Part B

The Analytical and Projective Hierarchies

Most of the results of Chapter III have in common that their proofs rely heavily on a direct analysis of the number and kind of quantifiers needed to define certain concepts. With these methods we were able to establish the main structural features of the arithmetical hierarchy, but were much less successful with the corresponding questions concerning the analytical hierarchy. In the two chapters of Part B we shall introduce several new techniques — prewellordering, uniformization, and construction of transfinite hierarchies — which will provide answers for many of the questions left open in the preceding chapter. These by no means exhaust the tools which are useful in studying the analytical and projective hierarchies and we shall return to these questions from time to time in Part C. On the other hand, the results of §§ V.2–3 indicate that some questions about analytical and projective relations cannot be decided on the basis of the current axioms of set theory.

Chapter IV
The First Level

We shall study in this chapter the classes Σ_1^1, Π_1^1, and Δ_1^1 and their relativized and boldface counterparts. Not only are these classes more amenable to analysis than the higher levels of the analytical hierarchy, but they have played a much larger role in the development both of descriptive set theory and of generalized recursion theory.

Among the early objects of study in Descriptive Set Theory were the Borel sets and the projective classes $\mathbf{\Sigma}_r^1$ and $\mathbf{\Pi}_r^1$. Although the Borel sets were easily proved to have many pleasant properties, the operation of projection ($\exists^1$) used in defining the projective classes seemed too non-constructive to allow much to be proved about these sets. In 1915 Suslin discovered that the $\mathbf{\Sigma}_1^1$ (*analytic*) sets could all be constructed by use of the much more explicit operation $\mathscr{A}$ (cf. Exercise III.2.19) and that $\mathbf{\Delta}_1^1$ coincides with the class of Borel sets (cf. Theorem IV.3.3). These results stimulated much of the later development of the theory. In particular, the structure of $\mathbf{\Pi}_1^1$ and $\mathbf{\Delta}_1^1$ is reflected in many other pairs of classes of relations to be studied in this book.

From the side of recursion theory, the notational analogy of Δ_1^1 with Δ_1^0 together with the transitivity of the relation "$\alpha \in \Delta_1^1[\beta]$", leads to the conjecture that the Δ_1^1 relations are in some sense "generalized recursive". This is supported by the results of §2 below which show a great similarity between the properties of Σ_1^0 and Π_1^1 (not Σ_1^1!). One such similarity has already been noted: for any monotone operator Γ, if $\Gamma \in \Sigma_1^0$ (Π_1^1), then also $\bar{\Gamma} \in \Sigma_1^0$ (Π_1^1). The conjecture will be verified in § VI.2, where we show that the (Π_1^1) Δ_1^1 relations are exactly those (semi-) recursive in the type-2 functional E, and in § VIII.3 where we see that the (Π_1^1) Δ_1^1 relations on numbers are exactly those which are ω_1-(semi-) recursive.

1. Π_1^1 *and Well-Orderings*

The basic tool in the study of Π_1^1 relations is a characterization of this class in terms of W, the set of well-ordering functions. Recall that for a total functional F of rank $(k+1, l)$, $\mathsf{F}[\mathbf{m}, \boldsymbol{\alpha}]$ denotes the function $\lambda p . \mathsf{F}(p, \mathbf{m}, \boldsymbol{\alpha})$ so that F serves also as a functional from ${}^{k,l}\omega$ into ${}^{\omega}\omega$. A relation R is called (*many-one*) *reducible* to a set A ($\mathsf{R} \leqslant \mathsf{A}$) iff for some recursive functional F,

$$\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \mathsf{F}[\mathbf{m}, \boldsymbol{\alpha}] \in \mathsf{A}.$$

We also say R is *reduced to* A *via* F. Note that by the Analytical Substitution Theorem (III.2.12), if $\mathsf{A} \in \Sigma^1_r$ (Π^1_r) and $\mathsf{R} \leqslant \mathsf{A}$, then also $\mathsf{R} \in \Sigma^1_r$ (Π^1_r).

1.1 Theorem. *For all* R, $\mathsf{R} \in \Pi^1_1 \leftrightarrow \mathsf{R} \leqslant \mathsf{W}$.

Proof. $\mathsf{W} \in \Pi^1_1$ by the examples of III.2.3, so if $\mathsf{R} \leqslant \mathsf{W}$ also $\mathsf{R} \in \Pi^1_1$. Suppose now that $\mathsf{R} \in \Pi^1_1$. Then there exists a recursive relation P such that

$$\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \forall \beta\, \exists n\, \mathsf{P}(\bar{\beta}(n), \mathbf{m}, \boldsymbol{\alpha}).$$

We shall associate with each $(\mathbf{m}, \boldsymbol{\alpha})$ a linear ordering $\trianglelefteq^{\mathsf{P}}_{\mathbf{m},\boldsymbol{\alpha}}$ such that

(1) $\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \trianglelefteq^{\mathsf{P}}_{\mathbf{m},\boldsymbol{\alpha}}$ is a well-ordering;

(2) the functional F defined by

$$\mathsf{F}(\langle s, t\rangle, \mathbf{m}, \boldsymbol{\alpha}) = \begin{cases} 0, & \text{if } s \trianglelefteq^{\mathsf{P}}_{\mathbf{m},\boldsymbol{\alpha}} t; \\ 1, & \text{otherwise} \end{cases}$$

is recursive.

Then R is reduced to W via F.

First, let $\trianglelefteq$ be the recursive linear ordering defined by:

$$s \trianglelefteq t \leftrightarrow s, t \in \mathrm{Sq} \wedge (t \subseteq s \vee (\exists n < \lg(s))[n < \lg(t) \wedge (\forall i < n)((s)_i = (t)_i) \wedge (s)_n < (t)_n]).$$

The ordering $\trianglelefteq^{\mathsf{P}}_{\mathbf{m},\boldsymbol{\alpha}}$ is a restriction of $\trianglelefteq$. Let

$$\mathsf{P}'(s, \mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow (\exists u \leqslant s)[u \in \mathrm{Sq} \wedge u \subseteq s \wedge \mathsf{P}(u, \mathbf{m}, \boldsymbol{\alpha})].$$

Then also P' is recursive and

$$\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \forall \beta\, \exists n\, \mathsf{P}'(\bar{\beta}(n), \mathbf{m}, \boldsymbol{\alpha}).$$

We set

$$s \trianglelefteq^{\mathsf{P}}_{\mathbf{m},\boldsymbol{\alpha}} t \leftrightarrow s \trianglelefteq t \wedge \sim\mathsf{P}'(s, \mathbf{m}, \boldsymbol{\alpha}) \wedge \sim\mathsf{P}'(t, \mathbf{m}, \boldsymbol{\alpha}).$$

Claim (2) is obvious. For the implication ($\leftarrow$) of (1), suppose $\sim\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha})$ so that for some β, $\forall n \sim \mathsf{P}'(\bar{\beta}(n), \mathbf{m}, \boldsymbol{\alpha})$. Then for all n, $\bar{\beta}(n+1) \trianglelefteq^{\mathsf{P}}_{\mathbf{m},\boldsymbol{\alpha}} \bar{\beta}(n)$ and by condition (4′) of I.1.6, $\trianglelefteq^{\mathsf{P}}_{\mathbf{m},\boldsymbol{\alpha}}$ is not a well-ordering.

Suppose now that $\trianglelefteq^{\mathsf{P}}_{\mathbf{m},\boldsymbol{\alpha}}$ is not a well-ordering and let $A = \{s : s \in \mathrm{Sq} \wedge \sim\mathsf{P}'(s, \mathbf{m}, \boldsymbol{\alpha})\}$. Then $A = \mathrm{Fld}(\trianglelefteq^{\mathsf{P}}_{\mathbf{m},\boldsymbol{\alpha}})$ and it suffices to show $\exists \beta\, \forall n . \bar{\beta}(n) \in A$. Note that by the definition of P', for any s and t,

$$(3) \qquad s \in A \wedge t \subseteq s \rightarrow t \in A.$$

By assumption there exists a non-empty set $B \subseteq A$ with no $\trianglelefteq^{\mathsf{P}}_{\mathbf{m},\boldsymbol{\alpha}}$-least element — that is, $(\forall s \in B)(\exists t \in B)\; t \vartriangleleft^{\mathsf{P}}_{\mathbf{m},\boldsymbol{\alpha}} s$. We may assume that (3) holds also with B in place of A. Let f be a partial function defined by

$$f(n) \simeq \text{least}\, p((\exists s \in B)[\bar{f}(n) \subsetneqq s \wedge (s)_n = p]).$$

Clearly either f is total or for some $\bar{n}$, $f(n)$ is defined exactly for $n < \bar{n}$. In the first case we have that for every n, $\bar{f}(n) \subsetneqq s$ for some $s \in B \subseteq A$. Hence by (3), $\forall n . \bar{f}(n) \in A$ and we have the desired conclusion: $\exists \beta\, \forall n . \bar{\beta}(n) \in A$. Suppose, on the other hand, that $f(n)$ is defined exactly for $n < \bar{n}$. As B has no $\trianglelefteq^{\mathsf{P}}_{\mathbf{m},\boldsymbol{\alpha}}$-least element, there is some $s \in B$ with $s \vartriangleleft^{\mathsf{P}}_{\mathbf{m},\boldsymbol{\alpha}} \bar{f}(\bar{n})$. If $\bar{f}(\bar{n}) \subsetneqq s$, then $f(\bar{n})$ would be defined, contrary to assumption. Hence for some $n < \bar{n}$,

$$(\forall i < n)((s)_i = (\bar{f}(\bar{n}))_i = f(i)) \wedge (s)_n < (\bar{f}(\bar{n}))_n = f(n).$$

But then $\bar{f}(n) \subsetneqq s$ so by the definition of f, $f(n) \leqslant (s)_n$, a contradiction. Thus this case does not arise. □

1.2 Corollary. $\mathsf{W} \notin \Sigma_1^1$.

Proof. If $\mathsf{W} \in \Sigma_1^1$, then by the Theorem every Π_1^1 relation is also Σ_1^1, contrary to the Analytical Hierarchy Theorem. □

From Theorem 1.1 we can already begin to see why Π_1^1 rather than Σ_1^1 plays the role of the class of generalized semi-recursive relations. If P is a recursive relation of rank $(k+1, l)$, let

$$|\mathbf{m}, \boldsymbol{\alpha}|_0^{\mathsf{P}} = \begin{cases} \text{least}\, p . \mathsf{P}(p, \mathbf{m}, \boldsymbol{\alpha}), & \text{if any;} \\ \omega, & \text{otherwise;} \end{cases}$$

$$|\mathbf{m}, \boldsymbol{\alpha}|_1^{\mathsf{P}} = \begin{cases} \text{order-type of } \trianglelefteq^{\mathsf{P}}_{\mathbf{m},\boldsymbol{\alpha}}, & \text{if this is a well-ordering;} \\ \aleph_1, & \text{otherwise.} \end{cases}$$

Then

$$\exists p\, \mathsf{P}(p, \mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow |\mathbf{m}, \boldsymbol{\alpha}|_0^{\mathsf{P}} < \omega$$

and

$$\forall \beta\, \exists p\, \mathsf{P}(\bar{\beta}(p), \mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow |\mathbf{m}, \boldsymbol{\alpha}|_1^{\mathsf{P}} < \aleph_1.$$

In these terms, the proof that the class of semi-recursive relations has the reduction property (II.4.17) proceeds by defining

$$\mathsf{R}^*(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \wedge |\mathbf{m}, \boldsymbol{\alpha}|_0^{\mathsf{P}} \leqslant |\mathbf{m}, \boldsymbol{\alpha}|_0^{\mathsf{Q}}, \quad \text{and}$$
$$\mathsf{S}^*(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \mathsf{S}(\mathbf{m}, \boldsymbol{\alpha}) \wedge |\mathbf{m}, \boldsymbol{\alpha}|_0^{\mathsf{Q}} < |\mathbf{m}, \boldsymbol{\alpha}|_0^{\mathsf{P}}.$$

To establish similarly the reduction property for Π_1^1, we need only evaluate the complexity of the relation $|\mathbf{m}, \boldsymbol{\alpha}|_1^{\mathsf{P}} \leqslant |\mathbf{m}, \boldsymbol{\alpha}|_1^{\mathsf{Q}}$.

We recall from Examples III.2.3 the Σ_1^1 relation $\preccurlyeq$ defined by:

$$\begin{aligned}\gamma \preccurlyeq \delta \leftrightarrow\ & \leqslant_\gamma \text{ and } \leqslant_\delta \text{ are linear orderings and } \leqslant_\gamma \text{ is isomorphic} \\ & \quad \text{to a subordering of } \leqslant_\delta \\ \leftrightarrow\ & \leqslant_\gamma \text{ and } \leqslant_\delta \text{ are linear orderings and} \\ & \quad \exists \alpha\, \forall p\, \forall q\, [\alpha \text{ is 1–1 on } \mathsf{Fld}(\gamma) \wedge (p \leqslant_\gamma q \rightarrow \alpha(p) \leqslant_\delta \alpha(q))].\end{aligned}$$

We define also a Σ_1^1 relation $\prec$ by

$$\begin{aligned}\gamma \prec \delta \leftrightarrow\ & \leqslant_\gamma \text{ and } \leqslant_\delta \text{ are linear orderings and } \leqslant_\gamma \text{ is isomorphic} \\ & \quad \text{to a subordering of a proper initial segment of } \leqslant_\delta \\ \leftrightarrow\ & \leqslant_\gamma \text{ and } \leqslant_\delta \text{ are linear orderings and} \\ & \quad \exists \alpha\, \exists r\, \forall p\, \forall q\, [\alpha \text{ is 1–1 on} \\ & \qquad \mathsf{Fld}(\gamma) \wedge (p \leqslant_\gamma q \rightarrow \alpha(p) \leqslant_\delta \alpha(q) <_\delta r)].\end{aligned}$$

If both γ and δ belong to W, then clearly

$$\gamma \preccurlyeq \delta \leftrightarrow \|\gamma\| \leqslant \|\delta\| \quad \text{and} \quad \gamma \prec \delta \leftrightarrow \|\gamma\| < \|\delta\|.$$

We shall need, however, relations which have this property whenever at least *one* of γ and δ belongs to W.

1.3 Definition. For all γ and δ,

(i) $\gamma \preccurlyeq_\Sigma \delta \leftrightarrow \gamma \preccurlyeq \delta \vee \delta \notin \mathsf{W}$;

(ii) $\gamma \prec_\Sigma \delta \leftrightarrow \gamma \prec \delta \vee \delta \notin \mathsf{W}$;

(iii) $\gamma \preccurlyeq_\Pi \delta \leftrightarrow \neg(\delta \prec_\Sigma \gamma)$;

(iv) $\gamma \prec_\Pi \delta \leftrightarrow \neg(\delta \preccurlyeq_\Sigma \gamma)$.

Note that $\preccurlyeq_\Sigma$ and $<_\Sigma$ are Σ_1^1 relations, whereas $\preccurlyeq_\Pi$ and $<_\Pi$ are Π_1^1 relations. None is Δ_1^1, but part of the import of the following theorem is that each is "Δ_1^1 on W".

1.4 Theorem. *For all* γ *and* δ, *if either* $\gamma \in \mathsf{W}$ *or* $\delta \in \mathsf{W}$, *then*
(i) $\gamma \preccurlyeq_\Sigma \delta \leftrightarrow [\gamma \in \mathsf{W} \wedge \|\gamma\| \leq \|\delta\|] \leftrightarrow \gamma \preccurlyeq_\Pi \delta$;
(ii) $\gamma <_\Sigma \delta \leftrightarrow [\gamma \in \mathsf{W} \wedge \|\gamma\| < \|\delta\|] \leftrightarrow \gamma <_\Pi \delta$.

Proof. Suppose first that $\gamma \preccurlyeq_\Sigma \delta$. If $\gamma \in \mathsf{W}$ then either $\delta \notin \mathsf{W}$, so $\|\gamma\| < \aleph_1 = \|\delta\|$, or $\delta \in \mathsf{W}$ and $\gamma \preccurlyeq \delta$, so $\|\gamma\| \leq \|\delta\| < \aleph_1$. If $\delta \in \mathsf{W}$, then $\gamma \preccurlyeq \delta$, so as any subordering of a well-ordering is also a well-ordering, also $\gamma \in \mathsf{W}$ and $\|\gamma\| \leq \|\delta\|$.

Suppose now that the middle condition of (i) holds: $\gamma \in \mathsf{W}$ and $\|\gamma\| \leq \|\delta\|$. Then either $\delta \notin \mathsf{W}$, or $\delta \in \mathsf{W}$ and $\preccurlyeq_\delta$ is a well-ordering of type at least $\|\gamma\|$. In either case $\neg\, \delta < \gamma$, which together with $\gamma \in \mathsf{W}$ implies $\gamma \preccurlyeq_\Pi \delta$.

Next suppose that $\gamma \preccurlyeq_\Pi \delta$ — i.e., $\neg(\delta < \gamma)$ and $\gamma \in \mathsf{W}$. If $\delta \notin \mathsf{W}$, then $\gamma \preccurlyeq_\Sigma \delta$. If $\delta \in \mathsf{W}$ then by the comparability of well-orderings, $\gamma \preccurlyeq \delta$ and again $\gamma \preccurlyeq_\Sigma \delta$.

The implications (ii) are proved similarly. □

1.5 Theorem. (i) Π_1^1 *has the reduction property but not the separation property*;
(ii) Σ_1^1 *has the separation property but not the reduction property.*

Proof. By Lemmas II.4.19 and II.4.21 it suffices to show that Π_1^1 has the reduction property. Let R and S be any two Π_1^1 relations of the same rank. By Theorem 1.1 there exist recursive functionals F and G such that

$$\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \mathsf{F}[\mathbf{m}, \boldsymbol{\alpha}] \in \mathsf{W} \quad \text{and} \quad \mathsf{S}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \mathsf{G}[\mathbf{m}, \boldsymbol{\alpha}] \in \mathsf{W}.$$

We set

$$\mathsf{R}^*(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \wedge \mathsf{F}[\mathbf{m}, \boldsymbol{\alpha}] \preccurlyeq_\Pi \mathsf{G}[\mathbf{m}, \boldsymbol{\alpha}];$$

$$\mathsf{S}^*(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \mathsf{S}(\mathbf{m}, \boldsymbol{\alpha}) \wedge \mathsf{G}[\mathbf{m}, \boldsymbol{\alpha}] <_\Pi \mathsf{F}[\mathbf{m}, \boldsymbol{\alpha}].$$

Observe that if $(\mathbf{m}, \boldsymbol{\alpha}) \in \mathsf{R} \cup \mathsf{S}$, then at least one of $\mathsf{F}[\mathbf{m}, \boldsymbol{\alpha}]$ and $\mathsf{G}[\mathbf{m}, \boldsymbol{\alpha}]$ belongs to W so that either $\mathsf{F}[\mathbf{m}, \boldsymbol{\alpha}] \preccurlyeq_\Pi \mathsf{G}[\mathbf{m}, \boldsymbol{\alpha}]$ or $\mathsf{G}[\mathbf{m}, \boldsymbol{\alpha}] <_\Pi \mathsf{F}[\mathbf{m}, \boldsymbol{\alpha}]$. With this it is straightforward to verify that $(\mathsf{R}^*, \mathsf{S}^*)$ reduces (R, S). □

In the proof of Theorem 1.1, the relation $\trianglelefteq^P_{\mathbf{m},\boldsymbol{\alpha}}$ is a well-ordering recursive in $\boldsymbol{\alpha}$. In particular, if P is a recursive relation and ω_1 denotes the least non-recursive ordinal (Definition III.3.11), then $|\mathbf{m}|_1^P$ is either a recursive ordinal or $\aleph_1$ so that

$$\forall \beta\, \exists p\, P(\bar{\beta}(p), \mathbf{m}) \leftrightarrow |\mathbf{m}|_1^P < \omega_1.$$

Hence if for any ordinal σ we set

$$\mathsf{W}_\sigma = \{\gamma : \gamma \in \mathsf{W} \wedge \|\gamma\| < \sigma\},$$

then in the notation of Theorem 1.1,

$$R(\mathbf{m}) \leftrightarrow \forall\beta\, \exists p\, P(\bar{\beta}(p), \mathbf{m}) \leftrightarrow \mathsf{F}[\mathbf{m}] \in \mathsf{W}_{\omega_1}.$$

Thus we have proved the implication ($\rightarrow$) of

1.6 Theorem. *For all* $R \subseteq {}^k\omega$, $R \in \Pi^1_1 \leftrightarrow R \leqslant \mathsf{W}_{\omega_1}$.

Proof. It suffices to show $\mathsf{W}_{\omega_1} \in \Pi^1_1$. This follows from the equivalences:

$$\begin{aligned} \gamma \in \mathsf{W}_{\omega_1} &\leftrightarrow \exists\delta\, [\delta \in \mathsf{W} \wedge \delta \text{ is recursive} \wedge \|\gamma\| \leqslant \|\delta\|] \\ &\leftrightarrow \exists c\, [\{c\} \text{ is a total function of rank } 1 \wedge \\ &\qquad \{c\} \in \mathsf{W} \wedge \gamma \leqslant_\Pi \{c\}]. \quad \square \end{aligned}$$

The same proof shows that every Π^1_1 relation on numbers is reducible to the denumerable set $\mathsf{W}_{\text{rec}} = \{\gamma : \gamma \in \mathsf{W} \wedge \gamma \text{ is recursive}\}$. It will, however, be more convenient to use the set of indices of recursive well-orderings.

1.7 Definition. $W = \{c : \{c\} \text{ is a total function of rank } 1 \wedge \{c\} \in \mathsf{W}\}$. For $c \in W$, $\|c\| = \|\{c\}\|$.

1.8 Theorem. *For all* $R \subseteq {}^k\omega$, $R \in \Pi^1_1 \leftrightarrow R \leqslant W$.

Proof. That W and hence all R reducible to it are Π^1_1 is immediate from the Analytical Substitution Theorem. If $R \in \Pi^1_1$ and F is a recursive function which reduces R to W, let a be an index such that $\{a\}(\mathbf{m}, p) = F(p, \mathbf{m})$. Then

$$R(\mathbf{m}) \leftrightarrow \lambda p.\{a\}(\mathbf{m}, p) \in \mathsf{W} \leftrightarrow \lambda p.\{\mathrm{Sb}_{k-1}(a, \mathbf{m})\}(p) \in \mathsf{W} \leftrightarrow \mathrm{Sb}_{k-1}(a, \mathbf{m}) \in W. \quad \square$$

1.9 Corollary. *Neither* W_{ω_1} *nor* W *is* Σ^1_1. $\square$

All of the preceding results have relativized and "boldface" extensions which we sketch below. In most cases the proofs are straightforward and are left to the Exercises.

1.10 Definition. For any $\beta, \mathsf{R}, \mathsf{A}$, and A,

(i) $\omega_1[\beta] = \sup^+\{\|\gamma\| : \gamma \in \mathsf{W} \wedge \gamma \text{ is recursive in } \beta\}$;

(ii) $W[\beta] = \{c : \lambda p.\{c\}(p, \beta) \in \mathsf{W}\}$;

(iii) for all $c \in W[\beta]$, $\|c\|_\beta = \|\lambda p.\{c\}(p, \beta)\|$;
(iv) $\mathsf{R} \leqslant_\beta A$ iff for some functional F recursive in β,

$$\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \mathsf{F}(\mathbf{m}, \boldsymbol{\alpha}) \in A;$$

(v) $\mathsf{R} \leqslant_\beta \mathsf{A}$ iff for some functional F recursive in β,

$$\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \mathsf{F}[\mathbf{m}, \boldsymbol{\alpha}] \in \mathsf{A};$$

(vi) $\mathsf{R} \blacktriangleleft \mathsf{A}$ iff for some continuous functional F,

$$\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \mathsf{F}[\mathbf{m}, \boldsymbol{\alpha}] \in \mathsf{A}.$$

1.11 Theorem. *For all β, R, and R,*
(i) $\mathsf{R} \in \Pi_1^1[\beta] \leftrightarrow \mathsf{R} \leqslant_\beta \mathsf{W}$;
(ii) $R \in \Pi_1^1[\beta] \leftrightarrow R \leqslant_\beta \mathsf{W}_{\omega_1[\beta]} \leftrightarrow R \leqslant W[\beta]$;
(iii) $\mathsf{R} \in \mathbf{\Pi}_1^1 \leftrightarrow \mathsf{R} \blacktriangleleft \mathsf{W}$;
(iv) *neither* $\mathsf{W}_{\omega_1[\beta]}$ *nor* $W[\beta]$ *is* $\Sigma_1^1[\beta]$;
(v) $\mathsf{W} \notin \mathbf{\Sigma}_1^1$. $\square$

1.12 Corollary. *For all β*
(i) $\Pi_1^1[\beta]$ *and* $\mathbf{\Pi}_1^1$ *have the reduction property but not the separation property*;
(ii) $\Sigma_1^1[\beta]$ *and* $\mathbf{\Sigma}_1^1$ *have the separation property but not the reduction property.* $\square$

1.13–1.25 Exercises

1.13. Show that $\trianglelefteq$ is a dense linear ordering with greatest element but no least element.

1.14. A set $A \subseteq \mathrm{Sq}$ is called a *tree* iff there is no sequence $s_0, s_1, \ldots$ such that for all i, $s_i \in A$ and $s_i \subsetneqq s_{i+1}$. Let

$$\mathsf{Tr} = \{A : A \text{ is a tree}\}.$$

Show that for all $\mathsf{R}, \mathsf{R} \in \Pi_1^1$ iff $\mathsf{R} \leqslant \mathsf{Tr}$.

1.15. For any tree A and any s, let

$$A_s = \{t : t \in A \wedge s \subsetneqq t\}.$$

Set

$$\mathsf{Tr}_{(\sigma)} = \bigcup \{\mathsf{Tr}_\tau : \tau < \sigma\} \cup \{\varnothing\}$$

and

$$A \in \mathsf{Tr}_\sigma \leftrightarrow \forall s.\, A_s \in \mathsf{Tr}_{(\sigma)}.$$

Show that $\mathsf{Tr} = \mathsf{Tr}_{(\aleph_1)}$, but that for all $\sigma < \aleph_1$, $\mathsf{Tr} \not\subseteq \mathsf{Tr}_{(\sigma)}$.

1.16. For $A \in \mathsf{Tr}$, let

$$|A| = \text{least } \sigma.\, A \in \mathsf{Tr}_\sigma.$$

Show that there exist relations $\lessdot_\Sigma$ and $\lessdot_\Pi$ which satisfy an analogue of Theorem 1.4. It follows that Tr could be used in place of W for all the results of this section.

1.17. For which ordinals σ is W_σ arithmetical? Classify as many W_σ as you can (cf. Exercise III.1.17).

1.18. Show that for every $r \geq 0$, there exists an ordinal $\sigma_r < \omega_1$ such that every Σ_r^0 relation is reducible to W_{σ_r}.

1.19. Show that a relation R is Π_1^1 iff it is reducible to $\{\gamma : \gamma \in \mathsf{W} \wedge \gamma \text{ is primitive recursive}\}$. Conclude that ω_1 is also the least ordinal which is not primitive recursive.

1.20. Show that there exists a recursive function F such that for all m,

$$F(m) \in W \quad \text{and} \quad \|F(m)\| = m.$$

1.21. Let

$$\mathrm{Pl}_W = \{\langle a, b, c\rangle : a, b, c \in W \wedge \|a\| + \|b\| = \|c\|\},$$

and

$$\mathrm{Ti}_W = \{\langle a, b, c\rangle : a, b, c \in W \wedge \|a\| \cdot \|b\| = \|c\|\}.$$

Show that Pl_W and Ti_W are Π_1^1 sets.

1.22. Show that there exists a Π_1^1 set $W^* \subseteq W$ such that for all $\sigma < \omega_1$ there is exactly one $a \in W^*$ such that $\|a\| = \sigma$.

1.23. Prove Theorem 1.11 and Corollary 1.12.

1.24. Show that $\mathsf{R} \in \Pi_1^1$ iff R is *uniformly reducible* to $W[\boldsymbol{\alpha}]$ — that is, there exists a recursive function G such that for all $(\mathbf{m}, \boldsymbol{\alpha})$

$$\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow G(\mathbf{m}) \in W[\boldsymbol{\alpha}].$$

1.25. Show that every Σ_2^1 relation is expressible in the form $(\exists \gamma \in \mathsf{W})\mathsf{P}(\mathbf{m}, \boldsymbol{\alpha}, \gamma)$ with $\mathsf{P} \in \Delta_1^1$.

1.26 Notes. The representation of Π_1^1 relations in terms of well-orderings is one of the oldest and most fundamental results of Descriptive Set Theory. It essentially originates with Lebesgue in 1905 and is studied further by Luzin, Suslin, and Sierpinski from 1915 on. Luzin [1930] is a good exposition of the state of knowledge then. Kleene [1955a] rediscovered the technique in his proof that the set O of notations for constructive ordinals (Definition 4.16) is a complete Π_1^1 set. Spector [1955] was the first explicit statement that every Π_1^1 set of numbers is reducible to W.

Alternative developments of the theory of this section are sketched in Exercises 1.14–16 and Exercises III.3.33–34.

2. *The Boundedness Principle and Other Applications*

2.1 Boundedness Theorem. *For any* $\mathsf{A} \subseteq {}^{\omega}\omega$ *and any* $A \subseteq \omega$,

(i) $\mathsf{A} \in \boldsymbol{\Sigma}_1^1 \wedge \mathsf{A} \subseteq \mathsf{W} \rightarrow \sup^+\{\|\gamma\| : \gamma \in \mathsf{A}\} < \aleph_1$;

(ii) $\mathsf{A} \in \Sigma_1^1 \wedge \mathsf{A} \subseteq \mathsf{W} \rightarrow \sup^+\{\|\gamma\| : \gamma \in \mathsf{A}\} < \omega_1$;

(iii) $A \in \Sigma_1^1 \wedge A \subseteq W \rightarrow \sup^+\{\|c\| : c \in A\} < \omega_1$.

Proof. Suppose, contrary to (i), that $\mathsf{A} \in \boldsymbol{\Sigma}_1^1$, $\mathsf{A} \subseteq \mathsf{W}$, but $\{\|\gamma\| : \gamma \in \mathsf{A}\}$ contains arbitrarily large countable ordinals. Thus any δ belongs to W just in case $\|\delta\| \leqslant \|\gamma\|$ for some $\gamma \in \mathsf{A}$. Since $\mathsf{A} \subseteq \mathsf{W}$, this yields

$$\delta \in \mathsf{W} \leftrightarrow \exists \gamma\, [\gamma \in \mathsf{A} \wedge \delta \leqslant_\Sigma \gamma]$$

which implies $\mathsf{W} \in \boldsymbol{\Sigma}_1^1$, contrary to Theorem 1.11.

Next, suppose that A were a counterexample to (iii). Then,

$$\delta \in \mathsf{W}_{\omega_1} \leftrightarrow \exists c\, [c \in A \wedge \delta \leqslant_\Sigma \{c\}],$$

which implies that $\mathsf{W}_{\omega_1} \in \Sigma_1^1$ contrary to Corollary 1.9.

Finally, if A were a counterexample to (ii), then

$$A = \{c : \{c\} \text{ is a total unary function} \wedge \exists \gamma\, [\gamma \in \mathsf{A} \wedge \{c\} \leqslant_\Sigma \gamma]\}$$

would be a counterexample to (iii). $\square$

For $\rho < \omega_1$, we set

$$W_\rho = \{c : c \in W \wedge \|c\| < \rho\}.$$

2.2 Theorem. *For all* R *and all* R,

(i) $\mathsf{R} \in \boldsymbol{\Delta}_1^1 \leftrightarrow \mathsf{R} \blacktriangleleft \mathsf{W}_\rho$ *for some* $\rho < \aleph_1$;

(ii) $\mathsf{R} \in \Delta_1^1 \leftrightarrow \mathsf{R} \preccurlyeq \mathsf{W}_\rho$ *for some* $\rho < \omega_1$;

(iii) $R \in \Delta_1^1 \leftrightarrow R \preccurlyeq W_\rho$ *for some* $\rho < \omega_1$.

Proof. If ρ is any countable ordinal, let δ be an element of W such that $\|\delta\| = \rho$. Then for all γ,

$$\gamma \in \mathsf{W}_\rho \leftrightarrow \gamma \prec_\Sigma \delta \leftrightarrow \gamma \prec_\Pi \delta.$$

Hence $\mathsf{W}_\rho \in \boldsymbol{\Delta}_1^1$, which yields the implication $(\leftarrow)$ of (i). If $\rho < \omega_1$, δ may be chosen to be recursive and we thus have $(\leftarrow)$ of (ii) and (iii).

For the implications $(\rightarrow)$, suppose first that $\mathsf{R} \in \boldsymbol{\Delta}_1^1$. Since $\mathsf{R} \in \boldsymbol{\Pi}_1^1$, there exists by Theorem 1.1 a continuous functional F such that $\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \mathsf{F}[\mathbf{m}, \boldsymbol{\alpha}] \in \mathsf{W}$. Let

$$\mathsf{A} = \{\gamma : \exists \mathbf{m} \exists \boldsymbol{\alpha}\, (\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \wedge \gamma = \mathsf{F}[\mathbf{m}, \boldsymbol{\alpha}])\}.$$

Since also $\mathsf{R} \in \boldsymbol{\Sigma}_1^1$ we have $\mathsf{A} \in \boldsymbol{\Sigma}_1^1$, and clearly $\mathsf{A} \subseteq \mathsf{W}$. Hence by the Boundedness Theorem there exists a $\rho < \aleph_1$ such that all $\gamma \in \mathsf{A}$ have $\|\gamma\| < \rho$. Hence $\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \mathsf{F}[\mathbf{m}, \boldsymbol{\alpha}] \in \mathsf{W}_\rho$ and thus $\mathsf{R} \blacktriangleleft \mathsf{W}_\rho$. The argument for (ii) and (iii) $(\rightarrow)$ is nearly identical. □

The import of this theorem is that the Δ_1^1 ($\boldsymbol{\Delta}_1^1$) relations may be naturally arranged in a sort of hierarchy of length ω_1 ($\aleph_1$). Let $X_\rho = \{\mathsf{R} : (\exists \tau \leq \rho) \mathsf{R} \preccurlyeq \mathsf{W}_\tau\}$. Then if $\rho \leq \sigma$, $X_\rho \subseteq X_\sigma$ and $\Delta_1^1 = \bigcup \{X_\rho : \rho < \omega_1\}$. Similarly there are classes $\mathbf{X}_\rho$ such that $\boldsymbol{\Delta}_1^1 = \bigcup \{\mathbf{X}_\rho : \rho < \aleph_1\}$. In §§ 3–4 we shall construct other hierarchies on Δ_1^1 and $\boldsymbol{\Delta}_1^1$. See also Exercise 2.24.

We aim next to show that the set of Δ_1^1 functions is Π_1^1 but not Δ_1^1. This will complete the proof of Lemma III.4.8 that Δ_1^1 is not a basis for Π_1^0. We need first some technical lemmas. We extend the relations $\preccurlyeq_\Sigma$, etc. to W in the obvious way:

$$c \preccurlyeq_\Sigma d \leftrightarrow \{c\} \text{ and } \{d\} \text{ are total functions of rank 1 and } \{c\} \preccurlyeq_\Sigma \{d\}$$

and similarly for $\preccurlyeq_\Pi$, $\prec_\Sigma$, and $\prec_\Pi$.

2.3 Lemma. *There exist relations* P^Σ *and* P^Π *in* Σ_1^1 *and* Π_1^1 *respectively, such that for any* α, b, *and* c, *if* $c \in W$, *then*

$$\mathrm{Gr}_\alpha \leqslant W_{\|c\|}\, via\{b\} \leftrightarrow \mathsf{P}^\Sigma(b, c, \alpha) \leftrightarrow \mathsf{P}^\Pi(b, c, \alpha).$$

Proof. Let

$$\begin{aligned} \mathsf{P}^\Sigma(b, c, \alpha) \leftrightarrow \{b\} &\text{ is a total function of rank } 2 \wedge \\ &\wedge \forall m\, \forall n\, [\alpha(m) = n \rightarrow \{b\}(m, n) <_\Sigma c] \wedge \\ &\wedge \forall m\, \forall n\, [\{b\}(m, n) <_\Pi c \rightarrow \alpha(m) = n]. \end{aligned}$$

Then for $c \in W$ and $\{b\}$ a total function of rank 2,

$$\mathsf{P}^\Sigma(b, c, \alpha) \leftrightarrow \forall m\, \forall n\, (\alpha(m) = n \leftrightarrow \|\{b\}(m, n)\| < \|c\|)$$

as required. P^Π is obtained by interchanging $<_\Sigma$ and $<_\Pi$. □

2.4 Lemma. *For every $\rho < \omega_1$, there exists a set $B \in \Delta^1_1$ such that $B \leqslant W_\sigma$ for no $\sigma \leqslant \rho$.*

Proof. Let ρ be any ordinal $< \omega_1$ and $d \in W$ such that $\|d\| = \rho$. Set

$$A = \{\langle a, c\rangle : a \in W \wedge \|a\| < \|c\| \wedge \|c\| \leqslant \|d\|\}.$$

Since for any a and c,

$$\langle a, c\rangle \in A \leftrightarrow a <_\Sigma c \wedge c \leqslant_\Sigma d \leftrightarrow a <_\Pi c \wedge c \leqslant_\Pi d,$$

$A \in \Delta^1_1$. Hence also $B = A^{\omega J}$, the jump of A, is Δ^1_1. Suppose that for some $\sigma \leqslant \rho$, $B \leqslant W_\sigma$. Then there is a recursive function f and a $c \in W$ such that $\|c\| = \sigma$ and

$$m \in B \leftrightarrow f(m) \in W_{\|c\|} \leftrightarrow \langle f(m), c\rangle \in A.$$

But then $B \leqslant A$, a contradiction, since $A^{\omega J}$ is not recursive in A, hence not reducible to A. □

2.5 Selection Theorem. *For any Π^1_1 relation* R, *there exists a partial functional* Sel_R *with Π^1_1 graph such that for all* $\mathbf{m}$ *and* $\boldsymbol{\alpha}$,

$$\exists p\, \mathsf{R}(p, \mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \mathsf{R}(\mathrm{Sel}_\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}), \mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \mathrm{Sel}_\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha})\downarrow.$$

Proof. Suppose $\mathsf{R} \in \Pi^1_1$ and is reducible to W via the recursive functional F. We define

$$\begin{aligned} \mathrm{Sel}_\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \simeq \text{least } p[&\mathsf{R}(p, \mathbf{m}, \boldsymbol{\alpha}) \\ &\wedge \|\mathsf{F}[p, \mathbf{m}, \boldsymbol{\alpha}]\| = \min\{\|\mathsf{F}[q, \mathbf{m}, \boldsymbol{\alpha}]\| : q \in \omega \wedge \mathsf{R}(q, \mathbf{m}, \boldsymbol{\alpha})\}]. \end{aligned}$$

It is clear that $\mathrm{Sel}_{\mathsf{R}}$ has the required property. That its graph is Π^1_1 follows from the easily proved equivalence:

$$\mathrm{Sel}_{\mathsf{R}}(\mathbf{m},\boldsymbol{\alpha})\simeq p \leftrightarrow \mathsf{R}(p,\mathbf{m},\boldsymbol{\alpha}) \wedge \forall q\,(\mathsf{F}[p,\mathbf{m},\boldsymbol{\alpha}] \leqslant_{\Pi} \mathsf{F}[q,\mathbf{m},\boldsymbol{\alpha}])$$
$$\wedge (\forall q<p)(\mathsf{F}[p,\mathbf{m},\boldsymbol{\alpha}] <_{\Pi} \mathsf{F}[q,\mathbf{m},\boldsymbol{\alpha}]). \quad \square$$

2.6 Theorem. $\{\alpha : \alpha \in \Delta^1_1\} \in \Pi^1_1 \sim \Delta^1_1$.

Proof. To show that the set is Π^1_1 we have, with the notation of Lemma 2.3,

$$\begin{aligned}\alpha \in \Delta^1_1 &\leftrightarrow (\exists \sigma < \omega_1)\,.\,\mathrm{Gr}_\alpha \leqslant W_\sigma\\ &\leftrightarrow (\exists c \in W)\,.\,\mathrm{Gr}_\alpha \leqslant W_{\|c\|}\\ &\leftrightarrow \exists b\, \exists c\, [c \in W \wedge \mathsf{P}^{\Pi}(b,c,\alpha)].\end{aligned}$$

Now for a contradiction suppose $\{\alpha\colon \alpha \in \Delta^1_1\} \in \Delta^1_1$. Let R be defined by:

$$\mathsf{R}(c,\alpha) \leftrightarrow c \in W \wedge ([\alpha \in \Delta^1_1 \wedge \exists b\, \mathsf{P}^{\Pi}(b,c,\alpha)]$$
$$\vee [\alpha \notin \Delta^1_1 \wedge \forall m\,.\{c\}(m) \simeq 1]).$$

Clearly $\mathsf{R} \in \Pi^1_1$, and by Theorem 2.2, $\forall \alpha\, \exists c\, \mathsf{R}(c,\alpha)$. Hence the functional $\mathrm{Sel}_{\mathsf{R}}$ is total and thus by Corollary III.2.7 has Δ^1_1 graph. Let $A = \mathrm{Im}\,\mathrm{Sel}_{\mathsf{R}}$. $A \in \Sigma^1_1$ and $A \subseteq W$ so by the Boundedness Theorem (2.1) there exists a $\rho < \omega_1$ such that for all $c \in A$, $\|c\| < \rho$. But then $\forall \alpha\, \exists c\, [\mathsf{R}(c,\alpha) \wedge \|c\| < \rho]$ from which it follows that for all $\alpha \in \Delta^1_1$, $\mathrm{Gr}_\alpha \leqslant W_\sigma$ for some $\sigma < \rho$. This contradicts Lemma 2.4. $\square$

By Theorem 1.6 any Π^1_1 relation on numbers can be represented in the form $(\exists \sigma < \omega_1)\mathsf{F}[\mathbf{m}] \in \mathsf{W}_\sigma$. Our next theorem gives another "existential" representation of Π^1_1 relations.

2.7 Lemma. *For any γ and $\delta \in \mathsf{W} \cap \Delta^1_1$, if* $\mathrm{Fld}(\delta) \neq \omega$, *then*

$$\|\gamma\| \leqslant \|\delta\| \leftrightarrow (\exists \beta \in \Delta^1_1)\forall p\, \forall q\, [p <_\gamma q \leftrightarrow \beta(p) <_\delta \beta(q)].$$

Proof. The implication $(\leftarrow)$ is immediate. Suppose $\gamma, \delta \in \mathsf{W} \cap \Delta^1_1$ and $\|\gamma\| \leqslant \|\delta\|$ and suppose $m \notin \mathrm{Fld}(\delta)$. Set

$$\mathsf{A} = \{\beta : \forall p\, \forall q\, [p <_\gamma q \leftrightarrow \beta(p) <_\delta \beta(q)] \wedge \beta \text{ maps } \mathrm{Fld}(\gamma)$$
$$\text{onto an intial segment of } \mathrm{Fld}(\delta) \wedge (\forall p \notin \mathrm{Fld}(\delta))\beta(p) = m\}.$$

A is arithmetical in γ and δ, hence is Δ^1_1. A has exactly one element β_0. By Corollary III.2.7(vii), $\beta_0 \in \Delta^1_1$. $\square$

2.8 Definition. $\Sigma_1^{1,\mathrm{Hyp}}$ is the class of relations R such that for some arithmetical relation P,

$$\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow (\exists \beta \in \Delta_1^1[\boldsymbol{\alpha}])\mathsf{P}(\mathbf{m}, \boldsymbol{\alpha}, \beta)$$

("Hyp" stands for "hyperarithmetical").

2.9 Spector–Gandy Theorem. $\Sigma_1^{1,\mathrm{Hyp}} = \Pi_1^1$.

Proof. We shall give the proof only for relations on numbers where the theorem takes the form: $R \in \Pi_1^1$ iff for some arithmetical relation P,

$$R(\mathbf{m}) \leftrightarrow (\exists \beta \in \Delta_1^1)\mathsf{P}(\mathbf{m}, \beta)$$

(cf. Exercise 2.28).

Suppose first that R and P satisfy this equivalence and let

$$\begin{aligned} S(b, c) \leftrightarrow c \in W \wedge \{b\} &\text{ is a total function of rank } 2 \wedge \\ &\forall m\, \exists!\, n(\{b\}(m, n) \in W_{\|c\|}). \end{aligned}$$

Then $S \in \Pi_1^1$ and

$$\begin{aligned} R(\mathbf{m}) &\leftrightarrow \exists \beta\, \exists c\, (c \in W \wedge \mathrm{Gr}_\beta \leqslant W_{\|c\|} \wedge \mathsf{P}(\mathbf{m}, \beta)) \\ &\leftrightarrow \exists b\, \exists c\, (S(b, c) \wedge \forall \beta\, [\mathsf{P}^\Sigma(b, c, \beta) \rightarrow \mathsf{P}(\mathbf{m}, \beta)]) \end{aligned}$$

with P^Σ from Lemma 2.3. Thus $R \in \Pi_1^1$.

For the converse implication, it suffices to show $W \in \Sigma_1^{1,\mathrm{Hyp}}$. By Theorems 2.6 and 1.1, there exists a recursive functional F such that

$$\alpha \in \Delta_1^1 \leftrightarrow \mathsf{F}[\alpha] \in \mathsf{W}.$$

Since $\{\alpha : \alpha \in \Delta_1^1\}$ is not Δ_1^1, it is not reducible to any W_ρ with $\rho < \omega_1$. Hence the ordinals $\|\mathsf{F}[\alpha]\|$ for $\alpha \in \Delta_1^1$ are unbounded in ω_1 and we have

$$\begin{aligned} c \in W &\leftrightarrow (\exists \alpha \in \Delta_1^1)(\|c\| \leqslant \|\mathsf{F}[\alpha]\|) \\ &\leftrightarrow (\exists \alpha \in \Delta_1^1)(\exists \beta \in \Delta_1^1)\forall p\, \forall q\, [p <_{\{c\}} q \leftrightarrow \beta(p) <_{\mathsf{F}[\alpha]} \beta(q)]. \end{aligned}$$

It is easy to verify that the last expression defines a set in $\Sigma_1^{1,\mathrm{Hyp}}$. □

We say $\mathsf{R} \in \Delta_1^{1,\mathrm{Hyp}}$ just in case both R and $\sim\mathsf{R} \in \Sigma_1^{1,\mathrm{Hyp}}$.

2.10 Corollary. *For all* R, $\mathsf{R} \in \Delta_1^1 \leftrightarrow \mathsf{R} \in \Delta_1^{1,\mathrm{Hyp}}$. □

These last two results have a natural interpretation in terms of the second-order comprehension axioms discussed in § III.5:

$$^{\omega}\omega \cap \Delta^1_1 \models \Delta^1_1\text{-Comprehension but } ^{\omega}\omega \cap \Delta^1_1 \not\models \exists^1_1\text{-Comprehension}.$$

Exercise 2.15 also asserts that

$$^{\omega}\omega \cap \Delta^1_1 \models \exists^1_1\text{-Choice}.$$

Turning now to the well-orderings of ω which occur in the various classes, we have the somewhat surprising

2.11 Theorem. (i) *For any R, if R is a Σ^1_1 well-ordering relation, then $\|R\| < \omega_1$;* (ii) *there exists a Π^1_1 well-ordering relation R such that $\|R\| = \omega_1$.*

Proof. Suppose, contrary to (i), that R is a Σ^1_1 well-ordering of type $\rho \geqslant \omega_1$. Then

$$\mathsf{W}_\rho = \{\gamma : \exists\alpha\, \forall p\, \forall q\, [p <_\gamma q \to R(\alpha(p), \alpha(q)) \wedge \alpha(p) \neq \alpha(q)]$$
$$\wedge \leqslant_\gamma \text{ is a linear ordering}\}$$

so that $\mathsf{W}_\rho \in \Sigma^1_1$, which contradicts the Boundedness Theorem.

For (ii), let

$$W^* = \{c : c \in W \wedge (\forall d < c)[\{c\} <_\Pi \{d\} \vee \{d\} <_\Pi \{c\}]\}.$$

Then $W^* \in \Pi^1_1$ and for each $\sigma < \omega_1$, there is a unique $c \in W^*$ such that $\|c\| = \sigma$. Hence if we set

$$R(c, d) \leftrightarrow c, d \in W^* \wedge \{c\} \leqslant_\Pi \{d\},$$

then R is a Π^1_1 well-ordering of type ω_1. □

Note as a consequence that δ^1_1, the least non-Δ^1_1 ordinal, is just ω_1 (cf. Corollary III.3.12).

2.12 Theorem. *For any R, A, and α,*

(i) $A \in \Pi^1_1 \sim \Delta^1_1 \wedge R \in \Pi^1_1 \to R \in \Delta^1_1[A]$;

(ii) $\omega_1[\alpha] > \omega_1 \wedge R \in \Pi^1_1 \to R \in \Delta^1_1[\alpha]$.

Proof. For (i), let F be a recursive function such that A is reducible to W_{ω_1} via F. Since $A \notin \Delta^1_1$, A is not reducible to any W_σ with $\sigma < \omega_1$ so the ordinals $\|F[m]\|$ with $m \in A$ are unbounded in ω_1. Hence, for any c,

$$c \in W \leftrightarrow (\exists m \in A)(\{c\} \leqslant_\Sigma F[m]).$$

From this follows $W \in \Sigma_1^1[A]$. If $R \in \Pi_1^1$, then $R \leqslant W$ so also $R \in \Sigma_1^1[A]$. But $\Pi_1^1 \subseteq \Pi_1^1[A]$ so also $R \in \Delta_1^1[A]$.

For (ii), if $\omega_1[\alpha] > \omega_1$, then there is a function δ recursive in α with $\|\delta\| = \omega_1$. Then for any c

$$c \in W \leftrightarrow \{c\} \text{ is a total unary function} \wedge \{c\} <_\Sigma \delta$$

and thus $W \in \Sigma_1^1[\alpha]$. The rest of the proof is as for (i). □

Part (i) of the theorem has a natural interpretation in terms of Δ_1^1-degrees, usually called *hyperdegrees* (cf. III.2.13 and the paragraph following). Let

$$A \leqslant_1^1 B \leftrightarrow A \in \Delta_1^1[B]$$

and

$$A <_1^1 B \leftrightarrow A \leqslant_1^1 B \wedge B \not\leqslant_1^1 A:'$$

so that

$$\text{hydg}(A) = \{B : A \leqslant_1^1 B \wedge B \leqslant_1^1 A\}.$$

The hyperdegrees inherit a natural partial ordering also denoted by $\leqslant_1^1$. The hyperdegree $\mathbf{0} = \{A : A \in \Delta_1^1\}$ is the least element of this partial ordering.

2.13 Corollary. *For all* $A, B \in \Pi_1^1 \sim \Delta_1^1$, $\text{hydg}(A) = \text{hydg}(B)$.

Proof. Immediate from 2.12(i). □

This points out an imperfection in the analogy of Δ_1^1 and Π_1^1 with the classes of recursive and semi-recursive relations. The Friedberg–Mučnik Theorem mentioned at the end of § II.5 asserts the existence of semi-recursive sets A and B such that $\text{dg}(A)$ and $\text{dg}(B)$ are not only different but incomparable. We shall return to this point in § VIII.3.

In § V.6 we shall need a relativized version of Theorem 2.12 which we state here but leave the proof to Exercise 2.33.

2.14 Theorem. *For any* R, A, α, *and* β, *if* $\beta \in \Delta_1^1[A]$ *and* $\beta \in \Delta_1^1[\alpha]$, *then*

(i) $A \in \Pi_1^1[\beta] \sim \Delta_1^1[\beta] \wedge R \in \Pi_1^1[\beta] \rightarrow R \in \Delta_1^1[A]$;

(ii) $\omega_1[\alpha] > \omega_1[\beta] \wedge R \in \Pi_1^1[\beta] \rightarrow R \in \Delta_1^1[\alpha]$;

(iii) $\omega_1[\alpha] > \omega_1[\beta] \rightarrow \beta <_1^1 \alpha$. □

We conclude this section by establishing a precise bound for the closure ordinals of Π^1_1 monotone operators and Π^0_1 inductive operators over ω (cf. Exercise III.3.34).

2.15 Theorem. *For any operator Γ over ω,*

$$\Gamma \text{ monotone} \wedge \Gamma \in \Pi^1_1 \rightarrow |\Gamma| \leqslant \omega_1.$$

Proof. Let Γ be any Π^1_1 monotone operator. It will suffice to show $\Gamma^{\omega_1} \subseteq \Gamma^{(\omega_1)}$. Let $\bar{m}$ be a fixed element of Γ^{ω_1}. We first observe that for any ρ,

$$(1) \qquad \begin{aligned} \bar{m} \in \Gamma^{\rho} &\leftrightarrow \forall A\,[\Gamma^{(\rho)} \subseteq A \rightarrow \bar{m} \in \Gamma(A)] \\ &\leftrightarrow \forall A\,[\bar{m} \notin \Gamma(A) \rightarrow (\exists \sigma < \rho)\Gamma^{\sigma} \nsubseteq A]. \end{aligned}$$

Hence if we define

$$\varphi(A) = \begin{cases} \text{least } \sigma < \omega_1 \,.\, \Gamma^{\sigma} \nsubseteq A, & \text{if} \quad \bar{m} \notin \Gamma(A); \\ 0, & \text{otherwise;} \end{cases}$$

then φ is a total function defined on all $A \subseteq \omega$. Let

$$\bar{\rho} = \sup{}^{+}\{\varphi(A) : A \subseteq \omega\}.$$

Clearly $\bar{\rho} \leqslant \omega_1$ and by (1), $\bar{m} \in \Gamma^{\bar{\rho}}$; we shall show $\bar{\rho} < \omega_1$.

If $\bar{m} \in \Gamma^0$ we are done. Otherwise, for any σ

$$\sigma < \bar{\rho} \leftrightarrow \exists A\,[\bar{m} \notin \Gamma(A) \wedge (\forall \tau < \omega_1)(\Gamma^{\tau} \nsubseteq A \rightarrow \sigma \leqslant \tau)].$$

In other words, if

$$\mathsf{A} = \{\gamma : \exists A\,[\bar{m} \notin \Gamma(A) \wedge (\forall \tau < \omega_1)(\Gamma^{\tau} \nsubseteq A \rightarrow \|\gamma\| \leqslant \tau)]\},$$

then $\bar{\rho} = \sup^{+}\{\|\gamma\| : \gamma \in \mathsf{A}\}$. Let V be the Π^1_1 "coding relation" whose existence is claimed by Theorem III.3.13. Then

$$\gamma \in \mathsf{A} \leftrightarrow \exists A\,[\bar{m} \notin \Gamma(A) \wedge \forall c\,(c \in W \wedge \exists p\,[\mathsf{V}(p, \{c\}) \wedge p \notin A] \rightarrow \gamma \leqslant_{\Sigma} \{c\})].$$

Thus $\mathsf{A} \in \Sigma^1_1$ so by the Boundedness Theorem, $\bar{\rho} < \omega_1$. $\square$

2.16 Theorem. *For any inductive operator Γ over ω,*

$$\Gamma \in \Pi^0_1 \rightarrow |\Gamma| \leqslant \omega_1.$$

Proof. Suppose $\Gamma \in \Pi_1^0$ and let R be a recursive relation such that

$$\mathsf{P}_\Gamma(m, \alpha) \leftrightarrow \forall n\, R(m, \bar{\alpha}(n)).$$

As in the proof of Theorem 1.1 we may assume that

$$(2) \qquad R(m, s) \wedge t \subseteq s \rightarrow R(m, t).$$

Let $\bar{m}$ be a fixed element of Γ^{ω_1}; we shall prove $\bar{m} \in \Gamma^{(\omega_1)}$.

For all ordinals σ, let (for typographical reaons) f_σ ($f_{(\sigma)}$) be the characteristic function of $\Gamma^\sigma(\Gamma^{(\sigma)})$. We note first that for any n and any limit ordinal ρ

$$(3) \qquad (\exists \sigma < \rho)(\forall \tau < \rho)[\sigma \leqslant \tau \rightarrow \bar{f}_\tau(n) = \bar{f}_{(\rho)}(n)].$$

To see this, let

$$\tau_i = \begin{cases} \text{least } \tau .\, i \in \Gamma^\tau, & \text{if} \quad i \in \Gamma^{(\rho)}; \\ 0, & \text{otherwise}; \end{cases}$$

and take $\sigma = \max\{\tau_i : i < n\}$.

From the assumption $\bar{m} \in \Gamma^{\omega_1}$, we have $\forall n\, R(\bar{m}, \bar{f}_{(\omega_1)}(n))$ and thus by (3), $\forall n\, (\exists \sigma < \omega_1) R(\bar{m}, \bar{f}_\sigma(n))$. Let

$$\varphi(n) = \text{least } \sigma < \omega_1 .\, R(\bar{m}, \bar{f}_\sigma(n))$$

and

$$\bar{\rho} = \sup{}^+\{\varphi(n) : n \in \omega\}.$$

As in the proof of the preceding theorem, $\bar{\rho} \leqslant \omega_1$ and $\bar{\rho} = \sup^+\{\|\gamma\| : \gamma \in \mathsf{A}\}$ where

$$\begin{aligned} \gamma \in \mathsf{A} &\leftrightarrow \exists n\, (\forall \tau < \omega_1)[R(\bar{m}, \bar{f}_\tau(n)) \rightarrow \|\gamma\| \leqslant \tau] \\ &\leftrightarrow \exists n\, \forall c\, [c \in W \wedge R(\bar{m}, \bar{f}_{\|c\|}(n)) \rightarrow \gamma \leqslant_\Sigma \{c\}]. \end{aligned}$$

First note that $\mathsf{A} \subseteq \mathsf{W}$ as if $\gamma \in \mathsf{A}$ because the condition is satisfied for n, then $\|\gamma\| \leqslant \varphi(n) < \omega_1$ so $\gamma \in \mathsf{W}$. To see that $\mathsf{A} \in \Sigma_1^1$, let V_Π and V_Σ be respectively the Π_1^1 and Σ_1^1 "coding relations" from Theorem III.3.9. Then for $c \in W$,

$$\begin{aligned} R(\bar{m}, \bar{f}_{\|c\|}(n)) \leftrightarrow \exists s\, (s \in \mathrm{Sq} \wedge \lg(s) = n \wedge (\forall i < n)((s)_i \leqslant 1) \wedge \\ R(\bar{m}, s) \wedge (\forall i < n)[(s)_i = 0 \rightarrow \mathsf{V}_\Pi(i, \{c\}) \\ \wedge \mathsf{V}_\Sigma(i, \{c\}) \rightarrow (s)_i = 0]). \end{aligned}$$

Thus this relation is Π^1_1 from which it easily follows that $\mathsf{A} \in \Sigma^1_1$ and thus by Boundedness that $\bar\rho < \omega_1$. To complete the proof we show $\bar m \in \Gamma^{\bar\rho}$.

We first observe that the function φ is monotone. If $n \leq p$, then $R(\bar m, \bar f_{\varphi(p)}(p))$ so by (2) also $R(\bar m, \bar f_{\varphi(p)}(n))$. As $\varphi(n)$ is the *least* σ such that $R(\bar m, \bar f_\sigma(n))$, it follows that $\varphi(n) \leq \varphi(p)$.

Case 1. $\bar\rho$ is a successor ordinal. Then for some n_0, $\bar\rho = \varphi(n_0) + 1$ and for all n, $\varphi(n) \leq \varphi(n_0)$. Because φ is monotone, $\varphi(n) = \varphi(n_0)$ for all $n \geq n_0$. Hence for all $n \geq n_0$, $R(\bar m, \bar f_{\varphi(n_0)}(n))$. But then it follows from (2) that this holds also for all $n < n_0$ and thus for all n. Thus $\mathsf{P}_\Gamma(\bar m, f_{\varphi(n_0)})$ and $\bar m \in \Gamma(\Gamma^{\varphi(n_0)}) = \Gamma^{\bar\rho}$.

Case 2. $\bar\rho$ is a limit ordinal. Let n be a fixed natural number and σ be as in (3) such that $\sigma < \bar\rho$ and

$$(4) \qquad (\forall \tau < \bar\rho)[\sigma \leq \tau \rightarrow \bar f_\tau(n) = \bar f_{(\bar\rho)}(n)].$$

Since φ is monotone and $\bar\rho = \sup^+\{\varphi(p) : p \in \omega\}$, there is an $n_0 \geq n$ such that $\sigma \leq \varphi(n_0)$. By the definition of φ, $R(\bar m, \bar f_{\varphi(n_0)}(n_0))$. But then by (2), $R(\bar m, \bar f_{\varphi(n_0)}(n))$ and by (4), $R(\bar m, \bar f_{(\bar\rho)}(n))$. As n was arbitrary, we have $\forall n R(\bar m, \bar f_{(\bar\rho)}(n))$ and thus $\bar m \in \Gamma(\Gamma^{(\bar\rho)}) = \Gamma^{\bar\rho}$. $\square$

2.17 Corollary. *For any* Π^0_1 *inductive operator* Γ, $\bar\Gamma \in \Pi^1_1$.

Proof. For any Π^0_1 inductive Γ, the following equivalences hold:

$$m \in \bar\Gamma \leftrightarrow (\exists \sigma < \omega_1)(m \in \Gamma^\sigma) \leftrightarrow (\exists c \in W)(m \in \Gamma^{\|c\|})$$
$$\leftrightarrow (\exists c \in W) \mathsf{V}_\Pi(m, \{c\}). \quad \square$$

Similar techniques lead also to a kind of boundedness theorem for inductive definitions:

2.18 Lemma. *For any inductive operator* Γ *over* ω *and any* Σ^1_1 *set* $A \subseteq \Gamma^{(\omega_1)}$, *if either* Γ *is* Δ^1_1 *or* Γ *is* Π^1_1 *and monotone, then for some* $\rho < \omega_1$, $A \subseteq \Gamma^\rho$.

Proof. Suppose A and Γ satisfy the hypotheses, define $|m|_\Gamma = \text{least } \sigma.\, m \in \Gamma^\sigma$ for $m \in \Gamma^{(\omega_1)}$, and set $\bar\rho = \sup^+\{|m|_\Gamma : m \in A\}$. Clearly $A \subseteq \Gamma^{\bar\rho}$ and by definition $\bar\rho \leq \omega_1$.

If $A = \varnothing$, then $A \subseteq \Gamma^0$. Otherwise $\bar\rho = \sup^+\{\|\gamma\| : \gamma \in \mathsf{A}\}$ where

$$\gamma \in \mathsf{A} \leftrightarrow (\exists m \in A)(\forall \tau < \omega_1)[m \in \Gamma^\tau \rightarrow \|\gamma\| \leq \tau]$$
$$\leftrightarrow (\exists m \in A)(\forall c \in W)[\mathsf{V}(m, \{c\}) \rightarrow \gamma \leq_\Sigma \{c\}],$$

with V as in III.3.13 if Γ is Π^1_1 monotone and $\mathsf{V} = \mathsf{V}_\Pi$ as in III.3.9 if Γ is Δ^1_1. Then $\mathsf{A} \in \Sigma^1_1$ and $\mathsf{A} \subseteq \mathsf{W}$, so by the Boundedness Theorem, $\bar\rho < \omega_1$. $\square$

2.19 Corollary. *For any inductive operator Γ over ω and any Σ_1^1 set $A \subseteq \bar{\Gamma}$, if either Γ is Π_1^0 or Γ is Π_1^1 and monotone, then for some $\rho < \omega_1$, $A \subseteq \Gamma^\rho$.* □

2.20 Corollary. *For any inductive operator Γ over ω, if either Γ is Π_1^0 or Γ is Π_1^1 and monotone, then*

(i) *if $\bar{\Gamma} \in \Sigma_1^1$, then $|\Gamma| < \omega_1$;*

(ii) *for any $R \in \Sigma_1^1$, if $R \leqslant \bar{\Gamma}$, then for some $\rho < \omega_1$, $R \leqslant \Gamma^\rho$.* □

2.21. Corollary. *For any inductive operator Γ over ω, if either Γ is Π_1^0 or Γ is Δ_1^1 and monotone, then*

$$\bar{\Gamma} \in \Delta_1^1 \quad \textit{iff} \quad |\Gamma| < \omega_1.$$

Proof. By Corollary 2.20 and Corollary III.3.12. □

2.22 Corollary. *For any $R \in \Delta_1^1$, there exists an implicitly Π_1^0 function δ such that R is recursive in δ.*

Proof. Suppose $R \in \Delta_1^1$. Then by Theorem III.3.2 there exists a Π_1^0 monotone operator Γ such that $R \leqslant \bar{\Gamma}$. By Corollary 2.20, $R \leqslant \Gamma^\rho$ for some $\rho < \omega_1$. Let S be the relation defined in the proof of Theorem III.3.9; as P_Γ is Π_1^0, S is arithmetical. Let γ be a recursive function such that $\|\gamma\| = \rho + 1$ and set $\varepsilon = \langle \alpha_\gamma, \beta_\gamma \rangle$. Then ε is the unique function which satisfies the arithmetical relation $\mathsf{S}((\varepsilon)_0, (\varepsilon)_1, \gamma)$, so ε is implicitly arithmetical. Furthermore, if p is the unique element of $\mathrm{Fld}(\gamma)$ such that $|p|_\gamma = \rho$, then $m \in \Gamma^\rho \leftrightarrow (\varepsilon)_1(\langle p, m \rangle) = 0$ so that Γ^ρ and hence R is recursive in ε. Finally, by Lemma III.4.4, ε is recursive in some δ which is implicitly Π_1^0. □

2.23–2.38 Exercises

2.23. Prove the following *Effective Boundedness Principle*: there exists a primitive recursive function h such that for any a, if a is a Σ_1^1 index for a set $A \subseteq W$ — that is

$$c \in A \leftrightarrow U_1^1(a, \langle c \rangle)$$

— then $h(a) \in W$ and $\|h(a)\| \geqslant \sup^+\{\|c\| : c \in A\}$. Formulate and prove a corresponding result for $\mathbf{\Sigma}_1^1$ subsets of W.

2.24. The hierarchy of Δ_1^1 relations described following Theorem 2.2 is deficient in that it may happen that for some $\rho < \sigma < \omega_1$, no new relations are reducible to some W_τ ($\tau \leqslant \sigma$) that are not already reducible to some W_τ ($\tau \leqslant \rho$) — that is, $X_\rho = X_\sigma$. This can be remedied by omitting superfluous levels in the hierarchy, and for relations on numbers, the resulting hierarchy may be seen still to have length ω_1. Let

$$Z = \{\sigma : \sigma < \omega_1 \wedge W_\sigma \not\preccurlyeq W_\rho \text{ for any } \rho < \sigma\}$$

and

$$\bar{\omega}_1 = \text{order-type of } Z.$$

Show that $\bar{\omega}_1 = \omega_1$.

Hint. Let

$$\bar{W} = \{c : c \in W \wedge \|c\| \in Z \wedge \neg \exists d\,(d < c \wedge \|c\| = \|d\|)\}$$

and suppose that for some $e \in W$, $\bar{\omega}_1 = \|e\| < \omega_1$. Set

$$\begin{aligned} P(c, d) \leftrightarrow (\|c\| \not< \|e\| \wedge d = 0) \vee (\|c\| < \|e\| \wedge d \in W \wedge \\ \forall\alpha\,[\text{if } \alpha \text{ is an ordinal-preserving map of} \\ \{a : \|a\| \leqslant \|c\|\} \text{ into } \{b : b \in \bar{W} \wedge \|b\| \leqslant \|d\|\}, \\ \text{then } \alpha(c) = d]). \end{aligned}$$

Show that $P \in \Pi^1_1$, there is a function $\beta \in \Delta^1_1$ such that $\forall c\, P(c, \beta(c))$ and

$$d \mathrel{\dot{\in}} \bar{W} \leftrightarrow \exists c\,(\|c\| \leqslant \|e\| \wedge \beta(c) = d).$$

2.25 (Kreisel [1962]). Prove the following two *effective choice principles*: for any Π^1_1 relation R,

(i) if $\forall \mathbf{m} \forall \boldsymbol{\alpha} \exists p\, \mathsf{R}(p, \mathbf{m}, \boldsymbol{\alpha})$, then there exists a Δ^1_1 functional F such that $\forall \mathbf{m} \forall \boldsymbol{\alpha}\, \mathsf{R}(\mathsf{F}(\mathbf{m}, \boldsymbol{\alpha}), \mathbf{m}, \boldsymbol{\alpha})$;

(ii) if $\forall \mathbf{m} \forall \boldsymbol{\alpha}\, (\exists \beta \in \Delta^1_1[\boldsymbol{\alpha}])\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}, \beta)$, then there exists a Δ^1_1 functional G such that $\forall \mathbf{m} \forall \boldsymbol{\alpha}\, \mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}, \lambda q\,.\,\mathsf{G}(q, \mathbf{m}, \boldsymbol{\alpha}))$. Conclude in particular that $\{\alpha : \alpha \in \Delta^1_1\}$ is a model for $\exists^1_1$-Choice.

2.26. Show that there exists a set $\mathsf{A} \in \Sigma^1_1$ such that $\mathsf{A} \cap \mathsf{W} = \mathsf{W}_{\omega_1}$.

2.27. Show that if a function $\theta : {}^\omega\omega \to {}^\omega\omega$ has Δ^1_1 graph, then $\mathrm{D\dot{m}}\,\theta \in \Delta^1_1$. If $\boldsymbol{\theta}$ is one-one, then also $\mathrm{Im}\,\theta \in \Delta^1_1$.

2.28. Prove the general case of the Spector–Gandy Theorem (2.9).

2.29 (Spector [1959]). Show that a relation R is Π^1_1 iff for some Π^0_1 relation P, $\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \exists! \beta\, \mathsf{P}(\mathbf{m}, \boldsymbol{\alpha}, \beta)$. (*Hint*: In one direction ($\leftarrow$) use the Spector–Gandy Theorem. For the other, P will have the property that $\forall \mathbf{m} \forall \boldsymbol{\alpha} \exists \beta\, \mathsf{P}(\mathbf{m}, \boldsymbol{\alpha}, \beta)$.)

2.30. Show that for any $A \in \Pi^1_1$, $A \in \Delta^1_1$ iff $\omega_1[A] = \omega_1$.

2.31. Show that if $\gamma \in \Delta^1_2$, then $\omega_1[\gamma]$ is the order-type of a Δ^1_2 well-ordering of ω.

2.32. For any $A \subseteq \omega$, the *hyperjump* of A is the set

$$A^{\mathsf{hJ}} = \{\langle a, \mathbf{m}\rangle : \mathsf{U}^1_1(a, \langle \mathbf{m}\rangle, \langle A\rangle)\}.$$

Show that if $A \leqslant^1_1 B$, then also $A^{\mathsf{hJ}} \leqslant^1_1 B^{\mathsf{hJ}}$ so that the hyperjump is well defined on hyperdegrees.

2.33. Prove Theorem 2.14.

2.34. Show that
(i) for any $\sigma < \omega_1$, there exists a Π^0_1 inductive operator Γ such that $|\Gamma| = \sigma$;
(ii) any $A \in \Delta^1_1$ is reducible to $\bar{\Gamma}$ for some Π^0_1 inductive operator Γ such that $|\Gamma| < \omega_1$.

2.35. Show that for any monotone arithmetical operator Γ, $\{\langle c, m\rangle : c \in W \wedge m \in \Gamma^{|c|}\}$ is implicitly Π^1_1 (cf. proof of Theorem III.3.7). Conclude that every Π^1_1 relation on numbers is reducible to an implicitly Π^1_1 set and hence that Δ^1_1 is not a basis for the class of Π^1_1 singletons.

2.36. Show that for any α, $\alpha \in \Delta^1_1$ iff for some function δ, $\langle \alpha, \delta\rangle$ is implicitly Π^0_2.

2.37. Construct an arithmetical monotone operator Γ such that for all $\sigma < \omega_1$, $\Gamma^\sigma = W_\sigma$.

2.38. The partial functionals with Π^1_1 graph may be indexed as follows. Let

$$\mathsf{V}(n, a, \langle \mathbf{m}\rangle, \langle \boldsymbol{\alpha}\rangle) \leftrightarrow {\sim}\mathsf{U}^1_1(a, \langle n, \mathbf{m}\rangle, \langle \boldsymbol{\alpha}\rangle)$$

and set

$$\{a\}^*(\mathbf{m}, \boldsymbol{\alpha}) \simeq \mathrm{Sel}_{\mathsf{V}}(a, \langle \mathbf{m}\rangle, \langle \boldsymbol{\alpha}\rangle).$$

Which properties of the class of partial recursive functionals are shared by the class of Π^1_1 functionals with this indexing? Does the Recursion Theorem hold?

2.39 Notes. The boundedness principle 2.1 (i) is also an old result of Descriptive Set Theory — the effective versions 2.1 (ii) and (iii) are implicitly in Spector [1955]. The Spector–Gandy Theorem was conjectured by Mostowski and proved independently by Spector [1959] and Gandy [1960]. The proof here is due to Moschovakis. Theorems 2.12 and 2.14 are due to Spector [1955] and 2.15 to Spector [1959]. Theorem 2.16 was proved by Gandy [unpublished] essentially as done here. Another quite different proof appears in Richter [1971].

3. The Borel Hierarchy

In the Introduction we discussed briefly the notion of a hierarchy as a decomposition of a class of objects into levels indexed by ordinals in such a way that members of higher levels are in some sense more complex than members of lower levels. The main examples to this point have been the arithmetical and analytical hierarchies and their relativized and boldface counterparts. The "hierarchy" discussed following Theorem 2.2 suffers from the fact that some levels with different indices coincide. In this and the next section we shall construct hierarchies for the $\boldsymbol{\Delta}^1_1$ and Δ^1_1 relations which do not have this failing (see also Exercise 2.24).

The first of these, the Borel hierarchy, is both the simplest and historically the first transfinite hierarchy. Recall that the class Bo of Borel relations is the smallest class containing the open relations (equivalently, the closed-open relations) and closed under countable union and intersection of relations of the same rank. This is an inductive definition and the corresponding hierarchy consists essentially of the stages. The actual definition will differ slightly from this idea as we shall define classes $\boldsymbol{\Sigma}^0_\rho$ and $\boldsymbol{\Pi}^0_\rho$ for all $\rho < \aleph_1$ such that $\mathrm{Bo} = \bigcup \{\boldsymbol{\Sigma}^0_\rho : \rho < \aleph_1\}$. Thus the Borel hierarchy is a natural extension of the boldface arithmetical hierarchy.

Although these results appear in this context to depend heavily on recursion theory, in fact they were first derived long before the development of recursion theory. It is only in hindsight that we see that the hierarchies of recursion theory are refinements of those of Descriptive Set Theory and that many of the techniques of recursion theory were known already in some form to the early descriptive set theorists. This section and the next have been arranged to emphasize the relationship between the classical hierarchy and its effective or recursion-theoretic counterpart. These constructions also serve as paradigms for several others in later parts of the book — those of §§ V.4, 5 which involve operations more complex than countable union and intersection, and those of § VI.5, § VI.6, and § VII.3.

3.1 Lemma. *The class of Borel relations is closed under complementation and composition and substitution of continuous functionals.*

Proof. Let F and G be continuous and consider

$$X = \{\mathsf{R} : \{(\mathbf{m}, \boldsymbol{\alpha}) : \mathsf{R}(\mathsf{F}(\mathbf{m}, \boldsymbol{\alpha}), \mathbf{m}, \boldsymbol{\alpha}, \lambda p . \mathsf{G}(p, \mathbf{m}, \boldsymbol{\alpha}))\} \text{ is Borel}\}.$$

By II.3.9 and II.5.5, X includes the class of closed-open relations and it is trivial to show that X is closed under countable union and intersection. Hence $\mathrm{Bo} \subseteq X$. The proof for complementation is similar. □

3.2 Lemma. *For all countable ordinals ρ, W_ρ is Borel.*

Proof. We proceed by induction on ρ. Clearly $\mathsf{W}_0 = \varnothing$ and W_1 are Borel. If ρ is a limit ordinal, then $\mathsf{W}_\rho = \bigcup \{\mathsf{W}_\sigma : \sigma < \rho\}$. If ρ is countable, then this is a countable union so if all W_σ $(\sigma < \rho)$ are Borel, so is W_ρ. If $\rho = \sigma + 1 > 1$, then $\gamma \in \mathsf{W}_\rho$ iff for all p, $\gamma \restriction p \in \mathsf{W}_\sigma$. Since W_σ is Borel, it follows from Lemma 3.1 that also for each $p, \mathsf{A}_p = \{\gamma : \gamma \restriction p \in \mathsf{W}_\sigma\}$ is Borel. Hence so is $\mathsf{W}_\rho = \bigcap \{\mathsf{A}_p : p \in \omega\}$. □

3.3 Theorem (Suslin). $\mathrm{Bo} = \boldsymbol{\Delta}^1_1$.

Proof. The inclusion $(\subseteq)$ is Corollary III.2.16. If $\mathsf{R} \in \boldsymbol{\Delta}^1_1$, then by Theorem 2.2, $\mathsf{R} \leqslant \mathsf{W}_\rho$ for some $\rho < \aleph_1$. Hence by Lemmas 3.1 and 3.2, R is Borel. □

3.4 Definition (The Borel Hierarchy). For all $\rho > 0$,

(i) $\boldsymbol{\Sigma}^0_0 = \boldsymbol{\Pi}^0_0 =$ the class of closed-open relations;
(ii) $\boldsymbol{\Sigma}^0_{(\rho)} = \bigcup \{\boldsymbol{\Sigma}^0_\tau : \tau < \rho\}$; $\boldsymbol{\Pi}^0_{(\rho)} = \bigcup \{\boldsymbol{\Pi}^0_\tau : \tau < \rho\}$;
(iii) $\boldsymbol{\Sigma}^0_\rho = \{\bigcup \{\mathsf{P}_p : p \in \omega\}$: all P_p have the same rank and belong to $\boldsymbol{\Pi}^0_{(\rho)}\}$;
(iv) $\boldsymbol{\Pi}^0_\rho = \{\bigcap \{\mathsf{P}_p : p \in \omega\}$: all P_p have the same rank and belong to $\boldsymbol{\Sigma}^0_{(\rho)}\}$;
(v) $\boldsymbol{\Delta}^0_\rho = \boldsymbol{\Sigma}^0_\rho \cap \boldsymbol{\Pi}^0_\rho$; $\boldsymbol{\Delta}^0_{(\rho)} = \bigcup \{\boldsymbol{\Delta}^0_\tau : \tau < \rho\}$.

It is immediate by induction on ρ that all of the classes $\boldsymbol{\Sigma}^0_\rho$ and $\boldsymbol{\Pi}^0_\rho$ are included in the class of Borel relations. The following lemma together with Theorem III.1.16 implies that this definition is consistent with the former definition of $\boldsymbol{\Sigma}^0_\rho$ for $\rho < \omega$ (III.1.15).

3.5 Lemma. *For all $\rho > 0$ and all R,*

(i) $\mathsf{R} \in \boldsymbol{\Sigma}^0_\rho \leftrightarrow {\sim}\mathsf{R} \in \boldsymbol{\Pi}^0_\rho$;
(ii) $\boldsymbol{\Sigma}^0_{(\rho)} \cup \boldsymbol{\Pi}^0_{(\rho)} \subseteq \boldsymbol{\Delta}^0_\rho$;
(iii) $\boldsymbol{\Sigma}^0_{(\rho+1)} = \boldsymbol{\Sigma}^0_\rho$; $\boldsymbol{\Pi}^0_{(\rho+1)} = \boldsymbol{\Pi}^0_\rho$.

Proof. The proof of (i) is an easy induction on ρ. If $\mathsf{R} \in \boldsymbol{\Sigma}^0_{(\rho)}$, then $\mathsf{R} \in \boldsymbol{\Sigma}^0_\tau$ for some $\tau < \rho$. Hence for some $\mathsf{P}_p \in \boldsymbol{\Pi}^0_{(\tau)}$, $\mathsf{R} = \bigcup \{\mathsf{P}_p : p \in \omega\}$. But $\boldsymbol{\Pi}^0_{(\tau)} \subseteq \boldsymbol{\Pi}^0_{(\rho)}$, so also $\mathsf{R} \in \boldsymbol{\Sigma}^0_\rho$. On the other hand, let $\mathsf{Q}_p = \mathsf{R}$ for all p. Then all $\mathsf{Q}_p \in \boldsymbol{\Sigma}^0_{(\rho)}$ so $\mathsf{R} = \bigcap \{\mathsf{Q}_p : p \in \omega\} \in \boldsymbol{\Pi}^0_\rho$. (ii) now follows from (i) and (iii) follows immediately. □

Many of the properties of the Borel hierarchy mirror those of the arithmetical hierarchy and we state in the following lemma some of the more important of these. The proofs are easy adaptations of previously used techiques and are omitted.

3.6 Lemma. *For all* $\rho > 0$,

(i) $\boldsymbol{\Sigma}^0_\rho$ *is closed under countable union, finite intersection, expansion, bounded quantification, existential number quantification, and composition and substitution of continuous functionals;*

(ii) $\boldsymbol{\Pi}^0_\rho$ *is closed under countable intersection, finite union, expansion, bounded quantification, universal number quantification, and composition and substitution of continuous functionals.* □

3.7 Theorem. $\mathrm{Bo} = \boldsymbol{\Delta}^0_{(\aleph_1)}$.

Proof. It suffices to show that $\boldsymbol{\Delta}^0_{(\aleph_1)}$ is closed under countable union and intersection. Suppose that for each $p \in \omega$, $\mathsf{P}_p \in \boldsymbol{\Delta}^0_{(\aleph_1)}$, say $\mathsf{P}_p \in \boldsymbol{\Delta}^0_{\sigma_p}$ with $\sigma_p < \aleph_1$. Then if $\rho = \sup\{\sigma_p : p \in \omega\}$, $\rho < \aleph_1$ and by Lemma 3.5 for all $p, \mathsf{P}_p \in \boldsymbol{\Delta}^0_\rho = \boldsymbol{\Sigma}^0_{(\rho+1)} \cap \boldsymbol{\Pi}^0_{(\rho+1)}$. Hence $\bigcup\{\mathsf{P}_p : p \in \omega\} \in \boldsymbol{\Sigma}^0_{\rho+1} \subseteq \boldsymbol{\Delta}^0_{\rho+2} \subseteq \boldsymbol{\Delta}^0_{(\aleph_1)}$. □

3.8 Corollary. $\boldsymbol{\Delta}^1_1 = \boldsymbol{\Delta}^0_{(\aleph_1)}$.

Proof. Immediate from 3.3 and 3.7. □

We want now to show that all of the levels of the Borel hierarchy are distinct. We proceed analogously to Theorem III.1.9 (the Arithmetical Hierarchy) and define relations U^0_ρ which are universal for $\boldsymbol{\Sigma}^0_\rho$ — that is, such that $\mathsf{U}^0_\rho \in \boldsymbol{\Sigma}^0_\rho$ and for every $\mathsf{R} \in \boldsymbol{\Sigma}^0_\rho$ there exists a $\beta \in {}^\omega\omega$ such that

$$\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \mathsf{U}^0_\rho(\langle \mathbf{m} \rangle, \langle \boldsymbol{\alpha} \rangle, \beta)$$

(cf. Exercise II.5.11).

We set

$$\mathsf{U}^0_1(\langle \mathbf{m} \rangle, \langle \boldsymbol{\alpha} \rangle, \beta) \leftrightarrow \mathsf{U}^0_1(\beta(0), \langle \mathbf{m} \rangle, \langle \boldsymbol{\alpha}, \lambda p\,.\,\beta(p+1) \rangle);$$

for $\rho > 0$,

$$\mathsf{U}^0_{\rho+1}(\langle \mathbf{m} \rangle, \langle \boldsymbol{\alpha} \rangle, \beta) \leftrightarrow \exists p \sim \mathsf{U}^0_\rho(\langle \mathbf{m} \rangle, \langle \boldsymbol{\alpha} \rangle, (\beta)^p);$$

and for limit ρ,

$$\mathsf{U}^0_\rho(\langle \mathbf{m} \rangle, \langle \boldsymbol{\alpha} \rangle, \beta) \leftrightarrow \exists p\, (\exists \sigma < \rho)[\|(\beta)^p_0\| = \sigma \wedge \sim \mathsf{U}^0_\sigma(\langle \mathbf{m} \rangle, \langle \boldsymbol{\alpha} \rangle, (\beta)^p_1)].$$

3.9 Lemma. *For all* $\rho < \aleph_1$, $\mathsf{W}_\rho \in \mathbf{\Delta}^0_{\rho+1}$.

Proof. Follow the proof of Lemma 3.2 with this stronger induction hypothesis. This shows in fact that for successor ρ, $\mathsf{W}_\rho \in \mathbf{\Pi}^0_\rho$ and for limit ρ, $\mathsf{W}_\rho \in \mathbf{\Sigma}^0_\rho$. □

3.10 Borel Indexing Theorem. *For all* ρ *such that* $0 < \rho < \aleph_1$,
(i) U^0_ρ *is universal for* $\mathbf{\Sigma}^0_\rho$;
(ii) $\sim\mathsf{U}^0_\rho$ *is universal for* $\mathbf{\Pi}^0_\rho$.

Proof. That $\mathsf{U}^0_1 \in \mathbf{\Sigma}^0_1$ is obvious. Suppose that $\mathsf{U}^0_\rho \in \mathbf{\Sigma}^0_\rho$. Then $\sim\mathsf{U}^0_\rho \in \mathbf{\Pi}^0_\rho$ and

$$\mathsf{U}^0_{\rho+1} = \bigcup\{\{(\langle \mathbf{m}\rangle, \langle \boldsymbol{\alpha}\rangle, \beta) : \sim\mathsf{U}^0_\rho(\langle \mathbf{m}\rangle, \langle \boldsymbol{\alpha}\rangle, (\beta)^p)\} : p \in \omega\}$$

so that $\mathsf{U}^0_{\rho+1} \in \mathbf{\Sigma}^0_{\rho+1}$. If ρ is a limit ordinal, then by the preceding lemma, $\{\gamma : \|\gamma\| = \sigma\} = \mathsf{W}_{\sigma+1} \sim \mathsf{W}_\sigma \in \mathbf{\Delta}^0_{\sigma+2}$, so that for all $\sigma < \rho$ this set is in $\mathbf{\Pi}^0_{(\rho)}$. As ρ is countable this shows that U^0_ρ is a countable union of $\mathbf{\Pi}^0_{(\rho)}$ relations, hence is $\mathbf{\Sigma}^0_\rho$.

Again it is obvious that U^0_1 is universal for $\mathbf{\Sigma}^0_1$. Suppose that U^0_ρ is universal for $\mathbf{\Sigma}^0_\rho$ and $\mathsf{R} \in \mathbf{\Sigma}^0_{\rho+1}$, so for some relations $\mathsf{P}_p \in \mathbf{\Pi}^0_\rho$, $\mathsf{R} = \bigcup\{\mathsf{P}_p : p \in \omega\}$. By hypothesis, there exist functions β_p such that

$$\mathsf{P}_p(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \sim\mathsf{U}^0_\rho(\langle \mathbf{m}\rangle, \langle \boldsymbol{\alpha}\rangle, \beta_p).$$

Then if β is a function such that for all p, $(\beta)^p = \beta_p$, then

$$\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \mathsf{U}^0_{\rho+1}(\langle \mathbf{m}\rangle, \langle \boldsymbol{\alpha}\rangle, \beta).$$

Finally, suppose that ρ is a limit ordinal, for all $\sigma < \rho$, U^0_σ is universal for $\mathbf{\Sigma}^0_\sigma$, and $\mathsf{R} = \bigcup\{\mathsf{P}_p : p \in \omega\} \in \mathbf{\Sigma}^0_\rho$. For each p there is an ordinal $\sigma_p < \rho$ such that $\mathsf{P}_p \in \mathbf{\Pi}^0_{\sigma_p}$ and thus a function β_p such that

$$\mathsf{P}_p(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \sim\mathsf{U}^0_{\sigma_p}(\langle \mathbf{m}\rangle, \langle \boldsymbol{\alpha}\rangle, \beta_p).$$

Then if β is a function such that for all p, $\|(\beta)^p_0\| = \sigma_p$ and $(\beta)^p_1 = \beta_p$, we again have

$$\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \mathsf{U}^0_\rho(\langle \mathbf{m}\rangle, \langle \boldsymbol{\alpha}\rangle, \beta)$$

so that U^0_ρ is universal for $\mathbf{\Sigma}^0_\rho$. □

3.11 Borel Hierarchy Theorem. *For all* ρ *such that* $0 < \rho < \aleph_1$,
(i) $\mathbf{\Sigma}^0_\rho \not\subseteq \mathbf{\Delta}^0_\rho$ *and* $\mathbf{\Pi}^0_\rho \not\subseteq \mathbf{\Delta}^0_\rho$;
(ii) $\mathbf{\Delta}^0_{\rho+1} \not\subseteq \mathbf{\Sigma}^0_\rho \cup \mathbf{\Pi}^0_\rho$.

Proof. It suffices for (i) to show that $\mathsf{U}^0_\rho \notin \mathbf{\Delta}^0_\rho$. Suppose the contrary and let

$$\mathsf{A} = \{\alpha : \mathsf{U}^0_\rho(\langle\ \ \rangle, \langle\alpha\rangle, \alpha)\}.$$

Then by Lemma 3.6, $\mathsf{A} \in \mathbf{\Delta}^0_\rho$ so in particular, $\sim\mathsf{A} \in \mathbf{\Sigma}^0_\rho$. Since U^0_ρ is universal, there exists a β such that for all α,

$$\alpha \notin \mathsf{A} \leftrightarrow \mathsf{U}^0_\rho(\langle\ \ \rangle, \langle\alpha\rangle, \beta).$$

In particular,

$$\beta \notin \mathsf{A} \leftrightarrow \mathsf{U}^0_\rho(\langle\ \ \rangle, \langle\beta\rangle, \beta) \leftrightarrow \beta \in \mathsf{A},$$

a contradiction.

For (ii), let

$$\mathsf{R}(m, \alpha) \leftrightarrow (m = 0 \wedge \alpha \in \mathsf{A}) \vee (m = 1 \wedge \alpha \notin \mathsf{A}).$$

Then as in the proof of Theorem III.1.9, R is $\mathbf{\Delta}^0_{\rho+1}$ but neither $\mathbf{\Sigma}^0_\rho$ nor $\mathbf{\Pi}^0_\rho$. □

By the homeomorphism discussed in § I.2, all of these results apply equally well to the space BIr of binary irrationals in the real interval (0, 1) with the induced topology: $Y \subseteq \mathrm{BIr}$ is open iff $Y = \mathrm{BIr} \cap Z$ for some open subset Z of (0, 1). Of course, it is not in general true that if $\mathrm{BIr} \cap Z$ is open in BIr, then Z is open in (0, 1). In fact, if $Z = \mathrm{BIr}$, then Z is not open or even an F_σ-set in (0, 1) (Exercise I.2.11), but $\mathrm{BIr} \cap Z = \mathrm{BIr}$ which is open in BIr.

Let BIr-$\mathbf{\Sigma}^0_\rho$, etc., denote the Borel hierarchy on BIr and (0, 1)-$\mathbf{\Sigma}^0_\rho$ denote the Borel hierarchy defined similarly over (0, 1), starting with (0, 1)-$\mathbf{\Sigma}^0_1$ = class of open subsets of (0, 1).

3.12 Theorem. *For all $\rho \geq 3$ and all $Z \subseteq (0, 1)$,*

$$\mathrm{BIr} \cap Z \textit{ is } \mathrm{BIr}\text{-}\mathbf{\Sigma}^0_\rho\ (\mathbf{\Pi}^0_\rho) \leftrightarrow Z \textit{ is } (0, 1)\text{-}\mathbf{\Sigma}^0_\rho\ (\mathbf{\Pi}^0_\rho).$$

Proof. The implication ($\leftarrow$) is immediate by induction on ρ (for *all* ρ). For ($\rightarrow$), it is first easy to prove by induction that for all $\rho > 0$,

$$X \in \mathrm{BIr}\text{-}\mathbf{\Sigma}^0_\rho\ (\mathbf{\Pi}^0_\rho) \rightarrow X = \mathrm{BIr} \cap Y \text{ for some } Y \in (0, 1)\text{-}\mathbf{\Sigma}^0_\rho\ (\mathbf{\Pi}^0_\rho).$$

Then if $\mathrm{BIr} \cap Z$ is BIr-$\mathbf{\Sigma}^0_\rho$, $\mathrm{BIr} \cap Z = \mathrm{BIr} \cap Y$ for some $Y \in (0, 1)$-$\mathbf{\Sigma}^0_\rho$. But then there exist (countable) sets of binary rationals Y_0 and Y_1 such that $Z = (Y \cup Y_0) \sim Y_1$. Since any countable set is F_σ ((0, 1)-$\mathbf{\Sigma}^0_2$), if $\rho \geq 3$, Z is also (0, 1)-$\mathbf{\Sigma}^0_\rho$. □

We can now see that (i) and (ii) of Corollary 3.11 hold also for $(0, 1)$-$\mathbf{\Sigma}^0_\rho$ for all $\rho \geq 1$. For $\rho = 1$, this is because $\varnothing$ and $(0, 1)$ are the only closed-open subsets of $(0, 1)$. For $\rho = 2$ we have that BIr is $(0, 1)$-$\mathbf{\Pi}^0_2$ but not $(0, 1)$-$\mathbf{\Sigma}^0_2$. For $\rho \geq 3$, it follows from Theorem 3.12.

Of course, the theorem applies to relations as well as sets. For any $\mathsf{R} \subsetneq {}^{k+1}(0,1)$, let $\exists^{(0,1)}\mathsf{R}$ denote the relation $\mathsf{P}(x) \leftrightarrow \exists y\, [y \in (0,1) \wedge \mathsf{R}(y, x)]$ and $\exists^{\mathrm{BIr}}\mathsf{R}$ the relation $\mathsf{Q}(x) \leftrightarrow \exists y\, [y \in \mathrm{BIr} \wedge \mathsf{R}(y, x)]$. The $(0, 1)$-$\mathbf{\Sigma}^1_1$ relations are those of the from $\exists^{(0,1)}\mathsf{R}$ for R in $(0,1)$-Borel and the BIr-$\mathbf{\Sigma}^1_1$ relations are those of the form $\exists^{\mathrm{BIr}}\mathsf{R}$ for R BIr-Borel. The other classes of the projective hierarchy are defined similarly. Clearly $(0, 1)$-$\mathbf{\Delta}^1_1$ contains all $(0,1)$-Borel relations.

3.13 Theorem. *For all $r > 0$ and all $Z \subseteq (0,1)$,*

$$\mathrm{BIr} \cap Z \textit{ is } \mathrm{BIr}\text{-}\mathbf{\Sigma}^1_r\ (\mathbf{\Pi}^1_r) \leftrightarrow Z \textit{ is } (0,1)\text{-}\mathbf{\Sigma}^1_r\ (\mathbf{\Pi}^1_r).$$

Proof. If $\mathrm{BIr} \cap Z$ is BIr-$\mathbf{\Sigma}^1_1$, then $\mathrm{BIr} \cap Z = \exists^{\mathrm{BIr}} R$ for some R in BIr-Borel. But since $\mathsf{R} \subseteq {}^2\mathrm{BIr}$, $\exists^{\mathrm{BIr}}\mathsf{R} = \exists^{(0,1)}\mathsf{R}$ and by Theorem 3.12, R is also $(0,1)$-Borel. Hence $\mathrm{BIr} \cap Z$ is also $(0, 1)$-$\mathbf{\Sigma}^1_1$, and since Z differs from $\mathrm{BIr} \cap Z$ by a countable set, also Z is $(0, 1)$-$\mathbf{\Sigma}^1_1$.

Conversely, if Z is $(0, 1)$-$\mathbf{\Sigma}^1_1$, then for some $(0,1)$-Borel relation R, $Z = \exists^{(0,1)}\mathsf{R}$. Then

$$\mathrm{BIr} \cap Z = \exists^{\mathrm{BIr}}[{}^2\mathrm{BIr} \cap \mathsf{R}] \cup (\mathrm{BIr} \cap \exists^{\mathrm{BRa}}\mathsf{R})$$

where $\exists^{\mathrm{BRa}}$ means "there exists a binary rational". By Theorem 3.12, ${}^2\mathrm{BIr} \cap \mathsf{R}$ is BIr-Borel so the first term is BIr-$\mathbf{\Sigma}^1_1$. $\exists^{\mathrm{BRa}}\mathsf{R}$ is a countable union of $(0,1)$-Borel relations, hence is $(0,1)$-Borel. Thus the second term is BIr-Borel and $\mathrm{BIr} \cap Z$ is BIr-$\mathbf{\Sigma}^1_1$.

The extension to larger r is by induction. □

3.14 Corollary. *For all $X \subseteq (0,1)$, X is $(0,1)$-Borel iff X is $(0, 1)$-$\mathbf{\Delta}^1_1$.* □

3.15–3.19 Exercises

3.15. Complete the following outline of an alternative proof that $\mathbf{\Delta}^1_1 \subseteq \mathrm{Bo}$ and that $\mathbf{\Sigma}^1_1$ has the separation property (cf. Exercises I.2.7 and II.4.32). For any $\mathsf{R} \subsetneq {}^\omega\omega \times {}^\omega\omega$, let

$$\mathsf{R}^{(s,t)} = \{(\alpha, \beta) : \mathsf{R}(s * \alpha, t * \beta)\}.$$

Two sets are called *Borel separable* iff they can be separated by a Borel set. Show

(i) if for all m, n, and p, $\exists^1\mathsf{R}^{(\langle m\rangle,\langle n\rangle)}$ and $\exists^1\mathsf{S}^{(\langle m\rangle,\langle p\rangle)}$ are Borel separable, then $\exists^1\mathsf{R}$ and $\exists^1\mathsf{S}$ are Borel separable;

(ii) if for all α, β, and γ, there exists an n such that $\exists^1 \mathsf{R}^{(\bar{\alpha}(n), \bar{\beta}(n))}$ and $\exists^1 \mathsf{S}^{(\bar{\alpha}(n), \bar{\gamma}(n))}$ are Borel separable, then $\exists^1 \mathsf{R}$ and $\exists^1 \mathsf{S}$ are Borel separable;

(iii) any two disjoint $\boldsymbol{\Sigma}_1^1$ sets are Borel separable;

(iv) every $\boldsymbol{\Delta}_1^1$ set is Borel.

3.16. For any indexed family $\langle \mathsf{P}_p : p \in \omega \rangle$ of relations of the same rank, let

$$\underline{\mathrm{Lim}}\langle \mathsf{P}_p \rangle(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \exists p\, (\forall q \geqslant p)\, \mathsf{P}_q(\mathbf{m}, \boldsymbol{\alpha})$$

and

$$\overline{\mathrm{Lim}}\langle \mathsf{P}_p \rangle(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \forall p\, (\exists q \geqslant p)\, \mathsf{P}_q(\mathbf{m}, \boldsymbol{\alpha}).$$

When $\underline{\mathrm{Lim}}\langle \mathsf{P}_p \rangle = \overline{\mathrm{Lim}}\langle \mathsf{P}_p \rangle$, we denote the common value by $\mathrm{Lim}\langle \mathsf{P}_p \rangle$; otherwise $\mathrm{Lim}\langle \mathsf{P}_p \rangle$ is undefined. For all ρ, let

$$\boldsymbol{\Delta}_0 = \text{the class of closed-open relations};$$

$$\boldsymbol{\Delta}_{(\rho)} = \bigcup\{\boldsymbol{\Delta}_\tau : \tau < \rho\};$$

$$\boldsymbol{\Delta}_\rho = \{\mathrm{Lim}\langle \mathsf{P}_p \rangle : \forall p\,.\,\mathsf{P}_p \in \boldsymbol{\Delta}_{(\rho)}\}.$$

Prove for all ρ:

(i) $\boldsymbol{\Delta}_\rho \subseteq \boldsymbol{\Delta}^0_{\rho+1}$;

(ii) $\boldsymbol{\Delta}_\rho$ is a Boolean algebra;

(iii) if $\forall p\,.\,\mathsf{P}_p \in \boldsymbol{\Delta}_{(\rho)}$, then both $\bigcup\{\mathsf{P}_p : p \in \omega\}$ and $\bigcap\{\mathsf{P}_p : p \in \omega\}$ belong to $\boldsymbol{\Delta}_\rho$;

(iv) if $\mathsf{R} = \bigcup\{\mathsf{P}_p : p \in \omega\} = \bigcap\{\mathsf{Q}_q : q \in \omega\}$ where for all p and q, P_p and Q_q belong to $\boldsymbol{\Delta}_\rho$, then also $\mathsf{R} \in \boldsymbol{\Delta}_\rho$;

(v) $\boldsymbol{\Delta}_\rho = \boldsymbol{\Delta}^0_{\rho+1}$.

Hint for (iv). Suppose $\mathsf{P}_p = \mathrm{Lim}\langle \mathsf{P}_{p,m} : m \in \omega \rangle$ and $\mathsf{Q}_q = \mathrm{Lim}\langle \mathsf{Q}_{q,n} : n \in \omega \rangle$ with all $\mathsf{P}_{p,m}, \mathsf{Q}_{q,n} \in \boldsymbol{\Delta}_{(\rho)}$. Let

$$\mathsf{S}_r = (\mathsf{P}_{0,r} \cap \mathsf{Q}_{0,r}) \cup (\mathsf{P}_{1,r} \cap \mathsf{Q}_{0,r} \cap \mathsf{Q}_{1,r}) \cup \ldots \cup (\mathsf{P}_{r,r} \cap \mathsf{Q}_{0,r} \cap \ldots \cap \mathsf{Q}_{r,r}).$$

Show that $\mathsf{R} \subseteq \underline{\mathrm{Lim}}\langle \mathsf{S}_r \rangle$ and $\sim \mathsf{R} \subseteq \sim \overline{\mathrm{Lim}}\langle \mathsf{S}_r \rangle$.

3.17. Let S and H be, respectively, a Borel relation and a Borel functional, and

$$\mathsf{Q}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \mathsf{S}(\mathbf{m}, \boldsymbol{\alpha}, \lambda p\,.\,\mathsf{H}(p, \mathbf{m}, \boldsymbol{\alpha})).$$

If $\mathsf{S} \in \boldsymbol{\Sigma}^0_\rho$ and $\mathsf{H} \in \boldsymbol{\Delta}^0_\sigma$, what can you conclude about the level of Q?

3.18. Show that for all $\rho < \aleph_1$, $\boldsymbol{\Sigma}^0_\rho$ has the reduction property.

3.19. Show that every $(0, 1)$-$\boldsymbol{\Sigma}_1^1$ set is the projection of a $(0, 1)$-$\boldsymbol{\Pi}_2^0$ relation.

3.20 Notes. It might fairly be said that Suslin's Thorem (3.3), proved in 1917, was the result that put Descriptive Set Theory on the map. It was a tremendous breakthrough in its time and is still today the model for most results which give equivalent characterizations of a class of relations "from below" and "from above".

4. *The Effective Borel and Hyperarithmetical Hierarchies*

We want now to define a similar extension of the arithmetical hierarchy to obtain classes Σ_ρ^0 and Π_ρ^0 $(\rho < \omega_1)$ such that $\Delta_{(\omega_1)}^0 = \Delta_1^1$. This construction is called the *effective* Borel hierarchy because it is derived from that in the preceding section by replacing the generating operation of countable union by *recursively enumerable* union. Roughly, the idea is to attach to each relation P as it is generated an index $\iota(\mathsf{P})$ and admit unions only of those families $\langle \mathsf{P}_p : p \in \omega \rangle$ such that the function $\lambda p \,.\, \iota(\mathsf{P}_p)$ is recursive. The details differ from this sketch in that we shall first define the set of indices and then assign relations to the indices.

In the second part of the section we consider an alternative method for constructing a hierarchy of the Δ_1^1 relations. This is based on a notion of iterating the ordinary jump operator oJ over a set of ordinal notations and should be considered as an extension of the ideas of Theorem III.1.13.

4.1 Definition. For each k and l, $N^{k,l}$ is the smallest subset of ω such that for all a and b,

(i) if $(b)_1 = k$ and $(b)_2 = l$, then $\langle 7, b \rangle \in N^{k,l}$;

(ii) if for all p, $\{a\}(p) \in N^{k,l}$, then $a \in N^{k,l}$.

As usual we denote by $N_{(\rho)}^{k,l}$ and $N_\rho^{k,l}$ the sets $\Gamma^{(\rho)}$ and Γ^ρ, where Γ is the monotone Π_1^0 operator such that $\bar{\Gamma} = N^{k,l}$. We also put $N = \bigcup\{N^{k,l} : k, l \in \omega\}$ and $N_\rho = \bigcup\{N_\rho^{k,l} : k, l \in \omega\}$. By Theorem 2.15, $|\Gamma| \leq \omega_1$, so $N^{k,l} = N_{(\omega_1)}^{k,l}$. It is easy to check that if $(k, l) \neq (k', l')$, then $N^{k,l} \cap N^{k',l'} = \emptyset$. Note that $N_0^{k,l}$ is recursive and if $a \in N^{k,l}$ by virtue of clause (ii), then $(a)_0 \neq 7$ so $a \notin N_0^{k,l}$.

We next assign to every $a \in N^{k,l}$ a relation $\mathsf{P}_a \subseteq {}^{k,l}\omega$. The assignment is recursive over $N^{k,l}$ and may be justified by an extension of the technique used in the proof of Theorem I.3.5.

4.2 Definition. For each k and l and any $a \in N^{k,l}$,

(i) if $a \in N_0^{k,l}$, then $\mathsf{P}_a = \mathrm{Dm}\{(a)_1\}$;

(ii) if $a \notin N_0^{k,l}$, then $\mathsf{P}_a = \bigcup\{\sim\mathsf{P}_{\{a\}(p)} : p \in \omega\}$.

4.3 Definition. For all ρ,

(i) $\Sigma^0_{1+\rho} = \{\mathsf{P}_a : a \in N_\rho\}$;

(ii) $\Pi^0_{1+\rho} = \{\sim\mathsf{P}_a : a \in N_\rho\}$;

(iii) $\Delta^0_\rho = \Sigma^0_\rho \cap \Pi^0_\rho$;

(iv) $\Sigma^0_{(\rho)} = \bigcup\{\Sigma^0_\tau : 0 < \tau < \rho\}$; $\Pi^0_{(\rho)} = \bigcup\{\Pi^0_\tau : 0 < \tau < \rho\}$;

$$\Delta^0_{(\rho)} = \bigcup\{\Delta^0_\tau : 0 < \tau < \rho\};$$

(v) $\mathrm{EfBo} = \Delta^0_{(\omega_1)} =$ the class of *effective Borel relations.*

Note that for $\rho < \omega$, $1 + \rho = \rho + 1$, while for $\rho \geqslant \omega$, $1 + \rho = \rho$. A relation is in Σ^0_ρ $(\rho \geqslant 2)$ just in case it is of the form $\bigcup\{\sim\mathsf{P}_{\{a\}(p)} : p \in \omega\}$ where all $\sim\mathsf{P}_{\{a\}(p)}$ are in $\Pi^0_{(\rho)}$. Similarly if $\mathsf{R} = \sim\mathsf{P}_a$ is in Π^0_ρ, then $\mathsf{R} = \bigcap\{\mathsf{P}_{\{a\}(p)} : p \in \omega\}$ and all $\mathsf{P}_{\{a\}(p)}$ are in $\Sigma^0_{(\rho)}$.

We want first to show that the notation here is not in conflict with that of Definition III.1.2 (arithmetical hierarchy) — that is, the classes Σ^0_ρ for $0 < \rho < \omega$ are just those previously defined.

4.4 Lemma. *For all $\rho > 0$,*

(i) $\Sigma^0_{(\rho)} \cup \Pi^0_{(\rho)} \subseteq \Delta^0_\rho$;

(ii) $\Sigma^0_{(\rho+1)} = \Sigma^0_\rho$; $\Pi^0_{(\rho+1)} = \Pi^0_\rho$.

Proof. $\Sigma^0_{(\rho)} \subseteq \Sigma^0_\rho$ by definition as $N_{(\rho)} \subseteq N_\rho$. If $\mathsf{R} \in \Pi^0_{(\rho)}$, then $\mathsf{R} = \sim\mathsf{P}_a$ for some $a \in N_{(\rho)}$. Thus if b is an index such that for all p, $\{b\}(p) = a$, then $b \in N_\rho$ and $\mathsf{R} = \bigcup\{\sim\mathsf{P}_{\{b\}(p)} : p \in \omega\} = \mathsf{P}_b$ which is in Σ^0_ρ. Thus $\Sigma^0_{(\rho)} \cup \Pi^0_{(\rho)} \subseteq \Sigma^0_\rho$ and (i) follows from the fact that $\Sigma^0_{(\rho)} \cup \Pi^0_{(\rho)}$ is closed under complementation. (ii) is then immediate. $\square$

4.5 Lemma. *For all $\rho > 0$, Σ^0_ρ is effectively closed under recursively enumerable union and finite intersection — that is, for each k and l there exist primitive recursive functions f and g such that for any $a, \mathbf{m}, \boldsymbol{\alpha}$, and any $\rho > 0$,*

(i) *if for all p, $\{a\}(p) \in N^{k,l}_\rho$, then $f(a) \in N^{k,l}_\rho$ and*

$$\mathsf{P}_{f(a)}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \exists p \,.\, \mathsf{P}_{\{a\}(p)}(\mathbf{m}, \boldsymbol{\alpha});$$

(ii) *if $a, b \in N^{k,l}_\rho$, then $g(a, b) \in N^{k,l}_\rho$ and*

$$\mathsf{P}_{g(a,b)} = \mathsf{P}_a \cap \mathsf{P}_b.$$

Proof. Let h be a primitive recursive function such that for all a, p, and q,

$$\{h(a, p, q)\}(\mathbf{m}, \boldsymbol{\alpha})\downarrow \text{ iff } \sim\mathsf{T}((\{a\}(p))_1, \langle\mathbf{m}\rangle, q, \langle\boldsymbol{\alpha}\rangle).$$

Then choose f to be a primitive recursive function such that

$$\{f(a)\}(\langle p,q\rangle)\simeq\begin{cases}\langle 7,h(a,p,q)\rangle, & \text{if}\quad \{a\}(p)\in N_0;\\ \{\{a\}(p)\}(q), & \text{otherwise};\end{cases}$$

and for r not of the form $\langle p,q\rangle, \{f(a)\}(r)\simeq\langle 7,a_0\rangle$, where $\mathrm{Dm}\{a_0\}={}^{k,l}\omega$.

Suppose that for all $p, \{a\}(p)\in N_\rho^{k,l}$. For any p such that $\{a\}(p)\in N_0^{k,l}$, clearly also $\{f(a)\}(\langle p,q\rangle)\in N_0^{k,l}$. For other p we have for all $q, \{\{a\}(p)\}(q)\in N_{(\rho)}^{k,l}$, hence also for all $q, \{f(a)\}(\langle p,q\rangle)\in N_{(\rho)}^{k,l}$. Thus for all r, $\{f(a)\}(r)\in N_{(\rho)}^{k,l}$ so $f(a)\in N_\rho^{k,l}$.

For p such that $\{a\}(p)\in N_0^{k,l}$ we have

$$\begin{aligned}\mathsf{P}_{\{a\}(p)}(\mathbf{m},\boldsymbol{\alpha}) &\leftrightarrow \{(\{a\}(p))_1\}(\mathbf{m},\boldsymbol{\alpha})\downarrow\\ &\leftrightarrow \exists q\ \mathsf{T}((\{a\}(p))_1,\langle\mathbf{m}\rangle,q,\langle\boldsymbol{\alpha}\rangle)\\ &\leftrightarrow \exists q\sim\mathsf{P}_{\{f(a)\}(\langle p,q\rangle)}(\mathbf{m},\boldsymbol{\alpha}).\end{aligned}$$

For other p,

$$\begin{aligned}\mathsf{P}_{\{a\}(p)}(\mathbf{m},\boldsymbol{\alpha}) &\leftrightarrow \exists q\sim\mathsf{P}_{\{\{a\}(p)\}(q)}(\mathbf{m},\boldsymbol{\alpha})\\ &\leftrightarrow \exists q\sim\mathsf{P}_{\{f(a)\}(\langle p,q\rangle)}(\mathbf{m},\boldsymbol{\alpha}).\end{aligned}$$

Hence

$$\begin{aligned}\exists p\,.\,\mathsf{P}_{\{a\}(p)}(\mathbf{m},\boldsymbol{\alpha}) &\leftrightarrow \exists p\ \exists q\sim\mathsf{P}_{\{f(a)\}(\langle p,q\rangle)}(\mathbf{m},\boldsymbol{\alpha})\\ &\leftrightarrow \exists r\sim\mathsf{P}_{\{f(a)\}(r)}(\mathbf{m},\boldsymbol{\alpha})\\ &\leftrightarrow \mathsf{P}_{f(a)}(\mathbf{m},\boldsymbol{\alpha}).\end{aligned}$$

We define g by the following four cases:

(1) if $a,b\in N_0$, then $g(a,b)=\langle 7,c\rangle$, where

$$\{c\}(\mathbf{m},\boldsymbol{\alpha})\simeq\{(a)_1\}(\mathbf{m},\boldsymbol{\alpha})+\{(b)_1\}(\mathbf{m},\boldsymbol{\alpha});$$

(2) if $a\notin N_0$ but $b\in N_0$, then $g(a,b)$ is an index c calculated from an index of the function f of (i) such that if $\{a\}(p)\in N$,

$$\mathsf{P}_{\{c\}(p)}=\mathsf{P}_{\{a\}(p)}\cup\sim\mathrm{Dm}\{(b)_1\};$$

(3) if $a\in N_0$ but $b\notin N_0$, then similarly

$$\mathsf{P}_{\{g(a,b)\}(p)}=\sim\mathrm{Dm}\{(a)_1\}\cup\mathsf{P}_{\{b\}(p)};$$

(4) if $a, b \notin N_0$, then

$$\mathsf{P}_{\{g(a,b)\}(p)} = \mathsf{P}_{\{a\}(p)} \cup \mathsf{P}_{\{b\}(p)}.$$

The proof that g satisfies (ii) is straightforward. For example, if $a \in N_0^{k,l}$ and $b \in N^{k,l} \sim N_0^{k,l}$,

$$\begin{aligned}\mathsf{P}_a \cap \mathsf{P}_b &= \mathrm{Dm}\{(a)_1\} \cap \bigcup\{\sim \mathsf{P}_{\{b\}(p)} : p \in \omega\}\\ &= \bigcup\{\sim [\sim \mathrm{Dm}\{(a)_1\} \cup \mathsf{P}_{\{b\}(p)}] : p \in \omega\}\\ &= \bigcup\{\sim \mathsf{P}_{\{g(a,b)\}(p)} : p \in \omega\}\\ &= \mathsf{P}_{g(a,b)}. \quad \square\end{aligned}$$

Before proceeding, we make a few remarks on the methods of proof we shall use in the remainder of this section and often in later parts of the book. In establishing closure properties of the classes Σ_ρ^0 we shall in most cases need to prove that the classes are *effectively* closed. In the preceding lemma the definition of g depends on the fact that f is primitive recursive, and this would be so even if we were not requiring that g be primitive recursive. In the next and many succeeding lemmas we shall need to define functions by *effective transfinite recursion*. In outline, the method is as follows. Suppose $<$ is a well-founded transitive relation on ω. For any function F and any $u \in \mathrm{Fld}(<)$, let $F \upharpoonright u$ denote the partial function g such that

$$g(v, p) \simeq \begin{cases} F(v, p), & \text{if} \quad v < u;\\ \text{undefined}, & \text{otherwise.}\end{cases}$$

The set-theoretic principle of definition by transfinite recursion asserts that for any function φ there is a function F such that for all $u \in \mathrm{Fld}(<)$,

$$F(u, m) \simeq \varphi(F \upharpoonright u, u, m),$$

and that the values of F for $u \in \mathrm{Fld}(<)$ are uniquely determined. Roughly speaking, the principle of effective transfinite recursion asserts that if φ is partial recursive, then F may also be chosen to be partial recursive.

In practice, when we apply the method φ will be such that there exists a partial recursive function H such that for all e and m and all $u \in \mathrm{Fld}(<)$,

$$\varphi(\{e\} \upharpoonright u, u, m) \simeq H(e, u, m)$$

and we shall apply the Recursion Theorem to obtain an index $\bar{e}$ such that

$$\{\bar{e}\}(u, m) \simeq H(\bar{e}, u, m).$$

Then clearly

$$\{\bar{e}\}(u, m) \simeq \varphi(\{\bar{e}\} \upharpoonright u, u, m)$$

and $F = \{\bar{e}\}$ is partial recursive.

A variant of this method defines a function f such that for $u \in \mathrm{Fld}(<)$, $\{f(u)\}(m) \simeq F(u, m)$. In a typical such situation there will be a primitive recursive function h such that

$$\varphi(\{\{e\} \upharpoonright u\}, u, m) \simeq \{h(e, u)\}(m),$$

where

$$\{\{e\} \upharpoonright u\}(v, m) \simeq \begin{cases} \{\{e\}(v)\}(m), & \text{if} \quad v < u; \\ \text{undefined}, & \text{otherwise}. \end{cases}$$

Then if $\bar{e}$ is chosen by the Primitive Recursion Theorem such that $\{\bar{e}\}(u) \simeq h(\bar{e}, u)$, the function $f = \{\bar{e}\}$ is in fact *primitive* recursive and has the desired property.

This leads also to an extension of the principle in which the value $F(u, m)$ may depend not only on the values $F(v, p)$ for $v < u$ but also on indices for the functions $\lambda p . F(v, p)$. In such a situation we have a function ψ such that $\psi(f \upharpoonright u, u, m)$ depends both on values $f(v)$ for $v < u$ and on the partial functions $\{f(v)\}$ that they index and a primitive recursive h such that

$$\psi(f \upharpoonright u, u, m) \simeq \{h(e, u)\}(m).$$

Again $\bar{e}$ is chosen so that $\{\bar{e}\}(u) \simeq h(\bar{e}, u)$, $f = \{\bar{e}\}$, and $F(u, m) \simeq \{f(u)\}(m)$.

There are many variations on these paradigms and mastery of the method comes only with practice. Rogers [1959] is also helpful.

4.6 Lemma. *There exists a primitive recursive function f such that for any ρ and any $a \in N_\rho^{k+1, l}$, for all p, $f(a, p) \in N_\rho^{k, l}$ and*

$$\mathsf{P}_{f(a, p)}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \mathsf{P}_a(p, \mathbf{m}, \boldsymbol{\alpha}).$$

Proof. Let h be a primitive recursive function such that

$$h(e, a, p) \simeq \begin{cases} \langle 7, \mathrm{Sb}_0((a)_1 p)\rangle, & \text{if} \quad a \in N_0; \\ \text{an index } c \text{ such that for all } q, & \\ \{c\}(q) \simeq \{e\}(\{a\}(q), p), & \text{otherwise}. \end{cases}$$

By the Primitive Recursion Theorem there exists an index $\bar{e}$ such that $h(\bar{e}, a, p) = \{\bar{e}\}(a, p)$. We take $f = \{\bar{e}\}$. If $a \in N_0$, then

$$\begin{aligned} \mathsf{P}_{f(a,p)}(\mathbf{m}, \boldsymbol{\alpha}) &\leftrightarrow \{\mathrm{Sb}_0((a)_1, p)\}(\mathbf{m}, \boldsymbol{\alpha})\downarrow \\ &\leftrightarrow \{(a)_1\}(p, \mathbf{m}, \boldsymbol{\alpha})\downarrow \\ &\leftrightarrow \mathsf{P}_a(p, \mathbf{m}, \boldsymbol{\alpha}). \end{aligned}$$

If $a \in N \sim N_0$ and we assume as induction hypothesis that for all q,

$$\mathsf{P}_{f(\{a\}(q), p)}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \mathsf{P}_{\{a\}(p)}(p, \mathbf{m}, \boldsymbol{\alpha}),$$

then

$$\begin{aligned} \mathsf{P}_{f(a,b)}(\mathbf{m}, \boldsymbol{\alpha}) &\leftrightarrow \exists q \sim \mathsf{P}_{f(\{a\}(q), p)}(\mathbf{m}, \boldsymbol{\alpha}) \\ &\leftrightarrow \exists q \sim \mathsf{P}_{\{a\}(q)}(p, \mathbf{m}, \boldsymbol{\alpha}) \\ &\leftrightarrow \mathsf{P}_a(p, \mathbf{m}, \boldsymbol{\alpha}). \quad \square \end{aligned}$$

4.7 Corollary. *For all $\rho > 0$, Σ^0_ρ is effectively closed under existential number quantification ($\exists^0$) and Π^0_ρ is effectively closed under universal number quantification ($\forall^0$) — that is, there exists a primitive recursive function f such that for all ρ and all $a \in N^{k+1,l}_\rho$, $f(a) \in N^{k,l}_\rho$ and*

$$\mathsf{P}_{f(a)}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \exists p \,.\, \mathsf{P}_a(p, \mathbf{m}, \boldsymbol{\alpha}).$$

Proof. It suffices to set $f(a) = f_5(\mathrm{Sb}_0(b_6, a))$, where f_5 is the function f of Lemma 4.5 and b_6 is an index for the function f of Lemma 4.6. $\square$

For the next corollary, let the finite levels of the effective Borel hierarchy be denoted by $\Sigma^{0'}_r$ and $\Pi^{0'}_r$ to distinguish them from the levels of the arithmetical hierarchy.

4.8 Corollary. *For all $r > 0$, $\Sigma^0_r = \Sigma^{0'}_r$ and $\Pi^0_r = \Pi^{0'}_r$.*

Proof. For each k, l, and r, and all $(\mathbf{m}, \boldsymbol{\alpha}) \in {}^{k,l}\omega$, let

$$\mathsf{V}^{k,l}_r(a, \mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow a \in N^{k,l}_{r-1} \wedge \mathsf{P}_a(\mathbf{m}, \boldsymbol{\alpha}).$$

We prove by induction on r the stronger assertion

$$(*) \qquad \Sigma^0_r = \Sigma^{0'}_r, \; N^{k,l}_r \in \Pi^0_1, \quad \text{and} \quad \mathsf{V}^{k,l}_r \in \Sigma^0_r.$$

Clearly $(*)$ holds for $r = 1$ as

$$\mathsf{V}_1^{k,l}(a, \mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow (a)_0 = 7 \wedge (a)_{1,0} = k \wedge (a)_{1,1} = l \wedge \{(a)_1\}(\mathbf{m}, \boldsymbol{\alpha}) \downarrow .$$

Suppose (*) holds for r. If $\mathsf{R} \in \Sigma^0_{r+1}$, then $\mathsf{R} = \exists^0 \mathsf{S}$ for some $\mathsf{S} \in \Pi^0_r$. Then $\mathsf{S} \in \Pi^{0'}_r \subseteq \Sigma^{0'}_{r+1}$ by the induction hypothesis and Lemma 4.4 so $\mathsf{R} \in \Sigma^{0'}_{r+1}$ by Corollary 4.7. Hence $\Sigma^0_{r+1} \subseteq \Sigma^{0'}_{r+1}$.

Next we observe that $a \in N^{k,l}_{r+1} \leftrightarrow \forall p . \{a\}(p) \in N^{k,l}_r$ so that if $N^{k,l}_r \in \Pi^0_1$, so is $N^{k,l}_{r+1}$. Then

$$\mathsf{V}^{k,l}_{r+1}(a, \mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow a \in N^{k,l}_r \wedge \exists p \sim \mathsf{V}^{k,l}_r(\{a\}(p), \mathbf{m}, \boldsymbol{\alpha}).$$

Hence $\mathsf{V}^{k,l}_{r+1} \in \Sigma^0_{r+1}$. Since $\mathsf{V}^{k,l}_{r+1}$ is universal for $\Sigma^{0'}_{r+1}$, this yields $\Sigma^{0'}_{r+1} \subseteq \Sigma^0_{r+1}$. □

We aim next to establish the equation $\Delta^1_1 = \Delta^0_{(\omega_1)}$.

4.9 Lemma. $\Delta^0_{(\omega_1)} \subseteq \Delta^1_1$.

Proof. For a fixed pair (k, l), let

$$\mathsf{V}(a, i, \mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow a \in N^{k,l} \wedge [i = 0 \wedge \mathsf{P}_a(\mathbf{m}, \boldsymbol{\alpha})] \vee [i = 1 \wedge \sim\mathsf{P}_a(\mathbf{m}, \boldsymbol{\alpha})].$$

It will suffice to show $\mathsf{V} \in \Pi^1_1$ as then for all $a \in N^{k,l}$, both P_a and $\sim\mathsf{P}_a$ are Π^1_1 so $\mathsf{P}_a \in \Delta^1_1$. For this we define a decomposable monotone arithmetical operator Γ such that $\bar{\Gamma} = \mathsf{V}$. Γ is essentially simply a combination of definitions 4.1 and 4.2. For any $\mathsf{R} \subseteq {}^{k+2,l}\omega$, all i, and all $(\mathbf{m}, \boldsymbol{\alpha}) \in {}^{k,l}\omega$,

(i) if $(b)_1 = k$ and $(b)_2 = l$, then
  (1) if $\{b\}(\mathbf{m}, \boldsymbol{\alpha}) \downarrow$, then $(\langle 7, b\rangle, 0, \mathbf{m}, \boldsymbol{\alpha}) \in \Gamma(\mathsf{R})$;
  (2) if $\{b\}(\mathbf{m}, \boldsymbol{\alpha}) \uparrow$, then $(\langle 7, b\rangle, 1, \mathbf{m}, \boldsymbol{\alpha}) \in \Gamma(\mathsf{R})$;
(ii) if for all p $(\exists i \leq 1)$ $(\{a\}(p), i, \mathbf{m}, \boldsymbol{\alpha}) \in \mathsf{R}$, then
  (1) if $\exists p . (\{a\}(p), 1, \mathbf{m}, \boldsymbol{\alpha}) \in \mathsf{R}$, then $(a, 0, \mathbf{m}, \boldsymbol{\alpha}) \in \Gamma(\mathsf{R})$;
  (2) if $\forall p . (\{a\}(p), 0, \mathbf{m}, \boldsymbol{\alpha}) \in \mathsf{R}$, then $(a, 1, \mathbf{m}, \boldsymbol{\alpha}) \in \Gamma(\mathsf{R})$.

Γ is clearly monotone, arithmetical, and decomposable; we leave it to the reader to verify that $\bar{\Gamma}$ is as claimed. □

See Exercise 4.25 for an alternative proof of Lemma 4.9. Toward the converse inclusion we establish some further closure properties of the effective Borel hierarchy.

4.10 Lemma. *For all $\rho > 0$, Σ^0_ρ and Π^0_ρ are effectively closed under composition and substitution of recursive functionals — that is, for any recursive functionals* F *and* G, *there exist primitive recursive functions f and g such that for all $\rho > 0$ and all a and b,*

(i) *if $a \in N^{k+1,l}_\rho$, then $f(a) \in N^{k,l}_\rho$ and*

$$\mathsf{P}_{f(a)}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \mathsf{P}_a(\mathsf{F}(\mathbf{m}, \boldsymbol{\alpha}), \mathbf{m}, \boldsymbol{\alpha});$$

(ii) *if* $b \in N_\rho^{k,l+1}$, *then* $g(b) \in N_\rho^{k,l}$ *and*

$$\mathsf{P}_{g(b)}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \mathsf{P}_b(\mathbf{m}, \boldsymbol{\alpha}, \lambda p\,.\,\mathsf{G}(p, \mathbf{m}, \boldsymbol{\alpha})).$$

Proof. For (i), let h be a primitive recursive function such that

$$h(e,a) = \begin{cases} \langle 7, b\rangle, \text{ where } \{b\}(\mathbf{m}, \boldsymbol{\alpha}) \simeq \{(a)_1\}(\mathsf{F}(\mathbf{m}, \boldsymbol{\alpha}), \mathbf{m}, \boldsymbol{\alpha}), \text{ if } a \in N_0; \\ \text{an index } c \text{ such that for all } p, \{c\}(p) \simeq \{e\}(\{a\}(p)), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{otherwise.} \end{cases}$$

By the Primitive Recursion Theorem choose $\bar{e}$ such that $h(\bar{e}, a) = \{\bar{e}\}(a)$ and set $f = \{\bar{e}\}$. A straightforward induction over $N^{k+1,l}$ shows that f satisfies (i). The construction of g is very similar. □

4.11 Lemma. *For all* $\rho < \omega_1$, W_ρ *is effective Borel* $(\Delta^0_{(\omega_1)})$.

Proof. Let ρ be any recursive ordinal and $\delta \in \mathsf{W}$ a recursive function such that $\|\delta\| = \rho + 1$. We shall show the existence of a primitive recursive function f such that for all $r \in \mathrm{Fld}(\delta)$, $f(r) \in N^{0,1}$ and $\mathsf{W}_{|r|_\delta} = \mathsf{P}_{f(r)}$.

Choose $a_0 \in N^{0,1}$ such that $\mathsf{P}_{a_0} = \varnothing$ and let G be a partial recursive function such that

$$G(e,p,r) \simeq \begin{cases} \{e\}(p), & \text{if } \; p <_\delta r; \\ a_0, & \text{otherwise.} \end{cases}$$

By Lemmas 4.5, 4.7, and 4.10, there is a primitive recursive function h defined as follows. If $|r|_\delta = 0$ or 1, $h(e,r)$ is any index such that $\mathsf{P}_{h(e,r)} = \mathsf{W}_{|r|_\delta}$. For all other r, $h(e,r)$ is such that if for all p, $G(e,p,r) \in N^{0,1}$, then also $h(e,r) \in N^{0,1}$ and

$$\mathsf{P}_{h(e,r)}(\gamma) \leftrightarrow \exists p\, \forall q\, \mathsf{P}_{G(e,p,r)}(\gamma \restriction q).$$

By the Primitive Recursion Theorem choose $\bar{e}$ such that $h(\bar{e}, r) = \{\bar{e}\}(r)$ and set $f = \{\bar{e}\}$. We prove by induction on $|r|_\delta$ that f has the required property. This is clear if $|r|_\delta = 0$ or 1 so suppose $|r|_\delta \geqslant 2$ and for all $p <_\delta r$, $f(p) \in N^{0,1}$ and $\mathsf{W}_{|p|_\delta} = \mathsf{P}_{f(p)}$. Then for all p, $G(\bar{e}, p, r) \in N^{0,1}$, so $f(r) = h(\bar{e}, r) \in N^{0,1}$, and for all γ,

$$\begin{aligned} \gamma \in \mathsf{W}_{|r|_\delta} &\leftrightarrow (\exists p <_\delta r) \forall q\,.\, \gamma \restriction q \in \mathsf{W}_{|p|_\delta} \\ &\leftrightarrow \exists p\, \forall q\, \mathsf{P}_{G(\bar{e},p,r)}(\gamma \restriction q) \\ &\leftrightarrow \mathsf{P}_{h(\bar{e},r)}(\gamma) \\ &\leftrightarrow \mathsf{P}_{f(r)}(\gamma). \quad \square \end{aligned}$$

4.12 Theorem. $\mathrm{EfBo} = \Delta_1^1$.

Proof. The inclusion ($\subseteq$) is Lemma 4.9. By Theorem 2.2, for any $\mathsf{R} \in \Delta_1^1$ there exists a $\rho < \omega_1$ such that $\mathsf{R} \leqslant \mathsf{W}_\rho$. By the preceding lemma, $\mathsf{W}_\rho \in \mathrm{EfBo}$ and by Lemma 4.10 so is R. $\square$

To complete the picture of the effective Borel hierarchy, we shall show that the levels are distinct. The method is again essentially the same as before — to find universal relations for each level.

4.13 Lemma. *For all $\rho > 0$, Σ_ρ^0 and Π_ρ^0 are effectively closed under expansion — that is, for all k' and l', there exist primitive recursive functions $f_{k',l'}$ such that for all $(\mathbf{n}, \boldsymbol{\beta}) \in {}^{k',l'}\omega$, all k, l and all $a \in N_\rho^{k,l}$, $f_{k',l'}(a) \in N_\rho^{k+k',l+l'}$ and*

$$\mathsf{P}_{f_{k',l'}(a)}(\mathbf{m}, \mathbf{n}, \boldsymbol{\alpha}, \boldsymbol{\beta}) \leftrightarrow \mathsf{P}_a(\mathbf{m}, \boldsymbol{\alpha}).$$

Proof. For each k' and l', let

$$h_{k',l'}(e, a) = \begin{cases} \langle 7, b\rangle \text{ such that } \{b\}(\mathbf{m}, \mathbf{n}, \boldsymbol{\alpha}, \boldsymbol{\beta}) \simeq \{(a)_1\}(\mathbf{m}, \boldsymbol{\alpha}), \text{ if } a \in N_1; \\ \text{an index } c \text{ such that for all } p, \\ \{c\}(p) \simeq \{e\}(\{a\}(p)), \quad \text{otherwise.} \end{cases}$$

Set $f_{k',l'}(a) = \{\bar{e}\}(a)$, where $\bar{e}$ is chosen by the Primitive Recursion Theorem so that $h_{k',l'}(\bar{e}, a) = \{\bar{e}\}(a)$. $\square$

4.14 Theorem. *For all ρ such that $0 < \rho < \omega_1$ and each (k, l), there exists a relation $\mathsf{U}_\rho^0 \in \Sigma_\rho^0$ which is universal for Σ_ρ^0.*

Proof. Let ρ be any recursive ordinal greater than 0 and fix (k, l). We shall construct a special recursive well-ordering δ such that $\|\delta\| > \rho$ and a primitive recursive function f such that for all $u \in \mathrm{Fld}(\delta)$, $f(u) \in N_{|u|_\delta}^{k+1,l}$ and $\mathsf{P}_{f(u)}$ is universal for $\Sigma_{1+|u|_\delta}^0$.

Let γ be any recursive well-ordering such that $\|\gamma\| > \rho$ and define δ by

$$\langle p, r\rangle \leqslant_\delta \langle q, s\rangle \leftrightarrow p \leqslant_\gamma q \wedge (p = q \rightarrow r \leqslant s).$$

Clearly δ is also a recursive well-ordering with $\|\delta\| > \rho$ — in fact, $\|\delta\| = \omega \cdot \|\gamma\|$ (ordinal multiplication). Furthermore, for all $u \in \mathrm{Fld}(\delta)$, $|u|_\delta$ is a limit ordinal or 0 iff $u = \langle p, 0\rangle$ for some $p \in \mathrm{Fld}(\gamma)$ and $|u|_\delta$ is a successor ordinal iff $u = \langle p, r+1\rangle$ for some $p \in \mathrm{Fld}(\gamma)$ and then $|\langle p, r+1\rangle|_\delta = |\langle p, r\rangle|_\delta + 1$.

Before proceeding to the definition of f, we define an auxiliary primitive recursive function g with the property that for all $u \in \mathrm{Fld}(\delta)$, $g(u) \in N_{|u|_\delta+1}^{1,0}$ and $N_{|u|_\delta}^{k,l} = {\sim}\mathsf{P}_{g(u)}$ — in short, g verifies that $N_\sigma^{k,l} \in \Pi_{1+\sigma+1}^0$. We obtain g via the

Primitive Recursion Theorem and a primitive recursive h defined as follows. If $|u|_\delta = 0$, then $h(e,u)$ is any index in $N_0^{1,0}$ for the recursive set $N_0^{k,l}$. If $u = \langle p, r+1\rangle$, then $h(e,u)$ is chosen by Lemmas 4.5–7, 10 such that whenever $\{e\}(\langle p,r\rangle) \in N_{(\sigma)}^{1,0}$, then $h(e,u) \in N_\sigma^{1,0}$ and

$$\sim\mathsf{P}_{h(e,u)}(a) \leftrightarrow \forall n\, [\{a\}(n)\downarrow \wedge \sim\mathsf{P}_{\{e\}(\langle p,r\rangle)}(\{a\}(n))].$$

If $u = \langle p, 0\rangle$ and $|u|_\delta > 0$, choose $a_1 \in N_0^{1,0}$ such that $\mathsf{P}_{a_1} = \omega$, let

$$G(e,v,u) \simeq \begin{cases} \{e\}(v), & \text{if } \quad v <_\delta u; \\ a_1, & \text{otherwise}; \end{cases}$$

and let $h(e,u)$ be chosen such that whenever for all v, $G(e,v,u) \in N_{(\sigma)}^{1,0}$, then $h(e,u) \in N_\sigma^{1,0}$ and

$$\sim\mathsf{P}_{h(e,u)}(a) \leftrightarrow \forall n\, [\{a\}(n)\downarrow \wedge \exists v \sim\mathsf{P}_{G(e,v,u)}(\{a\}(n))].$$

We leave to the reader the easy proof by induction on $|u|_\delta$ that if $\bar{e}$ is chosen by the Primitive Recursion Theorem such that $h(\bar{e},u) = \{\bar{e}\}(u)$, then $g = \{\bar{e}\}$ is as required.

We now define f via the Recursion Theorem and a primitive recursive function H defined as follows. If $|u|_\delta = 0$, then $H(e,u) = \langle 7, b\rangle$ where $\{b\}(a, \mathbf{m}, \boldsymbol{\alpha}) \simeq \{a\}(\mathbf{m}, \boldsymbol{\alpha})$ for all $(\mathbf{m}, \boldsymbol{\alpha}) \in {}^{k,l}\omega$. If $u = \langle p, r+1\rangle$, then $H(e,u)$ is chosen such that whenever $\{e\}(\langle p,r\rangle) \in N_{(\sigma)}^{k+1,l}$, then $H(e,u) \in N_\sigma^{k+1,l}$ and

$$\mathsf{P}_{H(e,u)}(a, \mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \forall n . \{a\}(n)\downarrow \wedge \exists n \sim \mathsf{P}_{\{e\}(\langle p,r\rangle)}(\{a\}(n), \mathbf{m}, \boldsymbol{\alpha}).$$

If $u = \langle p, 0\rangle$ and $|u|_\delta > 0$, choose $H(e,u)$ such that whenever for all v, $G(e,v,u) \in N_{(\sigma)}^{k+1,l}$, then $H(e,u) \in N_\sigma^{k+1,l}$ and

$$\begin{aligned} \mathsf{P}_{H(e,u)}(a, \mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \forall n . \{a\}(n)\downarrow \wedge \exists n\, \exists v\, [v <_\delta u \wedge \\ \{a\}(n) \in N_{|v|_\delta}^{k,l} \wedge \sim\mathsf{P}_{G(e,v,u)}(\{a\}(n), \mathbf{m}, \boldsymbol{\alpha})]. \end{aligned}$$

Now if $f = \{\bar{e}\}$, where $\bar{e}$ is chosen by the Primitive Recursion Theorem such that $H(\bar{e},u) = \{\bar{e}\}(u)$, then it is straightforward to prove by induction on $|u|_\delta$ that f is as required. $\square$

4.15 Effective Borel Hierarchy Theorem. *For all ρ such that $0 < \rho < \omega_1$,*

(i) $\Sigma_\rho^0 \not\subseteq \Delta_\rho^0$ *and* $\Pi_\rho^0 \not\subseteq \Delta_\rho^0$;

(ii) $\Delta_{\rho+1}^0 \not\subseteq \Sigma_\rho^0 \cup \Pi_\rho^0$.

Proof. Exactly as for Theorem III.1.9. $\square$

In the remainder of this section we sketch another approach to constructing a hierarchy for Δ_1^1. This method applies only to relations on numbers and has no descriptive-set-theoretic analogue. The idea arises from the fact established in Theorem III.1.13 that $R \in \Delta_{r+1}^0$ iff R is recursive in the set D_r obtained by applying the jump operator r times to a recursive set. We would like to extend this by defining a notion of iterating the jump operator a transfinite number of times and show that $R \in \Delta_1^1$ iff R is recursive in D_ρ for some $\rho < \omega_1$. This is not quite possible: although we may take $D_{\rho+1} = D_\rho^{\mathrm{J}}$, there is no canonical way to define D_ρ for limit ordinals ρ. What is missing is a way to "piece together" the sets D_σ ($\sigma < \rho$) to form a set D_ρ which contains essentially just the sum of the information contained in the D_σ's. To get around this difficulty we shall index our sets not by ordinals, but by natural numbers which serve as *notations* for ordinals. This will be arranged in such a way that to any notation for a limit ordinal ρ is canonically associated a recursive function whose values are notations for a sequence of smaller ordinals with limit ρ. Although the sets D_u and D_v assigned to two notations u and v for the same ordinal will not coincide, they will be recursive in each other.

4.16 Definition. *$<_O$ is the smallest subset of $\omega \times \omega$ such that for all a, u, and v,*

(i) $1 <_O 2$;

(ii) *if $u <_O v$, then $v <_O 2^v$;*

(iii) *if $\{a\}$ is a total unary function and for all p $\{a\}(p) <_O \{a\}(p+1)$, then for all p, $\{a\}(p) <_O 3^a$;*

(iv) *if $u <_O v$ and $v <_O w$, then $u <_O w$.*

This is clearly a monotone arithmetic definition and thus has at most ω_1 stages (Theorem 2.15). Furthermore, it is easy to check that $<_O$ is a well-founded partial ordering (Exercise 4.27). We denote the field of $<_O$ by O and assign ordinals $|u|_O$ to elements u of O according to their positions in this ordering. Thus, $|1|_O = 0$, $|2^u|_O = |u|_O + 1$, and $|3^a|_O = \sup^+\{|\{a\}(p)|_O : p \in \omega\}$. Except for $\sigma < \omega$, the σ-th stage of the inductive definition of $<_O$ consists of those u such that $|u|_O \leq \sigma$. In the following we write u^+ for 2^u.

4.17 Definition. For all $u \in O$,

(i) $D_1 = \{0\}$;

(ii) $D_{u^+} = D_u^{\mathrm{J}}$;

(iii) if $u = 3^a$, then $D_u = \{\langle m, p\rangle : m \in D_{\{a\}(p)}\}$.

4.18 Lemma. *For all $u \in O$, $D_u \in \Delta_1^1$.*

Proof. We proceed similarly as for Lemma 4.9. Let

$$V(u, i, m) \leftrightarrow u \in O \wedge ([i = 0 \wedge m \in D_u] \vee [i = 1 \wedge m \notin D_u]).$$

We can define a monotone arithmetical operator Γ such that

$$\bar{\Gamma} = \{(u, i, m) : V(u, i, m)\} \cup \{(u, 2, v) : u <_O v\}$$

by combining definitions 4.16 and 4.17. Once this is done, it follows that V is Π_1^1 and hence that each D_u is Δ_1^1.

The clauses (1)–(4) of the definition of Γ corresponding to 4.16 are straightforward and we leave it to the reader to write them out. Corresponding to 4.17 we have

(5) $(1,0,0) \in \Gamma(R)$; for all $m > 0$, $(1,1,m) \in \Gamma(R)$;

(6) (a) if $(u, w, u^+) \in R$ and $\exists w\, \exists s\, [(\forall n < \lg(s)) . (u, (s)_n, n) \in R \wedge T(a, \langle m \rangle, w, \langle s \rangle)]$

then $(u^+, 0, \langle a, m \rangle) \in \Gamma(R)$;

(b) if $(u, 2, u^+) \in R$ and $\forall w\, \forall s\, [(\forall n < \lg(s))(\forall i . (u, i, n) \in R \rightarrow \rightarrow (s)_n = i) \rightarrow \sim T(a, \langle m \rangle, w, \langle s \rangle)]$

then $(u^+, 1, \langle a, m \rangle) \in \Gamma(R)$;

(c) if n is not of the form $\langle a, m \rangle$, then $(u^+, 1, n) \in \Gamma(R)$;

(7) (a) if for all p, $(\{a\}(p), 2, 3^a) \in R$ and $(\{a\}(p), i, m) \in R$, then $(3^a, i, \langle m, p \rangle) \in \Gamma(R)$;

(b) if n is not of the form $\langle a, m \rangle$, then $(3^a, 1, n) \in \Gamma(R)$. □

To establish that each Δ_1^1 relation on numbers is recursive in some D_u, we shall define primitive recursive functions F and G such that for all $a \in N^{k,0}$, $F(a) \in O$ and P_a has index $G(a)$ from $D_{F(a)}$. The result then follows from Theorem 4.12. We need two technical lemmas.

4.19 Lemma. *There exists a partial recursive function $+_O$ such that for all $u, v \in O$, $u +_O v \in O$, $|u +_O v|_O = |u|_O + |v|_O$, and if $v \neq 1$, $u <_O (u +_O v)$.*

Proof. We define $+_O$ by the Recursion Theorem to satisfy the following conditions:

(1) $u +_O 1 \simeq u$;

(2) $u +_O v^+ \simeq (u +_O v)^+$;

(3) $u +_O 3^b \simeq 3^c$, where $\{c\}(p) \simeq u +_O \{b\}(p)$.

It follows easily by induction over O that $+_O$ has the required properties. □

4.20 Lemma. *There exists a primitive recursive function g such that for all $u, v \in O$, if $|u|_O \leq |v|_O$, then D_u is recursive in D_v with index $g(u, v)$.*

Proof. We shall define g by the Recursion Theorem simultaneously with an auxiliary function f such that for $u, v \in O$,

$$|u|_O < |v|_O \leftrightarrow \{f(v)\}(u, D_{v^+}) \simeq 0,$$

and

$$|u|_O \geqslant |v|_O \leftrightarrow \{f(v)\}(u, D_{v^+}) \simeq 1.$$

To increase legibility, we shall use the abbreviations:

$$A_{v,u} \quad \text{for} \quad \{p : \{g(u, v)\}(p, A) \simeq 0\}$$

and

$$A^{(p)} \quad \text{for} \quad \{m : \langle m, p\rangle \in A\}.$$

Thus if g is as in the statement of the lemma, we have

$(*)$ if $u, v \in O$, $|u|_O \leqslant |v|_O$, and $A = D_v$, then $A_{v,u} = D_u$;

and without any assumption,

$(**)$ if $3^b \in O$ and $A = D_{3^b}$, then for all p, $A^{(p)} = D_{\{b\}(p)}$.

We shall require that f and g satisfy the following conditions: for all u, v, a, b, m, and A,

(1) $\{f(1)\}(u, A) \simeq 1$; if $v \neq 1$, $\{f(v)\}(1, A) \simeq 0$;

(2) (a) $\{f(v^+)\}(u^+, A^{\omega J}) \simeq \{f(v)\}(u, A)$;

(b) $$\{f(v^+)\}(3^a, A^{\omega J}) \simeq \begin{cases} 1, & \text{if } \exists p . \{f(v)\}(\{a\}(p), A) \simeq 1; \\ 0, & \text{otherwise;} \end{cases}$$

(3) $$\{f(3^b)\}(u, A^{\omega J}) \simeq \begin{cases} 0, & \text{if } \exists p . \mathsf{G}(p, u, b, A) \simeq 0; \\ 1, & \text{otherwise,} \end{cases}$$

where $\mathsf{G}(p, u, b, A) \simeq \{f(\{b\}(p))\}(u, (A^{(p+1)})_{\{b\}(p+1), \{b\}(p)^+})$;

(4) $\{g(1, v)\}(m, A) \simeq \mathrm{sg}(m)$;

(5) (a) $\{g(u^+, v^+)\}(m, A) \simeq \{h(g(u, v))\}(m, A)$, where h is a primitive recursive function such that

$$\{h(a)\}(m, A^{\omega J}) = (\lambda n . \{a\}(n, A))^{\omega J}(m)$$

whenever both sides are defined;

(b) $\{g(3^a, v^+)\}(m, A^{\omega}) \simeq \{g(3^a, v)\}(m, A)$;

(6) (a) $\{g(u^+, 3^b)\}(m, A) \simeq \{g(u^+, \{b\}(\bar{p}))\}(m, A^{(\bar{p})})$,

where $\bar{p} \simeq$ "least" $p.\mathrm{G}(p, u^+, b, A) \simeq 0$ (G as in (3));

(b) $\{g(3^a, 3^b)\}(\langle m, p\rangle, A) \simeq \{g(\{a\}(p), 3^b)\}(m, A)$.

To see that such f and g exist, rewrite equations (1)–(6) with the following changes:

on the left-hand sides, replace $f(v)$ by $H(e, 0, 0, v)$

and $g(u, v)$ by $H(e, 1, u, v)$;

on the right-hand sides, replace $f(v)$ by $\{e\}(0, 0, v)$

and $g(u, v)$ by $\{e\}(1, u, v)$.

Then it is not hard to verify that there is a primitive recursive function H which satisfies these rewritten equations. Note that the quantifiers on the right-hand sides of (2)(b) and (3) are accounted for in the changing of A^{ω} to A. By the Primitive Recursion Theorem, there exists an $\bar{e}$ such that $H(\bar{e}, i, u, v) \simeq \{\bar{e}\}(i, u, v)$ and we set $f(v) = \{\bar{e}\}(0, 0, v)$ and $g(u, v) = \{\bar{e}\}(1, u, v)$.

We prove by induction on $|v|$ (dropping the subscript) that f and g have the required properties. If $|v| = 0$ this is obvious, so suppose it is true with v replaced by any w such that $|w| < |v^+|$. Then

(1) $|1| < |v^+|$ is true and $\{f(v^+)\}(1, D_{v^{++}}) \simeq 0$.

(2) (a) if $u^+ \in O$, then $u \in O$, so

$$|u^+| < |v^+| \leftrightarrow |u| < |v| \leftrightarrow \{f(v)\}(u, D_{v^+}) \simeq 0$$
$$\leftrightarrow \{f(v^+)\}(u^+, D_{v^{++}}) \simeq 0.$$

(b) if $3^a \in O$, then for all p, $\{a\}(p) \in O$ and

$$|3^a| < |v^+| \leftrightarrow \forall p . |\{a\}(p)| < |v|$$
$$\leftrightarrow \forall p . \{f(v)\}(\{a\}(p), D_{v^+}) \simeq 0$$
$$\leftrightarrow \{f(v^+)\}(3^a, D_{v^{++}}) \simeq 0.$$

(4) $\{g(1, v^+)\}(m, D_{v^+}) \simeq 0 \leftrightarrow m = 0 \leftrightarrow m \in D_1$.

(5) (a) If $|u^+| \leq |v^+|$, then $|u| \leq |v|$, so

$$m \in D_{u^+} \leftrightarrow m \in (D_u)^{\omega} \leftrightarrow (\lambda n . \{g(u, v)\}(n, D_v))^{\omega}(m) \simeq 0$$
$$\leftrightarrow \{h(g(u, v))\}(m, (D_v)^{\omega}) \simeq 0$$
$$\leftrightarrow \{g(u^+, v^+)\}(m, D_{v^+}) \simeq 0.$$

(b) If $|3^a| \leq |v^+|$, then $|3^a| \leq |v|$, so

$$m \in D_{3^a} \leftrightarrow \{g(3^a, v)\}(m, D_v) \simeq 0$$
$$\leftrightarrow \{g(3^a, v^+)\}(m, D_{v^+}) \simeq 0.$$

Now suppose that the result holds with v replaced by any w such that $|w| < |3^b|$.

(3) We first compute that as for all p, $|\{b\}(p)| < |3^b|$, for any $u \in O$,

$$|u| < |\{b\}(p)| \leftrightarrow \{f(\{b\}(p))\}(u, D_{\{b\}(p)^+}) \simeq 0$$
$$\leftrightarrow \{f(\{b\}(p))\}(u, \lambda n . \{g(\{b\}(p)^+, \{b\}(p+1))\}(n, D_{\{b\}(p+1)})) \simeq 0$$
$$\leftrightarrow \{f(\{b\}(p))\}(u, \lambda n . \{g(\{b\}(p)^+, \{b\}(p+1))\}(n, (D_{3^b})^{(p+1)})) \simeq 0$$
$$\leftrightarrow \{f(\{b\}(p))\}(u, ((D_{3^b})^{(p+1)})_{\{b\}(p+1), \{b\}(p)^+}) \simeq 0$$
$$\leftrightarrow \mathsf{G}(p, u, b, D_{3^b}) \simeq 0.$$

The second equivalence uses the induction hypothesis, the third uses (**).
Then if $u \in O$,

$$|u| < |3^b| \leftrightarrow \exists p . |u| < |\{b\}(p)|$$
$$\leftrightarrow \exists p . \mathsf{G}(p, u, b, D_{3^b}) \simeq 0$$
$$\leftrightarrow \{f(3^b)\}(u, D_{3^b}) \simeq 0.$$

(6) We proceed by induction on $|u|$. For $u = 1$, the result is obvious by (4).

(a) If $|u^+| \leq |3^b|$, then $|u^+| < |3^b|$ so for some (least) $\bar{p}$, $|u^+| < |\{b\}(\bar{p})|$. Then $\mathsf{G}(\bar{p}, u^+, b, D_{3^b}) \simeq 0$ and

$$n \in D_{u^+} \leftrightarrow \{g(u^+, \{b\}(\bar{p}))\}(m, D_{\{b\}(\bar{p})}) \simeq 0$$
$$\leftrightarrow \{g(u^+, \{b\}(\bar{p}))\}(m, (D_{3^b})^{(\bar{p})}) \simeq 0 \qquad \text{(by (**))}$$
$$\leftrightarrow \{g(u^+, 3^b)\}(m, D_{3^b}) \simeq 0.$$

(b) If $|3^a| \leq |3^b|$, then for all p, $|\{a\}(p)| < |3^a| \leq |3^b|$, so

$$\begin{aligned}\langle m,p\rangle \in D_{3^a} &\leftrightarrow m \in D_{\{a\}(p)}\\ &\leftrightarrow \{g(\{a\}(p),3^b)\}(m,D_{3^b})\simeq 0\\ &\leftrightarrow \{g(3^a,3^b)\}(\langle m,p\rangle, D_{3^b})\simeq 0. \quad \square\end{aligned}$$

4.21 Theorem. *For all $R \subseteq {}^k\omega$, R is Δ^1_1 iff R is recursive in D_u for some $u \in O$.*

Proof. We define primitive recursive functions F and G such that for all k and all $a \in N^{k,0}$, $F(a)\in O$ and P_a is recursive in $D_{F(a)}$ with index $G(a)$. As usual, F and G are defined by effective transfinite recursion and we shall give only an informal description of the construction.

If $a \in N_0$, let $F(a)=2$ and $G(a)$ be an index for the semi-recursive set P_a from $D_2=\{0\}^\omega$. If $a \notin N_0$, set $F(a)=(3^c)^+$, where c is an index such that

$$\{c\}(0)\simeq F(\{a\}(0)) \quad \text{and} \quad \{c\}(p+1)\simeq\{c\}(p)+_O F(\{a\}(p+1)).$$

The properties of $+_O$ ensure that if for all p, $F(\{a\}(p))\in O$, then $F(a)\in O$ and $|F(\{a\}(p))|_O<|3^c|_O<|F(a)|_O$. By 4.20 there exists a recursive function γ such that if for all p, $F(\{a\}(p))\in O$, then $D_{F(\{a\}(p))}$ is recursive in D_{3^c} with index $\gamma(p)$. We take $G(a)$ to be an index such that

$$\{G(a)\}(\mathbf{m},D_{F(a)})\simeq\begin{cases}0, & \text{if } \exists p\,.\{G(\{a\}(p))\}(\mathbf{m},\lambda n\,.\{\gamma(p)\}(n,D_{3^c}))\simeq 1,\\ 1, & \text{otherwise.}\end{cases}$$

For the induction step in the proof that F and G are as desired, we have as induction hypothesis that for all p,

$$\begin{aligned}\{G(\{a\}(p))\}(\mathbf{m},\lambda n\,.\{\gamma(p)\}(n,D_{3^c}))\simeq\{G(\{a\}(p))\}(\mathbf{m},D_{F(\{a\}(p))})\simeq 1\\ \leftrightarrow {\sim}\mathsf{P}_{\{a\}(p)}(\mathbf{m}).\end{aligned}$$

Hence

$$\{G(a)\}(\mathbf{m},D_{F(a)})\simeq 0 \leftrightarrow \exists p\,.{\sim}\mathsf{P}_{\{a\}(p)}(\mathbf{m})\leftrightarrow \mathsf{P}_a(\mathbf{m}). \quad \square$$

This characterization provides us with a new hierarchy of the Δ^1_1 relations on numbers known as the hyperarithmetical hierarchy. Let

$$\begin{aligned}&\Sigma^{0\prime}_\rho=\{R : R \text{ is many-one reducible to some } D_u \text{ with } |u|_O\leq\rho\};\\ &\Pi^{0\prime}_\rho=\{R : {\sim}R\in\Sigma^{0\prime}_\rho\},\\ &\Delta^{0\prime}_\rho=\Sigma^{0\prime}_\rho\cap\Pi^{0\prime}_\rho.\end{aligned}$$

If R is a relation recursive in D_u, then both R and $\sim R$ are many-one reducible to $D_u^{OJ} = D_{u^+}$. Hence

$$\Delta_1^1 = \bigcup \{\Delta_\rho^{0\prime} : \rho < \omega_1\}.$$

Furthermore, if $\rho < \sigma < \omega_1$, then $\Delta_\rho^{0\prime} \subsetneqq \Delta_\sigma^{0\prime}$ (Exercise 4.31). By Theorem III.1.13, $\Sigma_\rho^{0\prime} = \Sigma_\rho^0$ for $\rho < \omega$, but this is not in general true for all $\rho < \omega_1$.

4.22–4.32 Exercises

4.22. Formulate and prove a theorem which justifies recursive definitions over the sets $N^{k,l}$ of Definition 4.2.

4.23. Show that for all $\rho < \omega_1$, Σ_ρ^0 and Π_ρ^0 are closed under bounded number quantification ($\exists_<^0$ and $\forall_<^0$).

4.24. Show that for all $\rho < \omega_1$,

$$\Sigma_{\rho+1}^0 = \{\exists^0 \mathsf{Q} : \mathsf{Q} \in \Pi_\rho^0\}.$$

4.25. Give a different proof of Lemma 4.9 by defining by effective transfinite recursion a primitive recursive function f such that for all $a \in N^{k,l}$, $f(a)$ is an index for P_a as a Δ_1^1 relation — that is,

$$\mathsf{U}_1^1(((f(a))_0, \langle \mathbf{m}\rangle, \langle \boldsymbol{\alpha}\rangle) \leftrightarrow \mathsf{P}_a(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \sim \mathsf{U}_1^1(((f(a))_1, \langle \mathbf{m}\rangle, \langle \boldsymbol{\alpha}\rangle).$$

4.26. Show that $N^{k,l}$ (any k and l) and O are Π_1^1 complete.

4.27. Show that $\leqslant_O$ is a well-founded partial ordering.

4.28. Show that there exists a primitive recursive function h such that for any $u \in O$ and any d, if for all p, $\{d\}(p) \in O$, then $h(d) \in O$ and for all p, $|\{d\}(p)|_O < |h(d)|_O$.

4.29. Show that the ordinals $|u|_O$ for $u \in O$ are exactly the ordinals less than ω_1. (For each $u \in O$, show that the restriction of $<_O$ to $v <_O u$ is a well-ordering of type $|u|_O$. For the other direction, let $\gamma \in \mathsf{W}$ be recursive. Construct as in the proof of Theorem 4.14 a recursive $\delta \in \mathsf{W}$ such that $\|\delta\| = \omega \cdot \|\gamma\|$ and use effective transfinite recursion to define a recursive function f such that for all $p \in \mathrm{Fld}\,\delta$, $f(p) \in O$ and $|f(p)|_O \geqslant |p|_\delta$.)

4.30. Give a new proof of Corollary 2.22 by showing that for all $u \in O$, D_u is implicitly Π_2^0. (Construct by effective transfinite recursion a partial recursive function F such that for all $u \in O$ and all A

$$A = D_u \leftrightarrow \sim \mathsf{U}_2^0(F(u), \langle\ \ \rangle, \langle A \rangle).$$

Use an effective version of Exercise III.4.25 for defining $F(2^u)$ from $F(u)$.

4.31. Show that for all $\rho < \sigma < \omega_1$, $\Delta_\rho^{0\prime} \subsetneq \Delta_\sigma^{0\prime}$.

4.32. Recall that $U_{(\omega)}^0 = \{\langle r, a, \mathbf{m}\rangle : U_r^0(a, \langle \mathbf{m}\rangle)\}$. Show that for any R, the following are equivalent:

(i) $R \in \Sigma_\omega^0$;

(ii) $R(\mathbf{m}) \leftrightarrow \exists p . \langle f(p), a, \mathbf{m}\rangle \in U_{(\omega)}^0$ for some a and some primitive recursive f;

(iii) $R \leqslant \bar{\Gamma}$ for some Σ_1^0 inductive operator Γ.

(Cf. Theorems, III.3.6–7). How does this compare with $\Sigma_\omega^{0\prime}$?

4.33 Notes. The idea of an effective version of the Borel hierarchy was developed by Addison in his thesis, Addison [1954], and announced in Addison [1955], although at that time not all of the details of the transfinite levels had been worked out. Indeed, they were never published and may not have been completely written down until the Spring of 1964 when Addison conducted a seminar on the material at Berkeley. The hyperarithmetical hierarchy based on O and the set D_u and Theorem 4.21 are due to Kleene [1955b], but Lemma 4.20 is from Spector [1955].

5. Cardinality, Measurability and Category

One of the benefits for analysts in dealing with constructively defined sets and relations is that they are more likely to be "well-behaved". We consider here some of the pleasant properties of $\mathbf{\Sigma}_1^1$ and $\mathbf{\Pi}_1^1$ relations. To simplify some of the arguments we shall deal explicitly only with subsets of ${}^\omega\omega$ (relations of rank $(0,1)$), but all of the results hold also for relations of arbitrary rank.

5.1 Theorem. *Every $\mathbf{\Pi}_1^1$ or $\mathbf{\Sigma}_1^1$ set is both the union of an $\aleph_1$-sequence of Borel sets and the intersection of an $\aleph_1$-sequence of Borel sets.*

Proof. If suffices to prove the result for $A \in \mathbf{\Pi}_1^1$. Let P be a closed-open relation such that $\alpha \in \mathsf{A} \leftrightarrow \forall \beta\, \exists n\, \mathsf{P}(\bar{\beta}(n), \alpha)$, and $\trianglelefteq$ and $\trianglelefteq_\alpha^{\mathsf{P}}$ the relations defined in the proof of Theorem 1.1. For any t set

$$u \trianglelefteq_\alpha^{\mathsf{P},t}\, v \leftrightarrow u \trianglelefteq_\alpha^{\mathsf{P}} v \wedge v \trianglelefteq t$$

and

$$\mathsf{A}_\rho^t = \{\alpha : \trianglelefteq_\alpha^{\mathsf{P},t} \text{ has order-type} \leqslant \rho\}.$$

Thus as in the proof of Theorem 1.1,

$$\alpha \in \mathsf{A} \leftrightarrow \trianglelefteq_{\alpha}^{\mathsf{P},\langle\rangle} \text{ is a well-ordering}$$
$$\leftrightarrow (\exists \rho < \aleph_1)\, \alpha \in \mathsf{A}_{\rho}^{\langle\rangle}.$$

For any $\rho < \aleph_1$, choose γ such that $\|\gamma\| = \rho$ and let F be a continuous functional such that

$$\mathsf{F}(\langle u, v\rangle, t, \alpha) = \begin{cases} 0, & \text{if } u \trianglelefteq_{\alpha}^{\mathsf{P},t} v; \\ 1, & \text{otherwise.} \end{cases}$$

Then

$$\alpha \in \mathsf{A}_{\rho}^{t} \leftrightarrow \mathsf{F}[t, \alpha] \leqslant_{\Sigma} \gamma \leftrightarrow \mathsf{F}[t, \alpha] \leqslant_{\Pi} \gamma$$

so that for all t, A_{ρ}^{t} is Δ_1^1, i.e., Borel. In particular, $\mathsf{A} = \bigcup\{\mathsf{A}_{\rho}^{\langle\rangle} : \rho < \aleph_1\}$ is a representation of A as a union of $\aleph_1$ Borel sets.

For the intersection, let

$$\mathsf{B}_{\rho} = \mathsf{A}_{\rho}^{\langle\rangle} \cup \bigcup\{\mathsf{A}_{\rho+1}^{t} \sim \mathsf{A}_{\rho}^{t} : t \in \omega\}.$$

Again B_{ρ} is clearly Borel for all countable ρ, and we claim that $\mathsf{A} = \bigcap\{\mathsf{B}_{\rho} : \rho < \aleph_1\}$. Suppose first that $\alpha \in \mathsf{A}$, so that $\trianglelefteq_{\alpha}^{\mathsf{P},\langle\rangle}$ is a well-ordering, say of order-type σ. For any $\rho < \sigma$ there exists a t such that $\trianglelefteq_{\alpha}^{\mathsf{P},t}$ has order-type $\rho + 1$ so that $\alpha \in \mathsf{A}_{\rho+1}^{t} \sim \mathsf{A}_{\rho}^{t} \subseteq \mathsf{B}_{\rho}$. For $\rho \geqslant \sigma$, $\alpha \in \mathsf{A}_{\rho}^{\langle\rangle} \subseteq \mathsf{B}_{\rho}$. Hence $\alpha \in \mathsf{B}_{\rho}$ for all $\rho < \aleph_1$.

For the converse, suppose $\alpha \in \mathsf{B}_{\rho}$ for all $\rho < \aleph_1$. For each t let

$$\sigma_t = \begin{cases} \text{order-type of } \trianglelefteq_{\alpha}^{\mathsf{P},t}, \text{ if this is an ordinal;} \\ 0, \qquad\qquad \text{otherwise;} \end{cases}$$

and set $\bar{\rho} = \sup^{+}\{\sigma_t : t \in \omega\}$. Then $\bar{\rho} < \aleph_1$ and for all t, $\trianglelefteq_{\alpha}^{\mathsf{P},t}$ does not have order-type $\bar{\rho} + 1$ so $\alpha \notin \mathsf{A}_{\rho+1}^{t} \sim \mathsf{A}_{\rho}^{t}$. Since $\alpha \in \mathsf{B}_{\bar{\rho}}$, it follows that $\alpha \in \mathsf{A}_{\bar{\rho}}^{\langle\rangle} \subseteq \mathsf{A}$. □

5.2 Corollary. *Every Σ_2^1 set is the union of an $\aleph_1$-sequence of Borel sets.*

Proof. Suppose A is Σ_2^1 and $\alpha \in \mathsf{A} \leftrightarrow \exists\beta\, \mathsf{S}(\alpha, \beta)$ for some $\mathsf{S} \in \Pi_1^1$. By the preceding theorem (extended to relations) there exist Borel relations S_{σ} such that $\mathsf{S} = \bigcup\{\mathsf{S}_{\sigma} : \sigma < \aleph_1\}$. Then

$$\alpha \in \mathsf{A} \leftrightarrow \exists\beta\, (\exists\sigma < \aleph_1) \mathsf{S}_{\sigma}(\alpha, \beta)$$
$$\leftrightarrow (\exists\sigma < \aleph_1)\, \exists\beta\, \mathsf{S}_{\sigma}(\alpha, \beta).$$

By the theorem again there exist Borel sets $A_{\sigma\tau}$ such that

$$\{\alpha : \exists\beta\, S_\sigma(\alpha, \beta)\} = \bigcup\{A_{\sigma\tau} : \tau < \aleph_1\}$$

and thus $A = \bigcup\{A_{\sigma\tau} : \sigma, \tau < \aleph_1\}$. Since $\aleph_1 \times \aleph_1$ is of power $\aleph_1$, there is a pairing function on $\aleph_1$ with (say) inverses $(\ )_0$ and $(\ )_1$. Then $A = \bigcup\{A_{(\sigma)_0(\sigma)_1} : \sigma < \aleph_1\}$ as required. □

The converse of this Corollary is not provable without some additional set-theoretic assumption, since if $2^{\aleph_0} = \aleph_1$, then *every* set is the union of $\aleph_1$ singletons (and singletons are Borel). (Cf. Exercise V.3.25).

We turn next to the measurability of Σ_1^1 and Π_1^1 relations. We have discussed only Lebesgue measure on ${}^\omega\omega$, but the proof will depend only on the following four properties of the measure:

(1) The class of measurable sets contains the open sets and is closed under complementation and countable union and intersection;

(2) the union of countably many sets of measure 0 has measure 0;

(3) every subset of a set of measure 0 is measurable and has measure 0;

(4) in any family of $\aleph_1$ pairwise disjoint measurable sets, at most countably many sets have positive measure.

Properties (1) and (2) are direct consequences of the countable additivity of a measure and (3) is the completeness property. For (4), if $\langle A_\sigma : \sigma < \aleph_1\rangle$ is a family of pairwise disjoint measurable sets and $x_\sigma = \mathrm{mes}(\bigcup\{A_\tau : \tau < \sigma\})$, then $x_0 \leqslant x_1 \leqslant \cdots \leqslant x_\sigma \leqslant x_{\sigma+1} \ldots$. The strict inequality, $x_\sigma < x_{\sigma+1}$, holds just in case $\mathrm{mes}(A_\sigma) > 0$ and in this case there is a rational number y_σ such that $x_\sigma < y_\sigma < x_{\sigma+1}$. Clearly this can happen for at most countably many σ.

5.3 Theorem. *Every* Σ_1^1 *or* Π_1^1 *set is measurable with respect to any measure which satisfies* (1)–(4).

Proof. As the complement of a measurable set is measurable, it suffices to consider $A \in \Pi_1^1$. Let A_ρ^t be as in the proof of Theorem 5.1 and for each t consider the family $\langle A_{\rho+1}^t \sim A_\rho^t : \rho < \aleph_1\rangle$. This is a family of $\aleph_1$ pairwise disjoint Borel sets (hence measurable sets by (1)) so all but countably many have measure 0. Hence there is an ordinal σ_t such that for all $\rho \geqslant \sigma_t$, $\mathrm{mes}(A_{\rho+1}^t \sim A_\rho^t) = 0$. Let $\bar\rho = \sup^+\{\sigma_t : t \in \omega\}$. Then $\bar\rho < \aleph_1$ and for all t, $\mathrm{mes}(A_{\bar\rho+1}^t \sim A_{\bar\rho}^t) = 0$. In the proof of Theorem 5.1 we showed $A \subseteq B_{\bar\rho}$. Hence

$$A = A_{\bar\rho}^{\langle\,\rangle} \cup (A \cap \bigcup\{A_{\bar\rho+1}^t \sim A_{\bar\rho}^t : t \in \omega\}).$$

Thus A is the union of $A_{\bar\rho}^{\langle\,\rangle}$, which is Borel, hence measurable, and a subset of a countable union of sets of measure 0, which is measurable by (2) and (3). □

This result cannot be extended even to Δ_2^1 relations; we shall indicate a proof of this in §V.2. Of course, countable unions and intersections of $\mathbf{\Sigma}_1^1$ and $\mathbf{\Pi}_1^1$ relations are measurable. We shall discuss further extensions of this sort in § V.4.

We turn next to the topological analogue of measurability, the Baire property.

5.4 Definition. A set A has the *Baire property* iff there exists an open set B such that both $\mathsf{A} \sim \mathsf{B}$ and $\mathsf{B} \sim \mathsf{A}$ are meager.

We aim to show that all $\mathbf{\Sigma}_1^1$ and $\mathbf{\Pi}_1^1$ sets also have the Baire property. The proof is nearly identical to that of Theorem 5.3 once we establish the following analogues of (1)–(4):

(1′) The class of sets which have the Baire property contains the open sets and is closed under complementation and countable union and intersection;

(2′) the union of countably many meager sets is meager;

(3′) every subset of a meager set is meager and has the Baire property;

(4′) in any family of pairwise disjoint sets which have the Baire property, at most countably many sets are non-meager.

We denote the topological closure of a set A by $\bar{\mathsf{A}}$. Note that $\alpha \in \bar{\mathsf{A}}$ iff, for all n, $[\bar{\alpha}(n)] \cap \mathsf{A} \neq \varnothing$. A is dense in an interval $[s]$ iff $[s] \subseteq \bar{\mathsf{A}}$. Hence A is nowhere dense iff no interval is included in $\bar{\mathsf{A}}$ iff $\sim\bar{\mathsf{A}}$ is dense iff $\overline{\sim\bar{\mathsf{A}}} = {}^{\omega}\omega$. Properties (2′) and (3′) are immediate from the definitions.

5.5 Lemma. *For any open set* B, $\bar{\mathsf{B}} \sim \mathsf{B}$ *is nowhere dense.*

Proof. If B is open, then $\bar{\mathsf{B}} \sim \mathsf{B}$ is closed so by the preceding remarks, it suffices to show that $\sim(\bar{\mathsf{B}} \sim \mathsf{B})$ is dense. This is clear from:

$$\overline{\sim(\bar{\mathsf{B}} \sim \mathsf{B})} = \overline{\sim\bar{\mathsf{B}} \cup \mathsf{B}} = \overline{\sim\bar{\mathsf{B}}} \cup \bar{\mathsf{B}} \supseteq \sim\bar{\mathsf{B}} \cup \bar{\mathsf{B}} = {}^{\omega}\omega. \quad \square$$

5.6 Lemma. *The class of sets which have the Baire property is closed under complementation and countable union and intersection.*

Proof. Suppose first that A has the Baire property and B is an open set such that $\mathsf{A} \sim \mathsf{B}$ and $\mathsf{B} \sim \mathsf{A}$ are both meager. Then $\sim\bar{\mathsf{B}}$ is open,

$$(\sim\mathsf{A}) \sim (\sim\bar{\mathsf{B}}) = \bar{\mathsf{B}} \sim \mathsf{A} \subseteq (\mathsf{B} \sim \mathsf{A}) \cup (\bar{\mathsf{B}} \sim \mathsf{B}),$$

and

$$(\sim\bar{\mathsf{B}}) \sim (\sim\mathsf{A}) = \mathsf{A} \sim \bar{\mathsf{B}} \subseteq \mathsf{A} \sim \mathsf{B}.$$

By Lemma 5.5 both of these sets are meager and thus $\sim\mathsf{A}$ also has the Baire property.

Now suppose that for all n, A_n has the Baire property and B_n is an open set such that both $\mathsf{A}_n \sim \mathsf{B}_n$ and $\mathsf{B}_n \sim \mathsf{A}_n$ are meager. Then $\bigcup\{\mathsf{B}_n : n \in \omega\}$ is open,

$$\bigcup\{\mathsf{A}_n : n \in \omega\} \sim \bigcup\{\mathsf{B}_n : n \in \omega\} \subseteq \bigcup\{\mathsf{A}_n \sim \mathsf{B}_n : n \in \omega\},$$

and

$$\bigcup\{\mathsf{B}_n : n \in \omega\} \sim \bigcup\{\mathsf{A}_n : n \in \omega\} \subseteq \bigcup\{\mathsf{B}_n \sim \mathsf{A}_n : n \in \omega\},$$

so it follows by (2′) and (3′) that $\bigcup\{\mathsf{A}_n : n \in \omega\}$ also has the Baire property. □

Since all open sets obviously have the Baire property, this establishes (1′). It follows immediately that all Borel sets have the Baire property.

5.7 Lemma. *For any set* A *which has the Baire property and any* s, *there is some* $t \supseteq s$ *such that one of* $\mathsf{A} \cap [t]$ *and* $(\sim\mathsf{A}) \cap [t]$ *is meager.*

Proof. Suppose that A has the Baire property and B is an open set such that $\mathsf{A} \sim \mathsf{B}$ and $\mathsf{B} \sim \mathsf{A}$ are meager. For any set C let

$$\mathsf{C}^* = \{\alpha : \text{for all } n, \mathsf{C} \cap [\bar{\alpha}(n)] \text{ is not meager}\}.$$

Clearly $\mathsf{C}^* \subseteq \bar{\mathsf{C}}$. Since the meagerness of $\mathsf{A} \cap [\bar{\alpha}(n)]$ will not be affected by the addition or removal of a meager set, $\mathsf{A}^* = \mathsf{B}^*$ and $(\sim\mathsf{A})^* = (\sim\mathsf{B})^*$. Hence

$$\mathsf{A}^* \cap (\sim\mathsf{A})^* = \mathsf{B}^* \cap (\sim\mathsf{B})^* \subseteq \bar{\mathsf{B}} \cap \overline{\sim\mathsf{B}} \subseteq \bar{\mathsf{B}} \sim \mathsf{B},$$

which is nowhere dense by Lemma 5.5. In particular, for every s there exists a function $\alpha \in [s]$ such that either $\alpha \notin \mathsf{A}^*$ or $\alpha \notin (\sim\mathsf{A})^*$. If $\alpha \notin \mathsf{A}^*$, then for some n, $\mathsf{A} \cap [\bar{\alpha}(n)]$ is meager, and clearly n may be chosen such that $\bar{\alpha}(n) \supseteq s$. If $\alpha \notin (\sim\mathsf{A})^*$, then for some n, $(\sim\mathsf{A}) \cap [\bar{\alpha}(n)]$ is meager. □

5.8 Lemma. *For any* A, *if* A *is non-meager and has the Baire property, then for some* s, $(\sim\mathsf{A}) \cap [s]$ *is meager.*

Proof. Suppose that A has the Baire property but for all s, $(\sim\mathsf{A}) \cap [s]$ is non-meager. Then by the preceding lemma, for every s there exists a $t \supseteq s$ such that $\mathsf{A} \cap [t]$ is meager. Let $\mathsf{C} = \bigcup\{[t] : \mathsf{A} \cap [t] \text{ is meager}\}$. Clearly $\mathsf{A} \cap \mathsf{C}$ is meager. But $\sim\mathsf{C}$ is nowhere dense so $\mathsf{A} \cap (\sim\mathsf{C})$ is also meager and thus A is meager. □

5.9 Lemma. *In any family of pairwise disjoint sets which have the Baire property, at most countably many sets are non-meager.*

Proof. Let X be a family of pairwise disjoint sets which have the Baire property and Y the subfamily of X consisting of all non-meager sets. By the preceding lemma the function

$$\varphi(\mathsf{A}) = \text{least } s.(\sim\mathsf{A}) \cap [s] \text{ is meager}$$

is defined for all $\mathsf{A} \in Y$. It will suffice to show that φ is one-one as $\operatorname{Im}\varphi \subseteq \omega$ and hence is countable. Suppose $\varphi(\mathsf{A}) = \varphi(\mathsf{B}) = s$. Then $(\sim\mathsf{A}) \cap [s]$ and $(\sim\mathsf{B}) \cap [s]$ are both meager. Since $\mathsf{A} \cap \mathsf{B} = \varnothing$, the union of these sets is $[s]$, which is thus meager in contradiction with the Baire Category Theorem (I.2.2). □

5.10 Theorem. *Every* $\mathbf{\Sigma}_1^1$ *or* $\mathbf{\Pi}_1^1$ *set has the Baire property.*

Proof. Follow the proof of Theorem 5.3 substituting "has the Baire property" for "is measurable" and "is meager" for "has measure 0". □

Finally, we consider the question of the cardinality of $\mathbf{\Sigma}_1^1$ and $\mathbf{\Pi}_1^1$ sets. It is well known that the usual axioms of set theory, even including the Axiom of Choice, do not determine the power of the continuum. That is, it is consistent with these axioms (assuming that they alone are consistent) either that $2^{\aleph_0} = \aleph_1$ or that there are one or more distinct cardinals between $\aleph_0$ and $2^{\aleph_0}$. What we shall show is that these intermediate cardinalities cannot be realized by $\mathbf{\Sigma}_1^1$ sets nor, except for $\aleph_1$, by $\mathbf{\Sigma}_2^1$ sets.

5.11 Lemma. *For any uncountable set* $\mathsf{A} \subseteq {}^{\omega}\omega$, *there exist* s_0 *and* s_1 *such that* $[s_0] \cap [s_1] = \varnothing$ *and both* $\mathsf{A} \cap [s_0]$ *and* $\mathsf{A} \cap [s_1]$ *are uncountable.*

Proof. Let A be uncountable and set

$$B = \{s : s \in \mathrm{Sq} \wedge \mathsf{A} \cap [s] \text{ is uncountable}\}.$$

Clearly $\langle\ \rangle \in B$. Since $\mathsf{A} \cap [s] = \bigcup\{\mathsf{A} \cap [s * \langle n \rangle] : n \in \omega\}$, if $\mathsf{A} \cap [s]$ is uncountable, so is some $\mathsf{A} \cap [s * \langle n \rangle]$. Hence there there exists a unique function β such that for all k,

$$\beta(k) = \text{least } n[\bar{\beta}(k) * \langle n \rangle \in B].$$

Suppose that the conclusion of the lemma is false. Then for every s, if $\beta \notin [s]$, then $\mathsf{A} \cap [s]$ is countable. But

$$\mathsf{A} \subseteq \bigcup\{\mathsf{A} \cap [s] : \beta \notin [s]\} \cup \{\beta\}$$

and thus A is countable, contrary to assumption. □

At this point the reader should attempt to show for himself that any uncountable closed set has power $2^{\aleph_0}$. The proof of the following theorem is merely a two-dimensional version of this construction.

5.12 Theorem. *For all* $\mathsf{A} \in \Sigma_1^1$, *if* A *is uncountable, then* A *has power* $2^{\aleph_0}$.

Proof. Let A be an uncountable Σ_1^1 set. Obviously the power of A is at most $2^{\aleph_0}$ so it suffices to find a subset of A of power $2^{\aleph_0}$. Let R be a relation such that

$$\alpha \in \mathsf{A} \leftrightarrow \exists\beta\, \forall n\, R(\bar{\alpha}(n), \bar{\beta}(n)),$$

and

$$R(s, t) \wedge s' \subseteq s \wedge t' \subseteq t \to R(s', t').$$

By Lemma 5.11 choose s_0 and s_1 such that $\mathsf{A} \cap [s_0]$ and $\mathsf{A} \cap [s_1]$ are both uncountable and $[s_0] \cap [s_1] = \varnothing$. For $i = 0$ and 1 we may represent $\mathsf{A} \cap [s_i]$ in the form

$$\alpha \in \mathsf{A} \cap [s_i] \leftrightarrow \alpha \in [s_i] \wedge \exists t\, \exists \beta\, [\lg(t) = \lg(s_i) \wedge \beta \in [t] \wedge \forall n\, R(\bar{\alpha}(n), \bar{\beta}(n))].$$

As there are only countably many t of length $\lg(s_i)$, there is at least one, say t_i, such that

$$\mathsf{A}_i = \{\alpha : \alpha \in [s_i] \wedge (\exists \beta \in [t_i]) \forall n\, R(\bar{\alpha}(n), \bar{\beta}(n))\}$$

is uncountable.

For the next stage, choose for $i = 0$ and 1, $s_{i,0}$ and $s_{i,1}$ such that $\mathsf{A}_i \cap [s_{i,j}]$ is uncountable, and then $t_{i,0}$ and $t_{i,1}$ extending t_i such that

$$\mathsf{A}_{i,j} = \{\alpha : \alpha \in [s_{i,j}] \wedge (\exists \beta \in [t_{i,j}]) \forall n\, R(\bar{\alpha}(n), \bar{\beta}(n))\}$$

is uncountable. Continuing in this way, we define for each (code for a) finite sequence u of 0's and 1's, sequences s_u and t_u with the following properties:

(1) $R(s_u, t_u)$;

(2) $u \subseteq v \to s_u \subseteq s_v$ and $t_u \subseteq t_v$;

(3) $s_u \subseteq s_v \to u \subseteq v$;

(4) $\lg(s_u) = \lg(t_u) \geqslant \lg(u)$.

For each $\gamma \in {}^{\omega}2$, let α_γ be the limit of the sequences $s_{\bar{\gamma}(n)}$ and β_γ the limit of the $t_{\bar{\gamma}(n)}$ — that is, $\alpha_\gamma(n) = (s_{\bar{\gamma}(n+1)})_n$ and $\beta_\gamma(n) = (t_{\bar{\gamma}(n+1)})_n$. From (3) it follows that if $\gamma \neq \delta$, then $\alpha_\gamma \neq \alpha_\delta$, so the mapping $\gamma \mapsto \alpha_\gamma$ is 1–1. Hence $\mathsf{B} = \{\alpha_\gamma : \gamma \in {}^{\omega}2\}$ has the same power as ${}^{\omega}2$, that is, $2^{\aleph_0}$.

We claim that $\mathsf{B} \subseteq \mathsf{A}$. Note first that for every γ and n, $R(s_{\bar{\gamma}(n)}, t_{\bar{\gamma}(n)})$ so that also $R(\bar{\alpha}_\gamma(n), \bar{\beta}_\gamma(n))$. In particular, for every γ, $\exists\beta\, \forall n\, R(\bar{\alpha}_\gamma(n), \bar{\beta}(n))$ and thus $\alpha_\gamma \in \mathsf{A}$. $\square$

5.13 Corollary. *For all* $\mathsf{A} \in \mathbf{\Sigma}^1_2$, *if* A *is uncountable, then* A *has power either* $\aleph_1$ *or* $2^{\aleph_0}$.

Proof. By Corollary 5.2, every $\mathbf{\Sigma}^1_2$ set A is the union of $\aleph_1$ Borel sets A_ρ $(\rho < \aleph_1)$. The sets A_ρ may clearly be chosen pairwise disjoint. By the preceding theorem, each A_ρ is either countable or of power $2^{\aleph_0}$. There are three possible cases:

(1) for some ρ, A_ρ has power $2^{\aleph_0}$: then A has power $2^{\aleph_0}$;

(2) A_ρ is countable for all ρ and A_ρ is non-empty for uncountably many ρ: then A has power $\aleph_1$;

(3) A_ρ is countable for all ρ but A_ρ is non-empty for only countably many ρ: then A is countable. $\square$

By Theorem 3.13 all of the theorems of this section hold also for $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$ and $\mathbf{\Sigma}^1_2$ subsets of the real interval $(0, 1)$.

5.14–5.16 Exercises

5.14. Is the following "lightface" version of Theorem 5.1 true: every Σ^1_1 or Π^1_1 set is both the union of an ω_1-sequence of Δ^1_1 sets and the intersection of an ω_1-sequence of Δ^1_1 sets? Consider separately sets of functions and sets of numbers.

5.15. Show that for any α, $\{\beta : \alpha \in \Delta^1_1[\beta]\}$ has measure either 0 or 1 (use Exercise I.2.10).

5.16 (Harrison). Show that for any Σ^1_1 set A, if $A \not\subseteq \Delta^1_1$, then A is of power $2^{\aleph_0}$. (Apply a variant of the technique used for Theorem 5.12 to the Σ^1_1 set $\mathsf{A} \sim \Delta^1_1$.) Conclude that there is no largest countable Σ^1_1 subset of ${}^\omega\omega$. Note that the relativized version of this result implies Theorem 5.12.

5.17 Notes. An excellent and thorough account of the analogies and similarities between the theories of measure and category may be found in Oxtoby [1971].

As with many of the results of this chaptr, much effort went into attempts to extend Theorems 5.3, 5.10, and 5.12 to higher levels of the analytical and projective hierarchies. The attempts were futile because the extensions are independent of ZFC. Indeed, this was forseen as early as Luzin [1930] where we read as the final paragraph:

"Ou bien, les problèmes indiqués sur les ensembles projectifs (of measure, category, and power) resterons à jamais sans solutions augmentés de quantité de problèmes nouveaux aussi naturels et aussi inabordables. Dans ce cas il est clair que le jour serait venu de réformer nos idées sur le continu arithmétique."

In §§ V.2–3 we consider the situation under the additional hypotheses of Constructibility or Projective Determinacy. The hypothesis that there exist a measurable cardinal has the somewhat surprising effect of pushing the results of this section exactly one level further: all $\mathbf{\Sigma}^1_2$ sets are measurable, have the Baire property, and satisfy the Continuum Hypothesis (see Solovay [1969] and Shoenfield [1971b]).

6. Continuous Images

We have explored in §§1 and 2 the analogy between the classes Σ^0_1 and Π^1_1. There are, however, some ways in which Σ^0_1 resembles Σ^1_1. Consider the following two facts: for any non-empty $A \subseteq \omega$ and $\mathsf{A} \subseteq {}^{\omega}\omega$,

(1) $A \in \Sigma^0_1$ iff A is the image of a recursive set under a recursive function from ω into ω,

(1′) $\mathsf{A} \in \mathbf{\Sigma}^1_1$ iff A is the image of a $\begin{Bmatrix}\text{Borel}\\ \text{closed}\end{Bmatrix}$ set under a continuous functional from ${}^{\omega}\omega$ into ${}^{\omega}\omega$.

(1) is just a slight variant of part of Theorem II.4.15 and (1′) follows from Lemma 6.1 below and the fact that projection is a continuous functional together with the relativized version of Lemma III.2.8. We shall investigate here corresponding analogues for the following related facts:

(2) $A \in \Sigma^0_1$ and $A \neq \varnothing \leftrightarrow A$ is the image of a total recursive function;

(3) $A \in \Sigma^0_1 \leftrightarrow A$ is the image of a recursive set under a one–one recursive function;

(4) $A \in \Sigma^0_1$ and A is infinite $\leftrightarrow A$ is the image of a total one–one recursive function (Exercise II.4.30);

(5) $A \in \Sigma^0_1 \leftrightarrow A$ is the domain of a partial function with recursive graph.

We shall use letters θ, φ, χ, and ψ to denote (partial) functionals from ${}^{\omega}\omega$ into ${}^{\omega}\omega$. Such a functional is continuous iff $\theta^{-1}([s])$ is an open set for every interval $[s]$. Note that it is *not* true that the graph of a continuous functional θ is a closed-open subset of ${}^{\omega}\omega \times {}^{\omega}\omega$ as clearly no such functional has open graph.

6.1 Lemma. *For any θ, if $\mathsf{F}(p, \alpha) \simeq \theta(\alpha)(p)$, then θ is partial continuous iff F is partial continuous.*

Proof. Suppose first that F is partial continuous. Then

$$\theta^{-1}([s]) = \{\alpha : (\forall p < \lg(s))\mathsf{F}(p, \alpha) = (s)_p\},$$

and thus $\theta^{-1}([s])$ is a finite intersection of open sets and is open. On the other hand, if θ is partial continuous,

$$\mathsf{F}^{-1}(\{n\}) = \bigcup\{\{p\} \times \theta^{-1}(\{\alpha : \alpha(p) = n\}) : p \in \omega\},$$

so $\mathsf{F}^{-1}(\{n\})$ is open and thus F is partial continuous. $\square$

We first establish an analogue of (2). The following lemma is a refinement of Lemma 5.11.

6.2 Lemma. *For any uncountable set* $\mathsf{A} \subseteq {}^{\omega}\omega$, *there exists an infinite set* $A \subseteq \mathrm{Sq}$ *such that*

(i) *for all* $s \in A$, $\mathsf{A} \cap [s]$ *is uncountable;*
(ii) *for all* $s, t \in A$, *if* $s \neq t$, *then* $[s] \cap [t] = \varnothing$;
(iii) $\mathsf{A} \sim \bigcup\{\mathsf{A} \cap [s] : s \in A\}$ *is countable.*

Proof. Let B and β be defined as in the proof of Lemma 5.11. For each k set

$$A_k = \{s : s \in B \wedge \lg(s) = k + 1 \wedge (\forall i < k)[(s)_i = \beta(i)] \wedge \beta(k) < (s)_k\},$$

and

$$A = \bigcup\{A_k : k \in \omega\}.$$

By definition $A \subseteq B$ and thus satisfies (i). For (ii), suppose $s, t \in A$ and $s \neq t$. If s and t belong to the same A_k, then $(s)_k \neq (t)_k$ so $[s] \cap [t] = \varnothing$. Otherwise, for some $k < l$ (say), $s \in A_k$ and $t \in A_l$, so $(t)_k = \beta(k) < (s)_k$ and again $[s] \cap [t] = \varnothing$. Hence (ii) is satisfied. For (iii), suppose $\alpha \in \mathsf{A} \sim \bigcup\{\mathsf{A} \cap [s] : s \in A\}$. Either $\alpha = \beta$ or else for some k, $\bar{\alpha}(k) = \bar{\beta}(k)$ but $\alpha(k) \neq \beta(k)$. If $\alpha(k) < \beta(k)$, then $\bar{\alpha}(k+1) \notin B$ by the definition of β. If $\beta(k) < \alpha(k)$, then again $\bar{\alpha}(k+1) \notin B$, as otherwise $\bar{\alpha}(k+1) \in A$, contrary to the assumption that α belongs to no $\mathsf{A} \cap [s]$ with $s \in A$. Hence

$$\alpha \in \bigcup\{\mathsf{A} \cap [s] : \mathsf{A} \cap [s] \text{ is countable}\} \cup \{\beta\}.$$

This set is countable so (iii) is established.

Finally, suppose that A were finite so that for some k, $A_l = \varnothing$ for all $l > k$. Then for all s such that $\lg(s) > k$, if $\mathsf{A} \cap [s]$ is uncountable, then $\beta \in [s]$. But

$$\mathsf{A} \cap [\bar{\beta}(k)] \subseteq \{\beta\} \cup \bigcup\{\mathsf{A} \cap [s] : \lg(s) > k \wedge \beta \notin [s]\}.$$

The right-hand side is countable, but $\bar{\beta}(k) \in B$ so $\mathsf{A} \cap [\bar{\beta}(k)]$ is uncountable, a contradiction. $\square$

6.3 Lemma. *For any uncountable closed set* $\mathsf{A} \subseteq {}^{\omega}\omega$, *there exists a one–one total continuous functional* θ *such that* $\mathrm{Im}\theta \subseteq \mathsf{A}$ *and* $\mathsf{A} \sim \mathrm{Im}\,\theta$ *is countable.*

Proof. Let A be an uncountable closed set and set $\mathsf{A}_{\langle\,\rangle} = \mathsf{A}$. We shall define for each $u \in \mathrm{Sq}$ a set A_u such that for all u and v,

(i) A_u is uncountable;
(ii) $u \not\subseteq v \wedge v \not\subseteq u \to \mathsf{A}_u \cap \mathsf{A}_v = \varnothing$;
(iii) $\mathsf{A}_u \sim \bigcup\{\mathsf{A}_{u*\langle n\rangle} : n \in \omega\}$ is countable;
(iv) $u \subseteq v \to \mathsf{A}_v \subseteq \mathsf{A}_u$;
(v) for any β, $\bigcap\{\mathsf{A}_{\beta(k)} : k \in \omega\}$ contains a single element.

Suppose that A_u is defined and let $A_u \subseteq \mathrm{Sq}$ be as in the preceding lemma applied to the uncountable set A_u. If s_n denotes the n-th element of A_u enumerated in numerical order, set

$$\mathsf{A}_{u*\langle n\rangle} = \mathsf{A}_u \cap [s_n].$$

Properties (i)–(iii) now follow immediately from (i)–(iii) of the lemma, respectively, and (iv) is evident from the construction. (v) follows from the fact, easily proved by induction on the length of u, that A_u is closed and for all $s \in A_u$, $\lg(s) \geqslant \lg(u)$.

Now set $\theta(\beta) =$ the unique element of $\bigcap\{\mathsf{A}_{\bar\beta(k)} : k \in \omega\}$. For all β, $\theta(\beta) \in \mathsf{A} = \mathsf{A}_{\langle\,\rangle}$. That θ is one–one follows from (ii). If $\beta \in \theta^{-1}([s])$, then $\mathsf{A}_{\bar\beta(\lg(s))} \subseteq [s]$ so $[\bar\beta(\lg(s))] \subseteq \theta^{-1}([s])$ and thus θ is continuous. Finally, if $\alpha \in \mathsf{A} \sim \mathrm{Im}\,\theta$, then for some u, $\alpha \in \mathsf{A}_u \sim \bigcup\{\mathsf{A}_{u*\langle u\rangle} : n \in \omega\}$. Hence

$$\mathsf{A} \sim \mathrm{Im}\,\theta \subseteq \bigcup\{\mathsf{A}_u \sim \bigcup\{\mathsf{A}_{u*\langle n\rangle} : n \in \omega\} : u \in \mathrm{Sq}\}$$

which is countable by (iii). □

Note that the provision that $\mathrm{Im}\,\theta$ may differ from A by a countable set is essential as it is easy to see that $\mathrm{Im}\,\theta$ cannot contain isolated points (α is isolated in $\mathrm{Im}\,\theta$ iff for some n, α is the only member of $\mathrm{Im}\,\theta \cap [\bar\alpha(n)]$). Note that a set can have at most countably many isolated points.

6.4 Corollary. *Every non-empty closed set is the image of a total continuous functional.*

Proof. If A is countable, say $\mathsf{A} = \{\beta_n : n \in \omega\}$, we simply set $\chi(\alpha) = \beta_{\alpha(0)}$ and $\mathsf{A} = \mathrm{Im}\,\chi$. If A is uncountable, let θ be as in the preceding lemma a one–one continuous functional such that $\mathrm{Im}\,\theta \subseteq \mathsf{A}$ and $\mathsf{A} \sim \mathrm{Im}\,\theta$ is countable, say $\mathsf{A} \sim \mathrm{Im}\,\theta = \{\beta_n : n \in \omega\}$. Then if

$$\chi(\alpha) = \begin{cases} \theta(\lambda m\,.\,\alpha(m+1)), & \text{if } \alpha(0) = 0; \\ \beta_{\alpha(0)-1}, & \text{if } \alpha(0) > 0; \end{cases}$$

clearly χ is continuous and $\mathsf{A} = \mathrm{Im}\,\chi$. □

6.5 Theorem. *For any non-empty set* $A \subseteq {}^{\omega}\omega$, $A \in \Sigma_1^1$ *iff* A *is the image of a total continuous functional.*

Proof. First, if $A = \operatorname{Im}\theta$ for some total continuous θ, then

$$\alpha \in A \leftrightarrow \exists\beta\,\forall p\,.\,\theta(\beta)(p) = \alpha(p)$$

so it follows from Lemma 6.1 that $A \in \Sigma_1^1$. The converse is immediate from (1′) and Corollary 6.4. □

Consider now the following possible analogues for (3)–(5):

(3′) $A \in \Sigma_1^1 \overset{?}{\leftrightarrow} A$ is the image of a Borel set B under a continuous functional which is one–one on B;

(4′) $A \in \Sigma_1^1$ and A is uncountable $\overset{?}{\leftrightarrow}$ A is the image of a total one–one continuous functional;

(5′) $A \in \Sigma_1^1 \overset{?}{\leftrightarrow} A$ is the domain of a partial functional with Borel graph.

The implications $(\leftarrow)$ are trivially true in each case, but we shall show that all of the implications $(\rightarrow)$ are false. (This also follows from Exercise 2.27.) Of course, as we pointed out following 6.3, $(4)'(\rightarrow)$ fails for any A which has isolated points, but the following shows that it would be false even if we allowed for the exclusion of such points.

6.6 Theorem. *The image of any total one–one continuous functional is a Borel set.*

Proof. Let θ be a total one–one continuous functional. The graph of θ is Borel so there exists a Δ_1^1 relation R and a function γ such that for all α and β,

$$\theta(\alpha) = \beta \leftrightarrow R(\alpha, \beta, \gamma).$$

Then because θ is one–one,

$$\begin{aligned} \beta \in \operatorname{Im}\theta &\leftrightarrow \exists\alpha\, R(\alpha, \beta, \gamma) \\ &\leftrightarrow \exists!\,\alpha R(\alpha, \beta, \gamma) \\ &\leftrightarrow (\exists\alpha \in \Delta_1^1[\beta, \gamma])\, R(\alpha, \beta, \gamma). \end{aligned}$$

The first equivalence shows that $\operatorname{Im}\theta$ is Σ_1^1. The third shows that it is Π_1^1 (Theorem 2.9). □

The result of Lemma 6.3 can be extended to all Borel sets. First we give a new characterization of the class of Borel sets.

6.7 Lemma. *The class of Borel sets is the smallest class containing all closed sets and closed under countable intersection and countable disjoint union.*

Proof. Let X be the class described. Clearly $X \subseteq \mathrm{Bo}$ so it suffices to prove by induction that for all ρ, $\Sigma^0_\rho \subseteq X$. For $\rho = 0$ this is true by hypothesis so we assume $\rho > 0$ and $\Sigma^0_{(\rho)} \subseteq X$. As X is closed under countable intersection we have immediately that $\Pi^0_\rho \subseteq X$. Let $\mathsf{R} = \bigcup\{\mathsf{P}_p : p \in \omega\}$ be any element of Σ^0_ρ with all $\mathsf{P}_p \in \Pi^0_{(\rho)}$. For each p, let

$$\mathsf{Q}_p = \mathsf{P}_p \sim \bigcup\{\mathsf{P}_q : q < p\}.$$

By Lemmas 3.5 and 3.6, each $\mathsf{Q}_p \in \Delta^0_\rho \subseteq \Pi^0_\rho \subseteq X$. The Q_p are pairwise disjoint and $\mathsf{R} = \bigcup\{\mathsf{Q}_p : p \in \omega\}$ so $\mathsf{R} \in X$. □

6.8 Lemma. *Every Borel set is the image of a closed set* C *under a continuous functional which is one–one on* C.

Proof. Let X be the class of sets which are images of closed sets as described. Clearly all closed sets belong to X so it will suffice to show that X is closed under countable intersection and countable disjoint union.

Suppose that for all $p \in \omega$, $\mathsf{A}_p \in X$, C_p is closed, θ_p is a continuous functional which is one–one on C_p, and $\theta_p'' \mathsf{C}_p = \mathsf{A}_p$. If the sets A_p are pairwise disjoint and $\mathsf{B} = \bigcup\{\mathsf{A}_p : p \in \omega\}$, let $\mathsf{C} = \{\alpha : \lambda m \,.\, \alpha(m+1) \in \mathsf{C}_{\alpha(0)}\}$ and $\theta(\alpha) = \theta_{\alpha(0)}(\lambda m \,.\, \alpha(m+1))$. Then C is closed, θ is continuous and one–one on C, and $\theta''\mathsf{C} = \mathsf{B}$.

Now let $\mathsf{B} = \bigcap\{\mathsf{A}_p : p \in \omega\}$. Then we set

$$\mathsf{C} = \{\alpha \colon \forall p\, \forall q\, [(\alpha)^p \in \mathsf{C}_p \wedge \theta_p((\alpha)^p) = \theta_q((\alpha)^q)]$$
$$\wedge\, \forall t\, [t \notin \mathrm{Sq} \vee \lg(t) \neq 2 \rightarrow \alpha(t) = 0]\}$$

and $\theta(\alpha) = \theta_0((\alpha)^0)$ and claim that θ is continuous and one–one on C, C is closed, and $\theta''\mathsf{C} = \mathsf{B}$. That C is closed follows easily from Lemma 6.1. θ is clearly continuous. If $\beta \in \theta''\mathsf{C}$ then for some $\alpha \in \mathsf{C}$, $\beta = \theta_0((\alpha)^0) = \theta_p((\alpha)^p)$ for all p. Since $\theta_p((\alpha)^p) \in \theta_p''\mathsf{C}_p = \mathsf{A}_p$, $\beta \in \mathsf{B}$. On the other hand for any $\beta \in \mathsf{B}$ there exists for each p a function $\alpha_p \in \mathsf{C}_p$ such that $\theta_p(\alpha_p) = \beta$. But then there is an $\alpha \in \mathsf{C}$ such that $(\alpha)^p = \alpha_p$ for all p and thus $\beta \in \theta''\mathsf{C}$. Finally suppose for some $\alpha, \beta \in \mathsf{C}$ that $\theta(\alpha) = \theta(\beta)$. Then for all p, $\theta_p((\alpha)^p) = \theta_p((\beta)^p)$ which implies $(\alpha)^p = (\beta)^p$ since θ_p is one–one on C_p. Thus for all p and m, $\alpha(\langle p, m\rangle) = \beta(\langle p, m\rangle)$ and by the final condition on C, $\alpha = \beta$. Hence θ is one–one on C. □

We now have, in contrast with the proposed (3′) and (4′):

6.9 Theorem. *The following are equivalent for all* $\mathsf{A} \subseteq {}^\omega\omega$,

(i) A *is Borel*;

(ii) A *is the image of a Borel set* B *under a continuous functional which is one–one on* B;

(iii) A *is the image of a closed set* C *under a continuous functional which is one–one on* C;

(iv) A *is countable or for some one–one total continuous functional* θ, $\operatorname{Im}\theta \subseteq \mathsf{A}$ *and* $\mathsf{A} \sim \operatorname{Im}\theta$ *is countable.*

Proof. If A is Borel then it is the image of itself under the identity function, so (i) implies (ii). (iii) follows from (ii) by Lemma 6.8. Now suppose (iii) holds, say $\mathsf{A} = \chi''\mathsf{C}$, and by Lemma 6.3 let φ be a one–one total continuous functional such that $\operatorname{Im}\varphi \subseteq \mathsf{C}$ and $\mathsf{C} \sim \operatorname{Im}\varphi$ is countable. Then $\theta = \chi \circ \varphi$ clearly has the required properties. Finally, if A satisfies (iv), $\operatorname{Im}\theta$ is Borel by Theorem 6.6 and any countable set is Borel, so A is Borel. □

Finally, in contrast with the proposed (5′), we have

6.10 Corollary. *For any partial functional* θ, *if the graph of* θ *is a Borel relation, then the domain of* θ *is a Borel set.*

Proof. Let θ have Borel graph and set

$$\mathsf{B} = \{\langle\alpha, \beta\rangle : \theta(\alpha) \simeq \beta\} \quad \text{and} \quad \varphi(\gamma) = (\gamma)_0.$$

Then φ is continuous and one–one on the Borel set B and $\varphi''\mathsf{B}$ is the domain of θ. By the preceding theorem this is a Borel set. □

6.11–6.14 Exercises

6.11. Show that $\mathsf{A} \subseteq {}^{\omega}\omega$ is Σ^1_1 iff it is the image of some partial continuous functional with closed domain.

6.12. Prove that every Borel set is the domain of a functional with closed graph.

6.13. Show that if θ is continuous and 1–1 on A, then
(i) if $\mathsf{A} \in \Sigma^1_1$, then also $\theta''\mathsf{A} \in \Sigma^1_1$;
(ii) if $\mathsf{A} \notin \Delta^1_1$, then also $\theta''\mathsf{A} \notin \Delta^1_1$.

6.14. Establish the following analogue of Lemma 6.1: for any θ, if $\mathsf{F}(p, \alpha) \simeq \theta(\alpha)(p)$, then the following are equivalent:
(i) Gr_θ is Borel;
(ii) Gr_F is Borel;
(iii) for all s, $\theta^{-1}([s])$ is Borel.
What happens if "Borel" is replaced by 'Σ^1_1'? by 'Π^1_1'? (cf. Theorem 7.11 below).

6.15 Notes. The results of this section are all "classical" — that is, they appeared before the era of Recursion Theory. The proof of Theorem 6.6 is more modern, depending as it does on §2. There is, of course, a classical proof which goes in outline as follows: if θ is a total one–one continuous functional, use Corollary 1.12 to find for each $s \in \mathrm{Sq}$ a Borel set A_s such that

$$\theta''([s]) \subseteq \mathsf{A}_s \subseteq \sim \theta''(\sim [s])$$

and if $s \subseteq t$, then $\mathsf{A}_t \subseteq \mathsf{A}_s$. Then

$$\alpha \in \mathrm{Im}\,\theta \leftrightarrow \forall n\, \forall s\, [\mathrm{lg}(s) = n \wedge \alpha \in \mathsf{A}_s \cap \overline{\theta''([s])}].$$

For the flavor of this sort of Descriptive Set Theory as it was done in the good old days, peruse Ljapunov–Stschegolkov–Arsenin [1955].

7. *Uniformization*

The notion of a selection functional $\mathrm{Sel}_{\mathsf{R}}$ for a relation R played an important role in § II.4 and § 2 of this chapter. $\mathrm{Sel}_{\mathsf{R}}$ selects a number p such that $\mathsf{R}(p, \mathbf{m}, \boldsymbol{\alpha})$ holds whenever there is such a p. We consider here the analogous problem of selecting a β such that $\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}, \beta)$. The letters θ, φ, χ, and ψ denote functions from ${}^{k,l}\omega$ into ${}^{\omega}\omega$ in this section.

7.1 Definition. For any two classes X and Y of relations, X is *Y-uniformizable* (or Y *uniformizes* X) iff for every $\mathsf{R} \in X$, $\mathsf{R} \subseteq {}^{k,l+1}\omega$, there exists a partial functional $\theta : {}^{k,l}\omega \to {}^{\omega}\omega$ with $\mathrm{Gr}_\theta \in Y$ and such that for all $(\mathbf{m}, \boldsymbol{\alpha})$,

$$\exists \beta\, \mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}, \beta) \leftrightarrow \mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}, \theta(\mathbf{m}, \boldsymbol{\alpha})) \leftrightarrow \theta(\mathbf{m}, \boldsymbol{\alpha})\downarrow .$$

The functional θ is said to *uniformize* R. X has the *uniformization property* iff X is X-uniformizable.

The main result of this section is that Π^1_1, $\Pi^1_1[\beta]$, and $\boldsymbol{\Pi}^1_1$ all have the uniformization property. As background we examine uniformization for simpler classes.

7.2 Theorem. Σ^0_1 *is* Δ^0_2-*uniformizable.*

Proof. Suppose $\mathsf{R} \in \Sigma^0_1$, say $\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}, \beta) \leftrightarrow \exists p\, \mathsf{S}(\bar{\beta}(p), \mathbf{m}, \boldsymbol{\alpha})$ with S recursive. Then the functional θ defined by:

$$\theta(\mathbf{m}, \boldsymbol{\alpha}) \simeq \beta \leftrightarrow \exists s\, \mathsf{S}(s, \mathbf{m}, \boldsymbol{\alpha}) \wedge \forall s\, [\mathsf{S}(s, \mathbf{m}, \boldsymbol{\alpha}) \wedge (\forall t < s) \sim \mathsf{S}(t, \mathbf{m}, \boldsymbol{\alpha})$$
$$\rightarrow \bar{\beta}(\lg(s)) = s \wedge (\forall m \geq \lg(s))\beta(m) = 0]$$

has Δ_2^0 graph and uniformizes R. □

Of course Σ_1^0 cannot have the uniformization property since no non-empty functional θ has open graph. The same argument shows that $\Sigma_1^0[\beta]$ is $\Delta_2^0[\beta]$-uniformizable and hence $\boldsymbol{\Sigma}_1^0$ is $\boldsymbol{\Delta}_2^0$-uniformizable.

The following simple observation allows us to apply the results of § III.4.

7.3 Lemma. *For any X and any $r > 0$, if X is Σ_r^1-uniformizable, then Δ_r^1 is a basis for X.*

Proof. Suppose X is Σ_r^1-uniformizable and A is a non-empty set in X. Let $\mathsf{R}(p, \beta) \leftrightarrow \beta \in \mathsf{A}$ and θ uniformize R with $\mathrm{Gr}_\theta \in \Sigma_r^1$. Then if γ is defined by

$$\gamma(m) = n \leftrightarrow \exists \beta\, [\theta(0) = \beta \wedge \beta(m) = n]$$
$$\leftrightarrow \forall \beta\, [\theta(0) = \beta \rightarrow \beta(m) = n],$$

$\theta(0) = \gamma \in \Delta_r^1 \cap \mathsf{A}$. □

7.4 Theorem. *Π_1^0 is not Σ_1^1-uniformizable.*

Proof. By Theorem III.4.8, Δ_1^1 is not a basis for Π_1^0. □

Again this lemma and theorem can be easily extended to show that $\Pi_1^0[\beta]$ is not $\Sigma_1^1[\beta]$-uniformizable. However $\boldsymbol{\Delta}_1^1$ is trivially a basis for $\boldsymbol{\Pi}_1^0$ (since ${}^\omega\omega \subseteq \boldsymbol{\Delta}_1^0$), so we use a different approach to prove

7.5 Theorem. *$\boldsymbol{\Pi}_1^0$ is not $\boldsymbol{\Delta}_1^1$-uniformizable.*

Proof. Let A be any set in $\boldsymbol{\Sigma}_1^1 \sim \boldsymbol{\Delta}_1^1$ and R a $\boldsymbol{\Pi}_1^0$ relation such that $\alpha \in \mathsf{A} \leftrightarrow \exists \beta\, \mathsf{R}(\alpha, \beta)$. Suppose R were uniformizable by a $\boldsymbol{\Delta}_1^1$ functional θ. Then $\mathsf{A} = \mathrm{Dm}\, \theta$ is Borel by Corollary 6.10, a contradiction. □

7.6 Lemma. *For any functional $\theta \in \Sigma_1^1$ ($\boldsymbol{\Sigma}_1^1$), there exists a functional $\psi \in \Delta_1^1$ ($\boldsymbol{\Delta}_1^1$) such that $\theta \subseteq \psi$.*

Proof. Let θ be any Σ_1^1 functional and set

$$\mathsf{R}^*(\mathbf{m}, \boldsymbol{\alpha}, \beta) \leftrightarrow \exists \gamma\, \exists p\, [\theta(\mathbf{m}, \boldsymbol{\alpha}) \simeq \gamma \wedge \bar{\gamma}(p) = \bar{\beta}(p) \wedge \gamma(p) < \beta(p)];$$

$$\mathsf{R}^{**} = \mathsf{R}^* \cup \mathrm{Gr}(\theta);$$

$$\mathsf{S}^{**}(\mathbf{m}, \boldsymbol{\alpha}, \beta) \leftrightarrow \exists\gamma\ \exists p\ [\theta(\mathbf{m}, \boldsymbol{\alpha}) \simeq \gamma \wedge \bar{\gamma}(p) = \bar{\beta}(p) \wedge \beta(p) < \gamma(p)];$$

$$\mathsf{S}^* = \mathsf{S}^{**} \cup \mathrm{Gr}(\theta).$$

All of these relations are clearly Σ_1^1 and they satisfy

$$\mathsf{R}^* \cap \mathsf{S}^* = \emptyset = \mathsf{R}^{**} \cap \mathsf{S}^{**}.$$

By Theorem 1.5 there exist relations P^* and $\mathsf{P}^{**} \in \Delta_1^1$ such that

$$\mathsf{R}^* \subseteq \mathsf{P}^* \subseteq \sim\mathsf{S}^* \quad \text{and} \quad \mathsf{R}^{**} \subseteq \mathsf{P}^{**} \subseteq \sim\mathsf{S}^{**}.$$

Let $\mathsf{P} = \mathsf{P}^{**} \sim \mathsf{P}^*$. Clearly $\mathrm{Gr}(\theta) \subseteq \mathsf{R}^{**} \cap \mathsf{S}^* \subseteq \mathsf{P}$. P is not in general the graph of a functional, so let

$$\mathsf{U}(\mathbf{m}, \boldsymbol{\alpha}, \beta) \leftrightarrow P(\mathbf{m}, \boldsymbol{\alpha}, \beta) \wedge (\exists\gamma \neq \beta)\mathsf{P}(\mathbf{m}, \boldsymbol{\alpha}, \gamma).$$

$\mathsf{U} \in \Sigma_1^1$ and $\mathrm{Gr}(\theta) \cap \mathsf{U} = \emptyset$, so we again apply Theorem 1.5 to obtain a relation $\mathsf{Q} \in \Delta_1^1$ such that $\mathrm{Gr}(\theta) \subseteq \mathsf{Q} \sim \mathsf{U}$. Then $\mathsf{P} \cap \mathsf{Q}$ is the graph of a functional ψ as required. □

7.7 Corollary. $\boldsymbol{\Pi}_1^0$ *is not* $\boldsymbol{\Sigma}_1^1$*-uniformizable.*

Proof. By Theorem 7.5 choose R to be a $\boldsymbol{\Pi}_1^0$ relation which is not $\boldsymbol{\Delta}_1^1$-uniformizable and suppose $\theta \in \boldsymbol{\Sigma}_1^1$ uniformizes R. Then if ψ is any $\boldsymbol{\Delta}_1^1$ extension of θ, $\mathsf{R} \cap \mathrm{Gr}(\psi)$ is a $\boldsymbol{\Delta}_1^1$ function which uniformizes R, contrary to assumption. □

Thus the following is the best possible result.

7.8 Uniformization Theorem. Π_1^1 *and* $\boldsymbol{\Pi}_1^1$ *have the uniformization property.*

Proof. To simplify notation suppose that $\mathsf{R} \subseteq {}^{0,2}\omega$ is a Π_1^1 relation, say $\mathsf{R}(\alpha, \beta) \leftrightarrow \forall\gamma\ \exists n\ \mathsf{P}(\bar{\gamma}(n), \bar{\beta}(n), \alpha)$. Let P' be as in the proof of Theorem 1.1,

$$s \trianglelefteq^n_{\alpha,\beta} t \leftrightarrow s \trianglelefteq t \wedge t \trianglelefteq n \wedge \sim \mathsf{P}'(s, \bar{\beta}(\mathrm{lg}(s)), \alpha) \wedge \sim \mathsf{P}'(t, \bar{\beta}(\mathrm{lg}(t)), \alpha),$$

and

$$|n, \alpha, \beta| = \text{order-type of } \trianglelefteq^n_{\alpha,\beta}.$$

We define recursively relations R_n as follows: $\mathsf{R}_0 = \mathsf{R}$;

$$u_{\alpha,n} = \min\{\bar{\beta}(n) : \mathsf{R}_n(\alpha, \beta)\};$$

$$\sigma_{\alpha,n} = \min\{|n, \alpha, \beta| : \mathsf{R}_n(\alpha, \beta) \wedge \bar{\beta}(n) = u_{\alpha,n}\};$$

$$\mathsf{R}_{n+1}(\alpha, \beta) \leftrightarrow \mathsf{R}_n(\alpha, \beta) \wedge \bar{\beta}(n) = u_{\alpha,n} \wedge |n, \alpha, \beta| = \sigma_{\alpha,n}.$$

Let $\mathsf{Q}(\alpha, \beta) \leftrightarrow \forall n\, \mathsf{R}_n(\alpha, \beta)$. We shall show that $\mathsf{Q} \in \Pi_1^1$ and Q is the graph of a functional θ which uniformizes R. First note that if $\sim\exists\beta\, \mathsf{R}(\alpha, \beta)$, then also $\sim\exists\beta\, \mathsf{Q}(\alpha, \beta)$. Suppose that $\mathsf{R}(\alpha, \beta)$. Then for all n, $|n, \alpha, \beta|$ is an ordinal and there is a unique function γ_α such that for all n, $\bar{\gamma}_\alpha(n) = u_{\alpha,n}$. Clearly $\mathsf{Q}(\alpha, \beta) \rightarrow \beta = \gamma_\alpha$ so it suffices to prove that if $\exists\beta\, \mathsf{R}(\alpha, \beta)$, then $\mathsf{Q}(\alpha, \gamma_\alpha)$.

To this end we first establish the following technical lemmas:

(1) for any α such that $\exists\beta\, \mathsf{R}(\alpha, \beta)$, any p such that $\sim\mathsf{P}'(p, \bar{\gamma}_\alpha(\lg(p)), \alpha)$, and any q,

$$|p, \alpha, \gamma_\alpha| < |q, \alpha, \gamma_\alpha| \rightarrow \sigma_{\alpha,p} < \sigma_{\alpha,q};$$

(2) there exists a relation $\mathsf{S} \in \Pi_1^1$ such that for all n, α, and β,

$$\mathsf{R}_n(\alpha, \beta) \rightarrow [\mathsf{R}_{n+1}(\alpha, \beta) \leftrightarrow \mathsf{S}(n, \alpha, \beta)].$$

Suppose the hypotheses of (1) are satisfied, let $r = \max\{p, q\} + 1$, and let β be any function such that $\mathsf{R}_r(\alpha, \beta)$. Then $\bar{\beta}(\lg(p)) = \bar{\gamma}_\alpha(\lg(p))$ so also $\sim\mathsf{P}'(p, \bar{\beta}(\lg(p)), \alpha)$. Since $p \lhd q$, $|p, \alpha, \beta| < |q, \alpha, \beta|$. But since $\mathsf{R}_{p+1}(\alpha, \beta)$, $|p, \alpha, \beta| = \sigma_{\alpha,p}$ and as also $\mathsf{R}_{q+1}(\alpha, \beta)$, $|q, \alpha, \beta| = \sigma_{\alpha,q}$ and (1) is proved.

For (2) we first observe that for any n, α, and β such that $\mathsf{R}_n(\alpha, \beta)$,

$$\forall\gamma\, (\mathsf{R}_n(\alpha, \gamma) \leftrightarrow (\forall p < n)[\bar{\gamma}(p) \leqslant \bar{\beta}(p) \wedge |p, \alpha, \gamma| \leqslant |p, \alpha, \beta|])$$

and

$$\mathsf{R}_{n+1}(\alpha, \beta) \leftrightarrow \forall\gamma\, [\mathsf{R}_n(\alpha, \gamma) \rightarrow \bar{\beta}(n) \leqslant \bar{\gamma}(n) \wedge (\bar{\beta}(n) = \bar{\gamma}(n) \rightarrow |n, \alpha, \beta| \leqslant |n, \alpha, \gamma|)].$$

Using the relations $\leqslant_\Sigma$ and $\leqslant_\Pi$ of Theorem 1.4, we can define relations $\mathsf{U}_\Sigma \in \Sigma_1^1$ and $\mathsf{U}_\Pi \in \Pi_1^1$ such that if $|p, \alpha, \beta|$ and $|n, \alpha, \gamma|$ are ordinals, then

$$|p, \alpha, \gamma| \leqslant |p, \alpha, \beta| \leftrightarrow \mathsf{U}_\Sigma(p, \alpha, \beta, \gamma),$$

and

$$|n, \alpha, \beta| \leqslant |n, \alpha, \gamma| \leftrightarrow \mathsf{U}_\Pi(n, \alpha, \beta, \gamma).$$

Hence the requirements of (2) are met by

$$\mathsf{S}(n, \alpha, \beta) \leftrightarrow \forall\gamma\, [(\forall p < n)[\bar{\gamma}(p) \leqslant \bar{\beta}(p) \wedge \mathsf{U}_\Sigma(p, \alpha, \beta, \gamma)] \rightarrow \bar{\beta}(n) \leqslant \bar{\gamma}(n) \wedge (\bar{\beta}(n) = \bar{\gamma}(n) \rightarrow \mathsf{U}_\Pi(n, \alpha, \beta, \gamma))].$$

From (2) we have immediately that

$$\mathsf{Q}(\alpha,\beta) \leftrightarrow \mathsf{R}(\alpha,\beta) \wedge \forall n\, \mathsf{S}(n,\alpha,\beta)$$

and thus $\mathsf{Q} \in \Pi^1_1$. Suppose now that $\exists\beta\, \mathsf{R}(\alpha,\beta)$; we shall show $\mathsf{Q}(\alpha,\gamma_\alpha)$ and for this we first prove by induction on the ordinal $\sigma_{\alpha,n}$ that $|n,\alpha,\gamma_\alpha| \leqslant \sigma_{\alpha,n}$. Suppose this holds for all p such that $\sigma_{\alpha,p} < \sigma_{\alpha,n}$. Then by (1) and the induction hypothesis, if $|p,\alpha,\gamma_\alpha| < |n,\alpha,\gamma_\alpha|$, then $\sigma_{\alpha,p} < \sigma_{\alpha,n}$. Hence $|n,\alpha,\gamma_\alpha| \leqslant \sigma_{\alpha,n}$. Now $\mathsf{R}(\alpha,\gamma_\alpha)$ holds because $|\langle\ \rangle,\alpha,\gamma_\alpha| \leqslant \sigma_{\alpha,\langle\ \rangle}$ so $|\langle\ \rangle,\alpha,\gamma_\alpha|$ is an ordinal. Similarly, from $\mathsf{R}_n(\alpha,\gamma_\alpha)$ we conclude immediately that $\mathsf{R}_{n+1}(\alpha,\gamma_\alpha)$. Thus $\mathsf{Q}(\alpha,\gamma_\alpha)$. □

7.9 Basis Theorem. *Δ^1_2 is a basis for Σ^1_2.*

Proof. Immediate from the preceding theorem, Lemma 7.3, and Lemma III.4.7. □

We have given this result the important title, the Basis Theorem, because it will see extensive application in the latter four chapters of this book. Here we give two applications to questions arising in earlier sections. The first concerns closure ordinals for inductive operators and should be compared with Theorems 2.15 and 2.16. Here δ^1_2 is the least ordinal *not* the order-type of a Δ^1_2 wellordering of ω.

7.10 Theorem. *For any inductive operator Γ,*

(i) $\Gamma \in \Delta^1_2 \rightarrow |\Gamma| < \delta^1_2$;

(ii) Γ *monotone* $\wedge\, \Gamma \in \Sigma^1_2 \rightarrow |\Gamma| \leqslant \delta^1_2$.

Proof. For (i), suppose $\Gamma \in \Delta^1_2$. Using the relations defined in Theorem III.3.9 we have for all $\gamma \in \mathsf{W}$

$$|\Gamma| \leqslant \|\gamma\| \leftrightarrow \forall m\, [\mathsf{V}_\Pi(m,\gamma) \rightarrow \mathsf{V}^{()}_\Sigma(m,\gamma)].$$

Thus $\{\gamma : \gamma \in \mathsf{W} \wedge |\Gamma| \leqslant \|\gamma\|\}$ is a non-empty Σ^1_2 set and by the Basis Theorem contains a Δ^1_2 element γ_0. Thus $|\Gamma| \leqslant \|\gamma_0\| < \delta^1_2$.

For (ii), suppose Γ is monotone Σ^1_2 and $m \in \bar{\Gamma}$. Then using Theorem III.3.13,

$$\{\gamma : \gamma \in \mathsf{W} \wedge m \in \Gamma^{\|\gamma\|}\} = \{\gamma : \gamma \in \mathsf{W} \wedge \mathsf{V}(m,\gamma)\}$$

is a non-empty Σ^1_2 set which therefore has a Δ^1_2 element γ_0. Thus $m \in \Gamma^{\|\gamma_0\|} \subseteq \Gamma^{(\delta^1_2)}$, so $\bar{\Gamma} \subseteq \Gamma^{(\delta^1_2)}$. □

The second application is to continuous images. By the same resoning used to derive (1′) at the beginning of § 6, we have for any $\mathsf{A} \subseteq {}^\omega\omega$,

$\mathsf{A} \in \boldsymbol{\Sigma}_2^1 \leftrightarrow \mathsf{A}$ is the image of a $\boldsymbol{\Pi}_1^1$ set under a continuous functional from ${}^\omega\omega$ into ${}^\omega\omega$.

In contrast with Theorem 6.9 ((i)$\leftrightarrow$(ii)), this equivalence holds also if the functional is required to be one–one on the $\boldsymbol{\Pi}_1^1$ set (cf. also Corollary 6.10).

7.11 Theorem. *For all* $\mathsf{A} \subseteq {}^\omega\omega$, A *is* $\boldsymbol{\Sigma}_2^1$ *iff* A *is the domain of a functional* $\theta : {}^\omega\omega \to {}^\omega\omega$ *which has* $\boldsymbol{\Pi}_1^1$ *graph.*

Proof. The implication ($\leftarrow$) is obvious. Conversely, if A is $\{\alpha : \exists\beta\, \mathsf{R}(\alpha, \beta)\}$ with $\mathsf{R} \in \boldsymbol{\Pi}_1^1$, let θ be a functional with $\boldsymbol{\Pi}_1^1$ graph which uniformizes R. Then $\mathsf{A} = \mathrm{Dm}\,\theta$. $\square$

7.12 Corollary. *For all* $\mathsf{A} \subseteq {}^\omega\omega$, A *is* $\boldsymbol{\Sigma}_2^1$ *iff* A *is the image of a* $\boldsymbol{\Pi}_1^1$ *set* B *under a continuous functional which is one–one on* B.

Proof. If $\mathsf{A} = \mathrm{Dm}\,\theta$ with $\theta \in \boldsymbol{\Pi}_1^1$, it suffices to take $\mathsf{B} = \{\langle\alpha, \beta\rangle : \theta(\alpha) \simeq \beta\}$ and $\varphi(\gamma) = (\gamma)_0$. $\square$

The relativized version of the Basis Theorem also gives a trivial generalization of the Spector-Gandy Theorem IV.2.9: a relation R is Σ_2^1 iff for some relation $\mathsf{P} \in \Pi_1^1$,

$$\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow (\exists\beta \in \Delta_2^1[\boldsymbol{\alpha}])\, \mathsf{P}(\mathbf{m}, \boldsymbol{\alpha}, \beta).$$

Finally we have the following easy extension:

7.13 Theorem. Σ_2^1 *and* $\boldsymbol{\Sigma}_2^1$ *have the uniformization property.*

Proof. Let $\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}, \beta) \leftrightarrow \exists\gamma\, \mathsf{P}(\mathbf{m}, \boldsymbol{\alpha}, \beta, \gamma)$ be a Σ_2^1 relation with $\mathsf{P} \in \Pi_1^1$. Let

$$\mathsf{Q}(\mathbf{m}, \boldsymbol{\alpha}, \delta) \leftrightarrow \mathsf{P}(\mathbf{m}, \boldsymbol{\alpha}, (\delta)_0, (\delta)_1)$$

and by the Uniformization Theorem let φ be a functional with Π_1^1 graph which uniformizes Q. Then if

$$\theta(\mathbf{m}, \boldsymbol{\alpha}) \simeq \beta \leftrightarrow \exists\gamma\, [\varphi(\mathbf{m}, \boldsymbol{\alpha}) \simeq \langle\beta, \gamma\rangle],$$

θ has Σ_2^1 graph and uniformizes R. The proof for $\boldsymbol{\Sigma}_2^1$ is the same. $\square$

7.14–7.20 Exercises

7.14. Give an alternative proof for Corollary 7.7 by showing

(i) if $\boldsymbol{\Pi}_1^0$ were $\boldsymbol{\Sigma}_1^1$-uniformizable, then $\boldsymbol{\Sigma}_1^1$ would have the uniformization property;

(ii) if $\boldsymbol{\Sigma}_1^1$ had the uniformization property, then $\boldsymbol{\Sigma}_1^1$ would have the reduction property.

7.15. Give a simple proof that Π_1^1 uniformizes Π_1^0.

7.16. Find a Σ_1^1 subset of ${}^\omega\omega$ which has no non-empty Π_1^1 subsets.

7.17. Use the Basis Theorem to give a two-line proof of a weaker version of Exercise 5.16: every countable Σ_1^1 set of functions contains only Δ_2^1 elements.

7.18. Show that for any $\mathsf{A} \subseteq {}^\omega\omega$, if A is a model of Δ_2^1-Comprehension, then A is also a model of $\exists_2^1$-Choice. (Show that the Uniformization Theorem holds in any model of Δ_2^1-Comprehension.)

7.19. Show that Δ_2^1 is a model of the Δ_2^1-Comprehension schema (hence also of $\exists_2^1$-Choice by Exercise 7.18).

7.20. Show that Δ_2^1 has the uniformization property.

7.21 Notes. The Uniformization Theorem for $\boldsymbol{\Pi}_1^1$ is due to Kondô [1938]. The lightface version was announced by Addison in 1959 but never published. It has since appeared in (at least) Rogers [1967] and Shoenfield [1967].

Exercise III.3.34 suggests an alternate proof for Theorem 7.10(i).