kappa)\, \mathsf{P}(\pi, \boldsymbol{\mu}, f).$$

The characterization (27) of a particular $\mathfrak{M}$-Spector class in terms of definability over an admissible structure can be extended to give a representation for arbitrary $\mathfrak{M}$-Spector classes via the notion of a companion. For any $\mathfrak{M}$-Spector class X, let

$$\mathrm{Cmp}(X) = \bigcap \{N : N \text{ is an admissible set and } \Delta(X) \subseteq N\}.$$

$\mathrm{Cmp}(X)$ is called the *Companion Set* of X. By way of abbreviation, for any $C \subseteq \mathrm{Cmp}(X)$, let us write $\Sigma[C]$ ($\Delta[C]$) for the class of all relations on $\mathrm{Cmp}(X)$ which are Σ-definable (Δ-definable) over the structure $(\mathrm{Cmp}(X), \in, C)$. A structure of this type is called admissible iff it satisfies the axioms and schemata Pair, Union, Δ_0-Separation, and Δ_0-Collection formulated in the appropriate language, $\mathscr{L}_{\mathrm{ZF}}$ with an additional unary relation symbol. The *Companion Theorem*, Moschovakis [1974, 9E.1] asserts

(29) for any $\mathfrak{M}$-Spector class there exists a set $C_X \subseteq \mathrm{Cmp}(X)$ such that
(i) $(\mathrm{Cmp}(X), \in, C_X)$ is admissible;
(ii) $\mathrm{Cmp}(X)$ is C_X-projectible on M (that is, there exists a $\Delta[C_X]$ function f which maps a subset of M onto $\mathrm{Cmp}(X)$);
(iii) $\mathrm{Cmp}(X)$ is C_X-resolvable (that is, there exists a $\Delta[C_X]$ function $g : o(\mathrm{Cmp}(X)) \to \mathrm{Cmp}(X)$ such that $\mathrm{Cmp}(X) = \bigcup\{g(\sigma) : \sigma < o(\mathrm{Cmp}(X))\}$).
(iv) for any $S \subseteq {}^kM$, $S \in X$ iff $S \in \Sigma[C_X]$.

Furthermore, if C'_X is any other set satisfying (i)–(iv), then $\Sigma[C_X] = \Sigma[C'_X]$.

The class $\Sigma[C_X]$ is called the *Companion Class* of X and the pair $(\mathrm{Cmp}(X), \Sigma[C_X])$ is the *Companion* of X. Result (27) may now be restated as

(30) for any transitive $\in$-structure $\mathfrak{M}$ which admits an $\mathfrak{M}$-pos-Ind pairing function, the Companion of $\mathfrak{M}$-pos-Ind is $(\mathrm{HYP}(\mathfrak{M}), \Sigma[\varnothing])$.

Moschovakis [1974] and [1974b] computes the companions of many other Spector classes. We shall discuss here just one further class of examples which generalizes (6). A structure $\mathfrak{M}$ is *almost acceptable* iff there exist relations and functions N, $\preccurlyeq$, Seq, lh, and q as before which are $\mathfrak{M}$-pos-Hyp (instead of $\mathfrak{M}$-elementary). Let $\mathscr{L}^*$ be the first-order language for the expanded structure $\mathfrak{M}^* = (M, R_0, \ldots, R_{n-1}, \preccurlyeq, \mathrm{Seq}, \mathrm{lh}, \mathrm{q})$ together with names for the elements of M and second-order variables $U, V, W, \ldots$. The classes of $\exists_r^0$ and $\forall_r^0$ formulas of $\mathscr{L}^*$ are defined as in III.5.3 with $\preccurlyeq$ playing the role of $\leqslant$ for bounded quantifiers. If $\mathfrak{A}$ is a formula of $\mathscr{L}^*$ with $x_0, \ldots, x_{k-1}$, U free, then $\mathfrak{A}$ defines an inductive operator $\Gamma_{\mathfrak{A}}$ over ${}^kM$ by: for $R \subseteq {}^kM$,

$$\Gamma_{\mathfrak{A}}(R) = R \cup \{\mathbf{a} : \mathfrak{M}^* \models \mathfrak{A}[\mathbf{a}, R]\}.$$

Of course, $\Gamma_{\mathfrak{A}}$ is in general no longer monotone. A relation $S \subseteq {}^kM$ is called $\mathfrak{M}$-Π_r^0-*Inductive* ($\mathfrak{M}$-Σ_r^0-*Inductive*) iff for some $\forall_r^0$ ($\exists_r^0$) formula $\mathfrak{A}$ and some $\mathbf{b} \in {}^lM$, for all $\mathbf{a} \in {}^kM$,

$$S(\mathbf{a}) \leftrightarrow (\mathbf{a}, \mathbf{b}) \in \bar{\Gamma}_{\mathfrak{A}}.$$

Theorem 11 of Moschovakis [1974b] verifies that this is independent of the particular N, $\preccurlyeq$, Seq, lh, and q chosen. He further shows:

(31) for any almost acceptable structure $\mathfrak{M}$,
(i) $\mathfrak{M}$-Σ_2^0-Ind $=$ $\mathfrak{M}$-pos-Ind;
(ii) for all $r \geqslant 2$, $\mathfrak{M}$-Π_r^0-Ind $=$ $\mathfrak{M}$-Σ_{r+1}^0-Ind and is an $\mathfrak{M}$-Spector class.

In fact, $\mathfrak{M}$-Π_r^0-Ind is chatacterized as the smallest $\mathfrak{M}$-Spector class satisfying either one of two natural additional conditions. Note that the restriction to $r \geqslant 2$ is again necessary because by (13),

$$\mathfrak{N}_1\text{-}\Pi_1^0\text{-Ind} = \mathbf{\Pi}_1^1\text{-Ind} = \bigcup\{2\text{-env}(\mathbb{E}, \beta) : \beta \in {}^\omega\omega\}$$

is not a Spector class by VII.2.11. However by Moschovakis [1974b, Theorem 29]

(32) for any acceptable structure $\mathfrak{M}$, $\mathfrak{M}$-Π_1^0-Ind is the smallest $\mathfrak{M}$-semi-Spector class.

As for companions, Moschovakis [1974b, Theorem 23] shows

(33) for any almost acceptable transitive $\in$-structure $\mathfrak{M}$ and any $r \geq 2$, the companion of $\mathfrak{M}$-Π^0_r-Ind is $(\mathrm{HYP}(\mathfrak{M}, \Pi^0_r), \Sigma[\varnothing])$, where

$$\mathrm{HYP}(\mathfrak{M}, \Pi^0_r) = \bigcap\{N : N \text{ is a } \forall^0_{r+1}\text{-reflecting admissible set and } M, R_0, \ldots R_{n-1} \in N\}.$$

Similar but more complicated characterizations are given for $\mathfrak{M}$-Σ^m_r-Ind and $\mathfrak{M}$-Π^m_r-Ind for all m, $r \geq 1$.

At any given level of definability the class of monotone operators is intermediate between the class of positive operators and the class of arbitrary inductive operators. In many cases there are strictly more relations definable by arbitrary operators than by positive operators. For arithmetical operators over ω the relations definable by positive or by monotone operators coincide; a general version of this follows easily from (25):

(34) for any countable acceptable structure $\mathfrak{M}$, $\mathfrak{M}$-pos-Ind $=$ $\mathfrak{M}$-mon-Ind.

On the other hand it was known from (7) (and earlier) that over ω, Σ^1_1-Ind $= \Sigma^1_1$-mon-Ind. Harrington–Kechris [1976] show that this situation is not at all infrequent. They call a class Z of inductive operators over an almost acceptable structure $\mathfrak{M}$ *adequate* if it satisfies certain mild definability conditions, which in particular are satisfied by all of the classes $\mathfrak{M}$-Σ^k_r and $\mathfrak{M}$-Π^k_r $(k + r \geq 1)$. We say *well-foundedness is cZ over* $\mathfrak{M}$ iff the second-order relation "S is not well-founded" belongs to Z. Then Harrington–Kechris prove

(35) for any almost acceptable structure $\mathfrak{M}$ and any adequate class Z, if well-foundedness is cZ over $\mathfrak{M}$ and

$$cZ \subseteq \mathfrak{M}\text{-}Z\text{mon-Ind},$$

then

$$\mathfrak{M}\text{-}Z\text{-mon-Ind} = \mathfrak{M}\text{-}Z\text{-Ind}.$$

In particular, it follows that if well-foundedness is Π^0_1 over $\mathfrak{M}$, as it is for $\mathfrak{N}_1$, then for all $r \geq 2$, $\mathfrak{M}$-Π^0_r-Ind $=$ $\mathfrak{M}$-Π^0_r-mon-Ind, and $\mathfrak{M}$-mon-Ind $=$ $\mathfrak{M}$-Ind. As another application, note that for any almost acceptable $\mathfrak{M}$, well-foundedness is Π^1_1 over $\mathfrak{M}$. Thus if $\mathfrak{M}$ is also countable, it follows from (24) and (25) that $\mathfrak{M}$-Σ^1_1-mon-Ind $=$ $\mathfrak{M}$-Σ^1_1-Ind. A similar result holds for uncountable almost acceptable $\mathfrak{M}$ if we replace 'Π^1_1' by '$\mathfrak{M}$-pos-Ind'.

To extend the methods involving admissible sets to arbitrary relational structures $\mathfrak{M}$, Barwise [1975] develops the theory of admissible sets with urelements. The elements of M are treated as initially given objects and a class V_M of sets over M is defined recursively by:

$$V_M(0) = \varnothing;$$

$$V_M(\sigma + 1) = \mathbf{P}(M \cup V_M(\sigma));$$

$$V_M(\sigma) = \bigcup\{V_M(\tau) : \tau < \sigma\} \text{ for limit } \sigma.$$

Let $\mathscr{L}^*$ be a first-order language which includes the language of $\mathfrak{M}$, unary relation symbols U ("is an urelement") and S ("is a set"), and the membership symbol $\in$. Any set $A \in V_M$ determines a structure

$$\mathbb{A}_{\mathfrak{M}} = (M \cup A, R_0, \ldots, R_{n-1}, M, A, \in)$$

for $\mathscr{L}^*$. Such an A is called *admissible over* $\mathfrak{M}$ iff $M \cup A$ is transitive and $\mathbb{A}_{\mathfrak{M}}$ satisfies the axioms and schemata Pair, Union, Δ_0-Separation, and Δ_0-Collection formulated in $\mathscr{L}^*$. If in addition the set M of urelements belongs to A, then A is called *admissible above* $\mathfrak{M}$. Much of the theory of admissible sets carries over to admissible sets with urelements. As an example, define

$$\mathrm{HYP}_{\mathfrak{M}} = \bigcap\{A : A \text{ is admissible above } \mathfrak{M}\}.$$

Then $\mathrm{HYP}_{\mathfrak{M}}$ is itself admissible above $\mathfrak{M}$ and the following extension of (27) holds:

(36) for any structure $\mathfrak{M}$ which admits an $\mathfrak{M}$-pos-Ind pairing function and any $S \subseteq {}^kM$, S is $\mathfrak{M}$-pos-Ind iff S is Σ-definable over $\mathrm{HYP}_{\mathfrak{M}}$.

Still another approach to the study of definability over arbitrary structures is through abstract and axiomatic computation theories. Moschovakis [1969] develops a theory of computations over a structure $\mathfrak{M}$ by working in a larger set M^* obtained by closing $M \cup \{0\}$ under a pairing function. Since M^* may not have a natural wellordering, he is forced to deal with multiple-valued functions. A relation $\{e\}(\mathbf{x}) \rightarrow y$ is defined inductively and the functions so indexed are called *$\mathfrak{M}$-search computable* because of the inclusion of the following clause in the inductive definition:

$$\text{if} \quad \{b\}(y, \mathbf{x}) \rightarrow 0, \quad \text{then} \quad \{\langle 9, k, b\rangle\}(\mathbf{x}) \rightarrow y.$$

Gordon [1970] shows that $\mathfrak{M}$-search computability coincides with Σ-definability over $\mathrm{HF}_{\mathfrak{M}}$, the structure over $\mathfrak{M}$ whose sets are the hereditarily finite sets in V_M. If a schema of the form

$$\text{if} \quad \forall y\, \exists u\,.\{b\}(y, \mathbf{x}) \rightarrow \mathrm{u} \quad \text{and} \quad \exists y\,.\{b\}(y, \mathbf{x}) \rightarrow 0,$$

$$\text{then} \quad \{\langle 10, k, b\rangle\}(\mathbf{x}) \rightarrow 0;$$

$$\text{if} \quad \forall y\, \exists u\, [u \neq 0 \wedge \{b\}(y, \mathbf{x}) \rightarrow u], \quad \text{then} \quad \{\langle 10, k, b\rangle\}(\mathbf{x}) \rightarrow 1;$$

is added to those for search computability, the resulting functions are called $\mathfrak{M}$-*search computable* in E. Combining the results of Moschovakis [1969] and [1974] and Barwise [1975]

(37) for any structure $\mathfrak{M}$ which admits an $\mathfrak{M}$-pos-Ind pairing function and any $S \subseteq {}^kM$, S is $\mathfrak{M}$-semi-search computable in E iff S is Σ-definable over $\mathrm{HYP}_{\mathfrak{M}}$ iff S is $\mathfrak{M}$-pos-Ind.

If the E-schema is left in but the search schema is removed, the resulting functions are called $\mathfrak{M}$-*prime computable in* E. For $\mathfrak{N}_1$ this is equivalent to recursion in $\mathbb{E}$ (as opposed to $\mathbb{E}^{\#}$ for search computability in E) and in general Grilliot [1971a] shows essentially

(38) for any acceptable structure $\mathfrak{M}$, $\mathfrak{M}$-Π_1^0-Ind is the class of relations $\mathfrak{M}$-semi-prime computable in E.

Axiomatic computation theories have been studied by several people. Wagner [1969] and Strong [1968] developed much of the elementary theory. Moschovakis [1971] introduced axioms concerning the (ordinal) length of a computation and Fenstad [1974] adds the relation "is a subcomputation of". There is a natural notion of a Spector computation theory such that a class of relations is a Spector class iff it is the envelope of a Spector theory. Many of the deepest results of recursion theory on higher types such as the Grilliot Selection Theorem and the Plus-One and Plus-Two Theorems have been generalized to the axiomatic setting. The current state of the theory is surveyed in Fenstad [1975] and treated in detail in Moldestad [1977] and Fenstad [1980?].