Chapter III
Hierarchies and Definability

In the preceding chapter we saw that the semi-recursive relations are exactly those which arise from the recursive relations by existential number quantification (II.4.12). In this chapter we study the relations which arise from the recursive relations by all kinds of quantification: existential, universal, number, and function. After classifying these relations according to the number and type of quantifiers used and establishing the simplest combinatorial properties of this classification in §§ 1 and 2, we relate it to other notions of definability. In § 3 we compare the complexity of definition of an inductive operator Γ with that of the set $\bar{\Gamma}$. In § 4, we investigate the relationship between the complexity of a subset A of ${}^{\omega}\omega$ and that of its elements. In § 5 we show that the relations we are considering are exactly those definable in certain natural first- and second-order formal languages. Finally in § 6 we introduce the method of forcing to extend and complete some earlier results.

1. The Arithmetical Hierarchy

1.1 Definition. The class of *arithmetical relations* is the smallest class of relations containing the recursive relations and closed under number quantification ($\exists^0$ and $\forall^0$).

We next define a classification of the arithmetical relations based on the number of quantifiers needed to define a relation.

1.2 Definition (The Arithmetical Hierarchy). For all r,

(i) $\Sigma_0^0 = \Pi_0^0 =$ the class of recursive relations;

(ii) $\Sigma_{r+1}^0 = \{\exists^0\mathsf{P}\colon \mathsf{P} \in \Pi_r^0\}$;

(iii) $\Pi_{r+1}^0 = \{\forall^0\mathsf{P}\colon \mathsf{P} \in \Sigma_r^0\}$;

(iv) $\Delta_r^0 = \Sigma_r^0 \cap \Pi_r^0$;

(v) $\Delta_{(\omega)}^0 = \bigcup \{\Sigma_r^0 \cup \Pi_r^0\colon r \in \omega\}$.

It is immediate by induction on r that all of the classes Σ_r^0 and Π_r^0, and hence $\Delta_{(\omega)}^0$, are included in the class of arithmetical relations. The converse inclusion is immediate from Theorem 1.5 below.

1.3 Examples. Note first that by Theorem II.4.12, Σ_1^0 is exactly the class of semi-recursive relations and Π_1^0 is the class of co-semi-recursive relations. Hence by Corollary II.4.10, $\Delta_0^0 = \Delta_1^0 =$ the class of recursive relations.

Consider the set $\mathsf{A} = \{\alpha : \alpha \text{ is recursive}\}$. We have

$$\begin{aligned}\alpha \in \mathsf{A} &\leftrightarrow \exists a\, \forall m\, [\{a\}(m) \simeq \alpha(m)]\\ &\leftrightarrow \exists a\, \forall m\, \exists u\, [\mathsf{T}(a, \langle m \rangle, u, \langle\ \ \rangle) \wedge (u)_0 = \alpha(m)]\end{aligned}$$

and thus $\mathsf{A} \in \Sigma_3^0$. The set B of primitive recursive functions is Σ_2^0:

$$\alpha \in \mathsf{B} \leftrightarrow \exists a\, \forall m\, [a \in \mathrm{Pri} \wedge \mathsf{Ev}^{1,0}(a, m) = \alpha(m)].$$

Let C be the set of γ such that the relation $\leqslant_\gamma = \{(p, q) : \gamma(\langle p, q \rangle) = 0\}$ is a linear ordering. Then an easy computation shows $\mathsf{C} \in \Pi_1^0$ (try this now without using Theorem 1.5).

Let $\mathscr{L}$ be a first-order formal language with (for simplicity) only one binary relation symbol and no function symbols, and suppose Gödel numbers have been assigned to the symbols, formulas, and sequences of formulas in some standard way (as in Shoenfield [1967], for example). We henceforth identify these objects with their Gödel numbers. An analysis of the notion of formal proof shows that the relation

$$\mathsf{R}(p, m, A) \leftrightarrow p \text{ is a proof of } m \text{ from } A$$

is recursive. Hence, by the Completeness Theorem,

$$\mathsf{S}(m, A) \leftrightarrow m \text{ is a logical consequence of } A$$

is Σ_1^0. The set $\{A : A \text{ is consistent}\}$ is Π_1^0 and $\{A : A \text{ is complete}\}$ is Π_2^0. Every denumerable structure for $\mathscr{L}$ is isomorphic to one of the form (ω, R) with $R \subseteq {}^2\omega$. Analysis of any standard proof of the Completeness Theorem shows that if A is recursive and has infinite models, then A has a model (ω, R) with $R \in \Delta_2^0$.

We write $\mathsf{F} \in \Sigma_r^0$ (Π_r^0, Δ_r^0) to mean $\mathsf{Gr}_\mathsf{F} \in \Sigma_r^0$ (Π_r^0, Δ_r^0). Note that $\mathsf{R} \in \Sigma_r^0$ does not in general imply $\mathsf{K}_\mathsf{R} \in \Sigma_r^0$. We often use the terms Σ_r^0, etc. as adjectives and write, for example, "for any Σ_r^0 relation R" instead of "for any $\mathsf{R} \in \Sigma_r^0$".

1.4 Lemma. *For all r and* R,

$$\mathsf{R} \in \Sigma_r^0 \leftrightarrow {\sim}\mathsf{R} \in \Pi_r^0 \quad \textit{and} \quad \mathsf{R} \in \Pi_r^0 \leftrightarrow {\sim}\mathsf{R} \in \Sigma_r^0.$$

Proof. The case $r = 0$ is immediate. Suppose the result holds for r and suppose

$R \in \Sigma^0_{r+1}$. Then $R = \exists^0 P$ for some $P \in \Pi^0_r$ and $\sim R = \forall^0 {\sim} P$. Then $\sim P \in \Sigma^0_r$ so $\sim R \in \Pi^0_{r+1}$ by definition. The case for $R \in \Pi^0_{r+1}$ is similar. □

1.5 Theorem. *The classes of the arithmetical hierarchy have the following closure properties for all r:*

	Σ^0_r	Π^0_r	Δ^0_r	$\Delta^0_{(\omega)}$
Composition and substitution with recursive functions	√	√	√	√
Finite union and intersection	√	√	√	√
Expansion	√	√	√	√
Complementation			√	√
Bounded quantification ($\exists^0_<$ and $\forall^0_<$)	√	√	√	√
Existential number quantification ($\exists^0$)	√$(r>0)$			√
Universal number quantification ($\forall^0$)		√$(r>0)$		√

Proof. In the proofs we use the following equivalences and their duals (obtained by negating both sides):

$$\exists p\ P(p, \mathbf{m}, \boldsymbol{\alpha}) \vee \exists q\ Q(q, \mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \exists p\ [P(p, \mathbf{m}, \boldsymbol{\alpha}) \vee Q(p, \mathbf{m}, \boldsymbol{\alpha})];$$

$$\exists p\ P(p, \mathbf{m}, \boldsymbol{\alpha}) \wedge \exists q\ Q(q, \mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \exists p\ [P((p)_0, \mathbf{m}, \boldsymbol{\alpha}) \wedge Q((p)_1, \mathbf{m}, \boldsymbol{\alpha})];$$

$$(\exists q < s)\exists p\ P(p, q, s, \mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \exists p\ [(p)_1 < s \wedge P((p)_0, (p)_1, s, \mathbf{m}, \boldsymbol{\alpha})];$$

$$(\forall q < s)\exists p\ P(p, q, s, \mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \exists p\ (\forall q < s)\ P((p)_q, q, s, \mathbf{m}, \boldsymbol{\alpha});$$

$$\exists p\ \exists q\ P(p, q, \mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \exists p\ P((p)_0, (p)_1, \mathbf{m}, \boldsymbol{\alpha}).$$

For example, we prove by induction on r that Σ^0_r and Π^0_r are closed under bounded quantification. For $r = 0$, this is known (§ II.2). Suppose it holds for r and R is any Σ^0_{r+1} relation, say

$$R(q, s, \mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \exists p\ P(p, q, s, \mathbf{m}, \boldsymbol{\alpha})$$

with $P \in \Pi^0_r$. Using the third and fourth equivalences and the induction hypothesis, it follows that the relations defined by

$$(\exists q < s)\, R(q, s, \mathbf{m}, \boldsymbol{\alpha}) \quad \text{and} \quad (\forall q < s)\, R(q, s, \mathbf{m}, \boldsymbol{\alpha})$$

are also Σ^0_{r+1}. The other proofs are similar. $\square$

1.6 Corollary. *For all r,* $\Sigma^0_r \cup \Pi^0_r \subseteq \Delta^0_{r+1}$.

Proof. We observed in 1.3 that this holds for $r = 0$ and we proceed by induction. Given the conclusion for r, suppose $\mathsf{R} \in \Sigma^0_{r+1}$ and let S be defined by

$$\mathsf{S}(p, \mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}).$$

Then $\mathsf{R} = \exists^0 \mathsf{S} = \forall^0 \mathsf{S}$ and $\mathsf{S} \in \Sigma^0_{r+1}$ so immediately $\mathsf{R} \in \Pi^0_{r+2}$. But $\mathsf{S} = \exists^0 \mathsf{P}$ for some $\mathsf{P} \in \Pi^0_r \subseteq \Pi^0_{r+1}$ so also $\mathsf{R} \in \Sigma^0_{r+2}$ and thus $\mathsf{R} \in \Delta^0_{r+2}$. The proof for $\mathsf{R} \in \Pi^0_{r+1}$ is similar. $\square$

These last two results are often used together to "compute" where a given arithmetical relation falls in the hierarchy. For example, if

$$\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \exists p\, \exists q\, \forall r\, [\exists s\, \forall t\, \mathsf{P}(p, G(q, s, \mathbf{m}), t, \mathbf{m}, \boldsymbol{\alpha}) \vee \exists u\, \mathsf{Q}(p, u, r, \mathbf{m}, \boldsymbol{\alpha})],$$

with G, P, and Q recursive, then the relation described inside the brackets is the union of a Σ^0_2 and a Σ^0_1 relation, hence is Σ^0_2. Then by applying successively $\forall^0$, $\exists^0$, and $\exists^0$ we conclude $\mathsf{R} \in \Sigma^0_4$. Of course, such a computation does not always yield an optimal classification — in fact, in this example we have also

$$\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \exists p\, \exists q\, [\exists s\, \forall t\, \mathsf{P}(p, G(q, s, \mathbf{m}), t, \mathbf{m}, \boldsymbol{\alpha}) \vee \forall r\, \exists u\, \mathsf{Q}(p, u, r, \mathbf{m}, \boldsymbol{\alpha})]$$

from which we obtain $\mathsf{R} \in \Sigma^0_3$. We consider later in the chapter methods for showing that a relation does *not* belong to some class Σ^0_r or Π^0_r.

1.7 Corollary. *For any* $r > 0$, $\mathsf{F}\colon {}^{k,l}\omega \to \omega$, $F\colon {}^k\omega \to \omega$, *and* $\mathsf{R} \subseteq {}^{k,l}\omega$,

(i) *if* $\mathsf{F} \in \Sigma^0_r$ *and* F *is total, then also* $\mathsf{F} \in \Delta^0_r$;
(ii) *if* $\mathsf{F} \in \Sigma^0_r$ *then* $\mathrm{Dm}\,\mathsf{F} \in \Sigma^0_r$;
(iii) *if* $F \in \Sigma^0_r$ *then* $\mathrm{Im}\,F \in \Sigma^0_r$;
(iv) *if* $\mathsf{R} \in \Sigma^0_r \cup \Pi^0_r$, *then* $\mathsf{K}_\mathsf{R} \in \Delta^0_{r+1}$;
(v) Σ^0_r *and* Π^0_r *are closed under composition with total* Δ^0_r *functionals.*

Proof. For (i), if F is total, then

$$\mathsf{F}(\mathbf{m}, \boldsymbol{\alpha}) = n \leftrightarrow \forall n' [n' \neq n \to \mathsf{F}(\mathbf{m}, \boldsymbol{\alpha}) \neq n'].$$

Hence if Gr_F is Σ^0_r, it is also Π^0_r. (ii) and (iii) are immediate from the closure of Σ^0_r under $\exists^0$. For (iv) we have

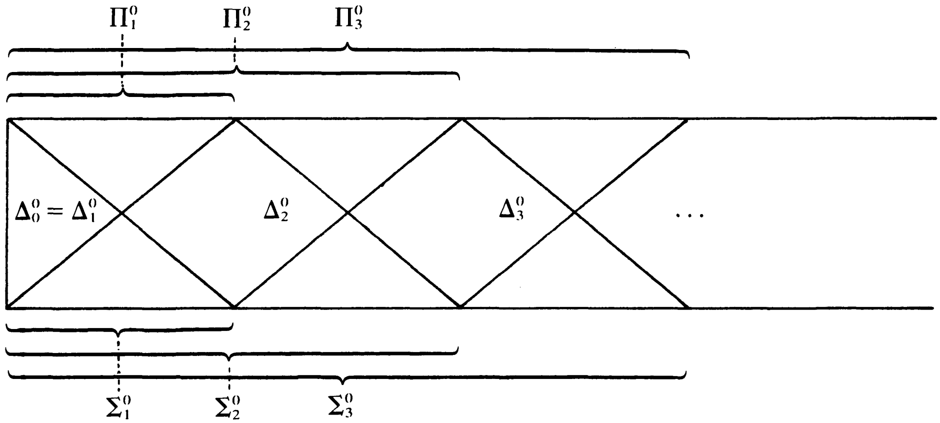
$$\mathsf{K}_\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) = n \leftrightarrow [\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \wedge n = 0] \vee [\sim \mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \wedge n = 1]$$

so $\mathsf{Gr}_{\mathsf{K_R}}$ is the union of a Σ^0_r and a Π^0_r relation, hence is Δ^0_{r+1}. For (v) we have for total G,

$$\mathsf{R}(\mathsf{G}(\mathbf{m},\boldsymbol{\alpha}),\mathbf{m},\boldsymbol{\alpha}) \leftrightarrow \exists n\,[\mathsf{G}(\mathbf{m},\boldsymbol{\alpha}) = n \wedge \mathsf{R}(n,\mathbf{m},\boldsymbol{\alpha})]$$
$$\leftrightarrow \forall n\,[\mathsf{G}(\mathbf{m},\boldsymbol{\alpha}) = n \rightarrow \mathsf{R}(n,\mathbf{m},\boldsymbol{\alpha})].$$

The first equivalence serves if $\mathsf{R} \in \Sigma^0_r$ and the second if $\mathsf{R} \in \Pi^0_r$. □

The following diagram exhibits the inclusions of Corollary 1.6:



To show that each of these inclusions is proper — that each space in the diagram represents a non-empty class of relations, we use a diagonal argument. For $r \geq 1$, let

$$\mathsf{U}^0_1(a,\langle\mathbf{m}\rangle,\langle\boldsymbol{\alpha}\rangle) \leftrightarrow \exists u\ \mathsf{T}(a,\langle\mathbf{m}\rangle,u,\langle\boldsymbol{\alpha}\rangle);$$

$$\mathsf{U}^0_{r+1}(a,\langle\mathbf{m}\rangle,\langle\boldsymbol{\alpha}\rangle) \leftrightarrow \exists p \sim \mathsf{U}^0_r(a,\langle p,\mathbf{m}\rangle,\langle\boldsymbol{\alpha}\rangle);$$

$$U^0_r(a,\langle\mathbf{m}\rangle) \leftrightarrow \mathsf{U}^0_r(a,\langle\mathbf{m}\rangle,\langle\ \rangle).$$

Recall (II.4.20) that U is universal for a class X iff $\mathsf{U} \in X$ and for every $\mathsf{R} \in X$ there exists an $a \in \omega$ such that

$$\mathsf{R}(\mathbf{m},\boldsymbol{\alpha}) \leftrightarrow \mathsf{U}(a,\langle\mathbf{m}\rangle,\langle\boldsymbol{\alpha}\rangle).$$

1.8 Arithmetical Indexing Theorem. *For all $r > 0$,*

(i) U^0_r *is universal for* Σ^0_r;
(ii) $\sim\mathsf{U}^0_r$ *is universal for* Π^0_r;
(iii) U^0_r *is universal for* $\{R\colon R \in \Sigma^0_r\}$;
(iv) $\sim U^0_r$ *is universal for* $\{R\colon R \in \Pi^0_r\}$.

Proof. A straightforward induction on r shows that U^0_r and U^0_r are Σ^0_r and $\sim\mathsf{U}^0_r$ and $\sim U^0_r$ are Π^0_r. For $r = 1$, (i)–(iv) are clear from the proof of Theorem II.4.12. Given (i)–(iv) for r, suppose $\mathsf{R} \in \Sigma^0_{r+1}$, say $\mathsf{R} = \exists^0\mathsf{P}$ with $\mathsf{P} \in \Pi^0_r$. Then for some a,

$$\mathsf{P}(p, \mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \sim\mathsf{U}^0_r(a, \langle p, \mathbf{m}\rangle, \langle\boldsymbol{\alpha}\rangle),$$

which implies

$$\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \exists p \sim\mathsf{U}^0_r(a, \langle p, \mathbf{m}\rangle, \langle\boldsymbol{\alpha}\rangle) \leftrightarrow \mathsf{U}^0_{r+1}(a, \langle \mathbf{m}\rangle, \langle\boldsymbol{\alpha}\rangle).$$

(ii)–(iv) are proved similarly. □

1.9 Arithmetical Hierarchy Theorem. *For all $r > 0$*

(i) $\Sigma^0_r \not\subseteq \Delta^0_r$ *and* $\Pi^0_r \not\subseteq \Delta^0_r$;

(ii) $\Delta^0_{r+1} \not\subseteq \Sigma^0_r \cup \Pi^0_r$.

Proof. Since $U^0_r \in \Sigma^0_r$, it will suffice for (i) to show $U^0_r \notin \Delta^0_r$. Suppose the contrary and let

$$A = \{a: U^0_r(a, \langle a\rangle)\}.$$

Then also $A \in \Delta^0_r$ and in particular $\sim A \in \Sigma^0_r$. Since U^0_r is universal, there is some $b \in \omega$ such that for all m,

$$m \notin A \leftrightarrow U^0_r(b, \langle m\rangle).$$

In particular,

$$b \notin A \leftrightarrow U^0_r(b, \langle b\rangle) \leftrightarrow b \in A,$$

a contradiction.

For (ii), let

$$B = \{2m: m \in A\} \cup \{2m + 1: m \notin A\}.$$

Clearly B is a union of a Σ^0_r set and a Π^0_r set, so $B \in \Delta^0_{r+1}$. Suppose, however, that $B \in \Sigma^0_r$. Then

$$m \notin A \leftrightarrow 2m + 1 \in B$$

and $\sim A \in \Sigma^0_r$ which was seen to be impossible. If $B \in \Pi^0_r$, then

$$m \in A \leftrightarrow 2m \in B$$

which implies $A \in \Pi^0_r$, hence $\sim A \in \Sigma^0_r$. Therefore $B \notin \Sigma^0_r \cup \Pi^0_r$. □

One application of the Hierarchy Theorem is to obtain precise classifications in the arithmetical hierarchy. Recall that a relation R is (*many-one*) *reducible* to a set A (in symbols, $\mathsf{R} \leqslant A$) iff for some recursive functional F,

$$\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \mathsf{F}(\mathbf{m}, \boldsymbol{\alpha}) \in A.$$

Clearly if $A \in \Sigma_r^0$ or Π_r^0, so does R. Suppose that A is such that *all* Σ_r^0 relations on numbers are reducible to A (such an A is called Σ_r^0-*complete*). In particular, if R is a relation which is Σ_r^0 but not Π_r^0, R is reducible to A. Hence $A \notin \Pi_r^0$. For example, let

$$\mathrm{Tot} = \{a : \{a\} \text{ is a total function of rank } 1\}.$$

Since for any a,

$$a \in \mathrm{Tot} \leftrightarrow \forall m\, \exists u\, \mathsf{T}(a, \langle m \rangle, u, \langle\ \rangle),$$

$\mathrm{Tot} \in \Pi_2^0$. In fact, this is an optimal estimate of the complexity of Tot — i.e., $\mathrm{Tot} \notin \Sigma_2^0$. To establish this we show that Tot is Π_2^0-complete.

If R is any Π_2^0 relation and P is a recursive relation such that $\forall p\, \exists q\, P(p, q, \mathbf{m}) \leftrightarrow R(\mathbf{m})$, let f be the partial recursive function defined by

$$f(\mathbf{m}, p) \simeq \text{least } q \,.\, P(p, q, \mathbf{m}).$$

Clearly, $R(\mathbf{m}) \leftrightarrow \forall p \,.\, f(\mathbf{m}, p)\downarrow$. Let b be an index for f. Then by Lemma II.2.5,

$$R(\mathbf{m}) \leftrightarrow Sb_{k-1}(b, \mathbf{m}) \in \mathrm{Tot}$$

and $R \leqslant \mathrm{Tot}$. Some other results of this type are given in the exercises.

To this point it appears that the properties of the classes Σ_r^0, Π_r^0, and Δ_r^0 and the relationships among them strongly resemble the corresponding properties and relationships for Σ_1^0, Π_1^0, and Δ_1^0. In the remainder of this section we shall examine how well this analogy holds up.

First the reduction and separation properties hold for $r > 1$ just as they do for $r = 1$:

1.10 Theorem. *For all* $r > 0$,

(i) Σ_r^0 *has the reduction property but not the separation property*;

(ii) Π_r^0 *has the separation property but not the reduction property.*

Proof. That Σ_r^0 has the reduction property is proved exactly as in Theorem II.4.17 using the closure properties of Theorem 1.5. The other results now follow from Lemmas II.4.19 and II.4.21. □

One flaw in the analogy appears in connection with function quantification. Σ_1^0 is closed under existential function quantification (II.4.14), but Σ_{r+2}^0 is not so closed for any r. Indeed, if R is any relation in $\Pi^0_{r+2} \sim \Delta^0_{r+2}$, say $\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \forall p\, \exists q\, \mathsf{S}(p, q, \mathbf{m}, \boldsymbol{\alpha})$ with $\mathsf{S} \in \Pi^0_r$, then

$$\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \exists \beta\, \forall p\, \mathsf{S}(p, \beta(p), \mathbf{m}, \boldsymbol{\alpha})$$

so that $\mathsf{R} = \exists^1 \mathsf{P}$ for a relation $\mathsf{P} \in \Pi^0_r \subseteq \Sigma^0_{r+2}$.

For relations on numbers, we shall see that the Σ^0_{r+1} relations are exactly those which are semi-recursive in a certain Σ^0_r set. To investigate this situation and for later use we prove first a general result on substitution of arithmetical functionals in arithmetical relations.

1.11 Arithmetical Substitution Theorem. *For any r and s, any $\mathsf{S} \in \Sigma^0_{r+1}$ (Π^0_{r+1}), and any total functionals $\mathsf{H}_0, \ldots, \mathsf{H}_n \in \Delta^0_s$, if*

$$\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \mathsf{S}(\mathbf{m}, \boldsymbol{\alpha}, \lambda p\,.\, \mathsf{H}_0(p, \mathbf{m}, \boldsymbol{\alpha}), \ldots, \lambda p\,.\, \mathsf{H}_n(p, \mathbf{m}, \boldsymbol{\alpha})),$$

then $\mathsf{R} \in \Sigma^0_{r+s}$ (Π^0_{r+s}).

Proof. We proceed by induction on r and take $n = 0$ for simplicity. Suppose first $\mathsf{S} \in \Sigma^0_1$. By Theorem II.4.12 there exists a recursive relation P such that

$$\mathsf{S}(\mathbf{m}, \boldsymbol{\alpha}, \beta) \leftrightarrow \exists p\, \mathsf{P}(\bar{\beta}(p), \mathbf{m}, \boldsymbol{\alpha}).$$

Then

$$\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \exists p\, \exists s\, [s = \bar{\mathsf{H}}_0(p, \mathbf{m}, \boldsymbol{\alpha}) \wedge \mathsf{P}(s, \mathbf{m}, \boldsymbol{\alpha})].$$

Since

$$s = \bar{\mathsf{H}}_0(p, \mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \mathrm{Sq}(s) \wedge \lg(s) = p \wedge (\forall i < p)((s)_i = \mathsf{H}(i, \mathbf{m}, \boldsymbol{\alpha})),$$

the relation inside the brackets is Δ^0_s, so $\mathsf{R} \in \Sigma^0_s$.

If $\mathsf{S} \in \Pi^0_1$, then for some recursive Q,

$$\mathsf{S}(\mathbf{m}, \boldsymbol{\alpha}, \beta) \leftrightarrow \forall p\, \mathsf{Q}(\bar{\beta}(p), \mathbf{m}, \boldsymbol{\alpha})$$

and we use the equivalence

$$\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \forall p\, \forall s\, [s = \bar{\mathsf{H}}_0(p, \mathbf{m}, \boldsymbol{\alpha}) \rightarrow \mathsf{Q}(s, \mathbf{m}, \boldsymbol{\alpha})].$$

The induction step is straightforward and is left to the reader. □

The next two results are known jointly as *Post's Theorem*.

1.12 Theorem. *For all r and all $R \subseteq {}^k\omega$,*

(i) *$R \in \Sigma^0_{r+1} \leftrightarrow R$ is semi-recursive in some set $A \in \Sigma^0_r$ (Π^0_r);*

(ii) *$R \in \Delta^0_{r+1} \leftrightarrow R$ is recursive in some set $A \in \Sigma^0_r$ (Π^0_r).*

Proof. The implications $(\leftarrow)$ are immediate from the preceding theorem together with Corollary 1.7(iv). Suppose now that $R = \exists^0 P$ with $P \in \Pi^0_r$. Let $A = \{\langle p, \mathbf{m}\rangle : \sim P(p, \mathbf{m})\}$. Then $A \in \Sigma^0_r$ and $R(\mathbf{m}) \leftrightarrow \exists p\, [\langle p, \mathbf{m}\rangle \notin A]$, so R is semi-recursive in A.

For (ii) ($\rightarrow$), if $R \in \Delta^0_{r+1}$ then both $R, \sim R \in \Sigma^0_{r+1}$ so by (i) there are $A, B \in \Sigma^0_r$ such that R is semi-recursive in A and $\sim R$ is semi-recursive in B. Let

$$C = \{2m: m \in A\} \cup \{2m+1: m \in B\}.$$

Then both R and $\sim R$ are semi-recursive in C, hence R is recursive in C (by the relativized version of Corollary II.4.10) and $C \in \Sigma^0_r$. Because a set is recursive in its complement, we may take as well $A \in \Pi^0_r$. $\square$

Let $D_0 = \{0\}$ and $D_{r+1} = (D_r)^{0J}$.

1.13 Theorem. *For all r and all $R \subseteq {}^k\omega$,*

(i) $R \in \Sigma^0_r \leftrightarrow R \leqslant D_r$ *(in particular, $D_r \in \Sigma^0_r$);*

(ii) $R \in \Sigma^0_{r+1} \leftrightarrow R$ *is semi-recursive in D_r;*

(iii) $R \in \Delta^0_{r+1} \leftrightarrow R$ *is recursive in D_r.*

Proof. We prove (i)–(iii) simultaneously by induction on r. For each r, (ii) and (iii) follow from (i) by Theorem 1.12. For $r = 0$, D_0 is recursive and hence so is any R which is many-one reducible to it. If $R \in \Sigma^0_0$, R is recursive and $R(\mathbf{m}) \leftrightarrow \mathsf{K}_R(\mathbf{m}) \in D_0$, so $R \leqslant D_0$.

Suppose (i)–(iii) hold for r. Then (i) for $r+1$ is immediate from (ii) for r and Theorem II.5.7. $\square$

There are two natural definitions for the relativized arithmetical hierarchy. We may either set:

$\Sigma^0_0[\boldsymbol{\beta}] = \Pi^0_0[\boldsymbol{\beta}] =$ the class of relations recursive in β;

$\Sigma^0_{r+1}[\boldsymbol{\beta}] = \{\exists^0 \mathsf{P}: \mathsf{P} \in \Pi^0_r[\boldsymbol{\beta}]\}$;

etc, as in Definition 1.2,

or we may define, for $r > 0$, $\mathsf{R} \in \Sigma^0_r[\boldsymbol{\beta}]$ ($\Pi^0_r[\boldsymbol{\beta}]$) iff $\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \mathsf{S}(\mathbf{m}, \boldsymbol{\alpha}, \boldsymbol{\beta})$ for some $\mathsf{S} \in \Sigma^0_r$ (Π^0_r). Fortunately, these two definitions are equivalent (Exercise 1.24). Note that $\mathsf{R} \in \Delta^0_r[\boldsymbol{\beta}]$ is *not* in general equivalent to the condition that $\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \mathsf{S}(\mathbf{m}, \boldsymbol{\alpha}, \boldsymbol{\beta})$ for some $\mathsf{S} \in \Delta^0_r$ (cf. remarks following II.5.1). It is easy to check that R is (semi-) recursive in $\boldsymbol{\beta}$ just in case $\mathsf{R} \in \Delta^0_1[\boldsymbol{\beta}]$ ($\Sigma^0_1[\boldsymbol{\beta}]$). $\square$

With appropriate changes, the results of this section all hold for the relativized arithmetical hierarchy with essentially the same proofs. In particular, there exists for each $r > 0$ a relation $\mathsf{U}^0_r[\boldsymbol{\beta}]$ universal for $\Sigma^0_r[\boldsymbol{\beta}]$ and thus not in $\Delta^0_r[\boldsymbol{\beta}]$. (*Relativized Arithmetical Hierarchy Theorem*)

Since α is recursive in β just in case $\alpha \in \Delta^0_1[\beta]$, it is natural to enquire the properties of the relations '$\alpha \in \Delta^0_r[\beta]$'. Surprisingly they are not transitive for $r > 1$ (Exercise 1.23). The best one can say in general is

1.14 Theorem. *For any r and s, if $\alpha \in \Delta^0_{r+1}[\beta]$ and $\beta \in \Delta^0_s[\gamma]$, then $\alpha \in \Delta^0_{r+s}[\gamma]$.*

Proof. Immediate from the relativized version of Theorem 1.11. □

Finally, motivated by Theorem II.5.5, we define

1.15 Definition. For all $r > 0$,

(i) $\mathbf{\Sigma}_r^0 = \bigcup\{\Sigma_r^0[\beta]: \beta \in {}^\omega\omega\}$;

(ii) $\mathbf{\Pi}_r^0 = \bigcup\{\Pi_r^0[\beta]: \beta \in {}^\omega\omega\}$;

(iii) $\mathbf{\Delta}_r^0 = \mathbf{\Sigma}_r^0 \cap \mathbf{\Pi}_r^0$.

Thus $\mathbf{\Sigma}_1^0$ is the class of open relations, $\mathbf{\Pi}_1^0$ is the class of closed relations, and $\mathbf{\Delta}_1^0$ is the class of closed-open relations.

1.16 Theorem. *For all r and* R, $\mathsf{R} \in \mathbf{\Sigma}_{r+1}^0 \leftrightarrow \mathsf{R}$ *is the union of countably many $\mathbf{\Pi}_r^0$ relations of the same rank.*

Proof. Suppose first that $\mathsf{R} \in \mathbf{\Sigma}_{r+1}^0$, so for some β and some $\mathsf{P} \in \mathbf{\Pi}_r^0[\beta]$, $\mathsf{R} = \exists^0\mathsf{P}$. If for each $p \in \omega$,

$$\mathsf{P}_p(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \mathsf{P}(p, \mathbf{m}, \boldsymbol{\alpha}),$$

then $\mathsf{P}_p \in \Pi_r^0[\beta] \subseteq \mathbf{\Pi}_r^0$ and $\mathsf{R} = \bigcup\{\mathsf{P}_p: p \in \omega\}$.

For the converse, let $\mathsf{R} = \bigcup\{\mathsf{P}_p: p \in \omega\}$ with each $\mathsf{P}_p \in \mathbf{\Pi}_r^0$. For each p choose β_p such that $\mathsf{P}_p \in \Pi_r^0[\beta_p]$. Then for each p there exists a_p such that

$$\mathsf{P}_p(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \sim \mathsf{U}_r^0((\gamma(\langle p, 0\rangle))_0, \langle\mathbf{m}\rangle, \langle\boldsymbol{\alpha}, (\gamma)_1^p\rangle)$$

Let γ be a function such that for all p and q, $\gamma(\langle p, q\rangle) = \langle a_p, \beta_p(q)\rangle$. Then for all p,

$$\mathsf{P}_p(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \sim \mathsf{U}_r^0(\gamma(\langle p, 0\rangle)_0, \langle\mathbf{m}\rangle, \langle\boldsymbol{\alpha}, (\gamma)_1^p\rangle)$$

and thus for some relation $\mathsf{Q} \in \Pi_r^0[\gamma]$,

$$\mathsf{P}_p(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \mathsf{Q}(p, \mathbf{m}, \boldsymbol{\alpha}).$$

Then $\mathsf{R} = \exists^0\mathsf{Q} \in \Sigma_{r+1}^0[\gamma] \subseteq \mathbf{\Sigma}_{r+1}^0$. □

Thus, for example, $\mathbf{\Sigma}_2^0$ is the class of countable unions of closed relations, commonly called F_σ in analysis, $\mathbf{\Pi}_2^0$ is the class G_δ, $\mathbf{\Sigma}_3^0 = G_{\delta\sigma}$, etc. These comprise the finite levels of the *Borel Hierarchy* which will be studied further in the next section and in § IV.3. (See also Exercises 1.25–30.)

1.17–1.30 Exercises

1.17. Show that the following relations are arithmetical and estimate the level at which they occur in the arithmetical hierarchy:

$\mathsf{P}_1(p, \gamma) \leftrightarrow \leqslant_\gamma$ is a linear ordering and p is the $\leqslant_\gamma$-least element of $\mathrm{Fld}(\gamma)$;

$P_2(\gamma) \leftrightarrow \leqslant_\gamma$ is a discrete linear ordering (every element of $\mathrm{Fld}(\gamma)$ has an immediate $\leqslant_\gamma$-successor);
$P_3(\gamma) \leftrightarrow \leqslant_\gamma$ is a well-ordering of type ω;
$P_4(n, \gamma) \leftrightarrow \leqslant_\gamma$ is a well-ordering of type $\leqslant \omega \cdot n$.

1.18. Show that for all r, $\{\alpha: \alpha \in \Delta_r^0\}$ is arithmetical and estimate its level in the arithmetical hierarchy.

1.19. Write out the proofs for two of the other parts of Theorem 1.5.

1.20. Let α be a function such that $\{\alpha\}$ is a Π_1^0 set (α is *implicitly* Π_1^0). Show that if $\mathrm{Im}\,\alpha$ is bounded, then α is recursive, but otherwise α may be non-recursive.

1.21. Let $X_0 = \Sigma_1^0$, $\bar{X}_0 = \Pi_1^0$, and for all r,

$$X_{r+1} = \{\mathrm{R} \cup \mathrm{S}: \mathrm{R} \in X_r,\ \mathrm{S} \in \bar{X}_r,\ \mathrm{R} \text{ and } \mathrm{S} \text{ of the same rank}\}$$

and

$$\bar{X}_{r+1} = \{\mathrm{R} \cap \mathrm{S}: \mathrm{R} \in X_r,\ \mathrm{S} \in \bar{X}_r,\ \mathrm{R} \text{ and } \mathrm{S} \text{ of the same rank}\}.$$

Show that for all r,
(i) $\mathrm{R} \in X_r \leftrightarrow {\sim}\mathrm{R} \in \bar{X}_r$;
(ii) $X_r \cup \bar{X}_r \subseteq X_{r+1} \cap \bar{X}_{r+1}$;
(iii) $X_r \nsubseteq \bar{X}_r$ and $\bar{X}_r \nsubseteq X_r$;
(iv) $\bigcup \{X_r: r \in \omega\}$ is a proper subclass of Δ_2^0.

1.22. From Theorem 1.11 we see that if $\beta \in \Delta_s^0$, then $\Sigma_{r+1}^0[\beta] \subseteq \Sigma_{r+s}^0$. Is this ever an equality?

1.23. Let $\alpha \leqslant_r^0 \beta \leftrightarrow \alpha \in \Delta_r^0[\beta]$. Show that for $r > 1$, $\leqslant_r^0$ is not transitive.

1.24. Show that the two characterizations of $\Sigma_{r+1}^0[\boldsymbol{\beta}]$ are equivalent.

1.25. Verify that
(i) R is semi-recursive in $\boldsymbol{\beta} \leftrightarrow \mathrm{R} \in \Sigma_1^0[\boldsymbol{\beta}]$;
(ii) $\boldsymbol{\Sigma}_{r+1}^0 = \{\exists^0\mathrm{P}: \mathrm{P} \in \boldsymbol{\Pi}_r^0\}$;
(iii) $\boldsymbol{\Delta}_r^0 = \bigcup \{\Delta_r^0[\beta]: \beta \in {}^\omega\omega\}$.

1.26. Show that for all $r \geqslant 1$, $\boldsymbol{\Sigma}_r^0$ and $\boldsymbol{\Pi}_r^0$ are parametrizable (cf. Exercise II.5.11).

1.27. Show that for all $r > 0$ and all relations R,
(i) $\mathrm{R} \in \boldsymbol{\Sigma}_r^0 \leftrightarrow {\sim}\mathrm{R} \in \boldsymbol{\Pi}_r^0$;

(ii) $\boldsymbol{\Sigma}_r^0 \cup \boldsymbol{\Pi}_r^0 \subseteq \boldsymbol{\Delta}_{r+1}^0$;
(iii) $\boldsymbol{\Sigma}_r^0 \not\subseteq \boldsymbol{\Delta}_r^0$ and $\boldsymbol{\Pi}_r^0 \not\subseteq \boldsymbol{\Delta}_r^0$;
(iv) $\boldsymbol{\Delta}_{r+1}^0 \not\subseteq \boldsymbol{\Sigma}_r^0 \cup \boldsymbol{\Pi}_r^0$.

1.28. Show that the Arithmetical Hierarchy Theorem may be improved to: for all $r > 0$, $\Sigma_r^0 \not\subseteq \boldsymbol{\Delta}_r^0$ and $\Pi_r^0 \not\subseteq \boldsymbol{\Delta}_r^0$.

1.29. Show that for all r,
(i) $\boldsymbol{\Sigma}_r^0$ has the reduction property but not the separation property;
(ii) $\boldsymbol{\Pi}_r^0$ has the separation property but not the reduction property.

1.30. Let $\mathsf{V}_1^0 = \mathsf{U}_1^0$, and for $r > 0$,

$$\mathsf{V}_{r+1}^0(a, \langle \mathbf{m} \rangle, \langle \boldsymbol{\alpha} \rangle) \leftrightarrow \exists p \sim \mathsf{V}_r^0(\{a\}(p), \langle \mathbf{m} \rangle, \langle \boldsymbol{\alpha} \rangle).$$

Set

$$\mathsf{P}_a^r(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \mathsf{V}_r^0(a, \langle \mathbf{m} \rangle, \langle \boldsymbol{\alpha} \rangle).$$

Show that for all $r > 0$,

$$\Sigma_r^0 = \{\mathsf{P}_a^r\colon a \in \omega\}.$$

(Note: P_a^{r+1} is the *recursive union* of the relations $\sim \mathsf{P}_{\{a\}(p)}^r$ so that Σ_{r+1}^0 consists exactly of recursive unions of Π_r^0 relations. The arithmetical hierarchy is thus sometimes called the (finite) *effective Borel hierarchy*. Cf. § IV.4.)

2. The Analytical Hierarchy

We take up next the relations which are obtained from the recursive relations by application of both number and function quantifiers. The basic structure we develop in this section is parallel in most points to that of § 1, but we shall see in Part B that these are indeed much more complex relations. Proofs will be omitted when they are very similar to corresponding proofs in § 1.

2.1 Definition. The class of *analytical relations* is the smallest class of relations containing the arithmetical relations and closed under function quantification ($\exists^1$ and $\forall^1$).

2.2 Definition (The Analytical Hierarchy). For all r,
(i) $\Sigma_0^1 = \Pi_0^1 =$ the class of arithmetical relations;

(ii) $\Sigma^1_{r+1} = \{\exists^1 \mathsf{P} \colon \mathsf{P} \in \Pi^1_r\}$;
(iii) $\Pi^1_{r+1} = \{\forall^1 \mathsf{P} \colon \mathsf{P} \in \Sigma^1_r\}$;
(iv) $\Delta^1_r = \Sigma^1_r \cap \Pi^1_r$;
(v) $\Delta^1_{(\omega)} = \bigcup \{\Sigma^1_r \cup \Pi^1_r \colon r \in \omega\}$.

Clearly every relation in $\Delta^1_{(\omega)}$ is analytical; the converse inclusion follows from Theorem 2.6 below. As before, we write $\mathsf{F} \in \Sigma^1_r$, etc., to mean $\mathrm{Gr}_{\mathsf{F}} \in \Sigma^1_r$, etc., and use the terms Σ^1_r, Π^1_r, and Δ^1_r as adjectives.

2.3 Examples. Recall that

$$\mathsf{W} = \{\gamma \colon \leqslant_\gamma \text{ is a well-ordering}\}.$$

Then using (4′) of I.1.6,

$$\begin{aligned}\gamma \in \mathsf{W} \leftrightarrow {\leqslant_\gamma} &\text{ is a linear ordering} \wedge \\ &\forall \alpha\, (\forall m\, [\alpha(m+1) \leqslant_\gamma \alpha(m)] \rightarrow \exists m\,.\, \alpha(m) \leqslant_\gamma \alpha(m+1)) \\ &\leftrightarrow \forall \alpha\, [\leqslant_\gamma \text{ is a linear ordering} \wedge (\forall m\,.\, \gamma(\langle \alpha(m+1), \alpha(m)\rangle) = 0 \\ &\qquad \rightarrow \exists m\,.\, \gamma(\langle \alpha(m), \alpha(m+1)\rangle) = 0)].\end{aligned}$$

The part inside the brackets is easily seen to be arithmetical so that $\mathsf{W} \in \Pi^1_1$.

Let

$$\begin{aligned}\gamma \preccurlyeq \delta \leftrightarrow {\leqslant_\gamma} &\text{ and } \leqslant_\delta \text{ are linear orderings and} \\ &\leqslant_\gamma \text{ is isomorphic to a subordering of } \leqslant_\delta.\end{aligned}$$

Then the second conjunct is equivalent to

$$\exists \alpha\, \forall p\, \forall q\, [\alpha \text{ is } 1\text{–}1 \text{ on } \mathrm{Fld}(\gamma) \wedge (p \leqslant_\gamma q \rightarrow \alpha(p) \leqslant_\delta \alpha(q))]$$

so that the relation $\preccurlyeq$ is Σ^1_1. Note that if $\gamma, \delta \in \mathsf{W}$,

$$\gamma \preccurlyeq \delta \leftrightarrow \|\gamma\| \leqslant \|\delta\|.$$

A similar argument shows that

$$\{(R, S) \colon \text{the relational structures } (\omega, R) \text{ and } (\omega, S) \text{ are isomorphic}\}$$

is Σ^1_1.

Suppose A is an arithmetical set of functions with just one element, α. Then α is Δ^1_1 since for any m and n

$$\alpha(m) = n \leftrightarrow \exists\beta\,[\beta \in \mathsf{A} \wedge \beta(m) = n]$$
$$\leftrightarrow \forall\beta\,[\beta \in \mathsf{A} \rightarrow \beta(m) = n].$$

Let Γ be a monotone operator over ω such that the relation '$m \in \Gamma(\{n\colon \alpha(n) = 0\})$' is arithmetical. By Theorem I.3.3,

$$m \in \bar{\Gamma} \leftrightarrow \forall B\,[\Gamma(B) \subseteq B \rightarrow m \in B]$$
$$\leftrightarrow \forall\alpha\,[\forall p\,(p \in \Gamma(\{n\colon \alpha(n) = 0\}) \rightarrow \alpha(p) = 0) \rightarrow \alpha(m) = 0].$$

Hence $\bar{\Gamma} \in \Pi^1_1$.

Two other examples which will be treated in § V.2 but may serve to orient the reader familiar with other parts of logic are:

$\{(m, R)\colon m$ is the Gödel number of a formula valid in $(\omega, R)\}$ is Δ^1_1;

$\{\alpha\colon \alpha$ is constructible (in the sense of Gödel)$\}$ is Σ^1_2.

2.4 Lemma. *For all r and* R,

$$\mathsf{R} \in \Sigma^1_r \leftrightarrow \sim\mathsf{R} \in \Pi^1_r \quad \textit{and} \quad \mathsf{R} \in \Pi^1_r \leftrightarrow \sim\mathsf{R} \in \Sigma^1_r. \quad \square$$

2.5 Theorem. *The classes of the analytical hierarchy have the following closure properties for all r:*

	Σ^1_r	Π^1_r	Δ^1_r	$\Delta^1_{(\omega)}$
Composition and substitution with recursive functionals	$\surd$	$\surd$	$\surd$	$\surd$
Finite union and intersection	$\surd$	$\surd$	$\surd$	$\surd$
Expansion	$\surd$	$\surd$	$\surd$	$\surd$
Complementation			$\surd$	$\surd$
Bounded quantification ($\exists^0_<$ and $\forall^0_<$)	$\surd$	$\surd$	$\surd$	$\surd$
Existential number quantification ($\exists^0$)	$\surd$	$\surd$	$\surd$	$\surd$
Universal number quantification ($\forall^0$)	$\surd$	$\surd$	$\surd$	$\surd$
Existential function quantification ($\exists^1$)	$\surd(r>0)$			$\surd$
Universal function quantification ($\forall^1$)		$\surd(r>0)$		$\surd$

Proof. In the proofs we use the following equivalences and their duals (obtained by negating both sides):

$$\exists\beta\,\mathsf{P}(\mathbf{m},\boldsymbol{\alpha},\beta)\vee\exists\gamma\,\mathsf{Q}(\mathbf{m},\boldsymbol{\alpha},\gamma)\leftrightarrow\exists\beta\,[\mathsf{P}(\mathbf{m},\boldsymbol{\alpha},\beta)\vee\mathsf{Q}(\mathbf{m},\boldsymbol{\alpha},\beta)];$$
$$\exists\beta\,\mathsf{P}(\mathbf{m},\boldsymbol{\alpha},\beta)\wedge\exists\gamma\,\mathsf{Q}(\mathbf{m},\boldsymbol{\alpha},\gamma)\leftrightarrow\exists\beta\,[\mathsf{P}(\mathbf{m},\boldsymbol{\alpha},(\beta)_0)\wedge\mathsf{Q}(\mathbf{m},\boldsymbol{\alpha},(\beta)_1)];$$
$$(\exists q<r)\exists\beta\,\mathsf{P}(q,r,\mathbf{m},\boldsymbol{\alpha},\beta)\leftrightarrow\exists\beta\,(\exists q<r)\mathsf{P}(q,r,\mathbf{m},\boldsymbol{\alpha},\beta);$$
$$(\forall q<r)\exists\beta\,\mathsf{P}(q,r,\mathbf{m},\boldsymbol{\alpha},\beta)\leftrightarrow\exists\beta\,(\forall q<r)\mathsf{P}(q,r,\mathbf{m},\boldsymbol{\alpha},(\beta)_q);$$
$$\exists p\,\exists\beta\,\mathsf{P}(p,\mathbf{m},\boldsymbol{\alpha},\beta)\leftrightarrow\exists\beta\,\mathsf{P}(\beta(0),\mathbf{m},\boldsymbol{\alpha},\lambda q\,.\,\beta(q+1));$$
$$\forall p\,\exists\beta\,\mathsf{P}(p,\mathbf{m},\boldsymbol{\alpha},\beta)\leftrightarrow\exists\beta\,\forall p\,\mathsf{P}(p,\mathbf{m},\boldsymbol{\alpha},(\beta)^p);$$
$$\exists\beta\,\exists\gamma\,\mathsf{P}(\mathbf{m},\boldsymbol{\alpha},\beta,\gamma)\leftrightarrow\exists\beta\,\mathsf{P}(\mathbf{m},\boldsymbol{\alpha},(\beta)_0,(\beta)_1).$$

For example, we prove by induction on r that Σ^1_r and Π^1_r are closed under $\exists^0$ and $\forall^0$. For $r=0$ this is contained in Theorem 1.5. Suppose it holds for r and R is any Σ^1_{r+1} relation, say

$$\mathsf{R}(p,\mathbf{m},\boldsymbol{\alpha})\leftrightarrow\exists\beta\,\mathsf{P}(p,\mathbf{m},\boldsymbol{\alpha},\beta)$$

with $\beta\in\Pi^1_r$. Using the fifth and sixth equivalences and the induction hypothesis, it follows that the relations $\exists^0\mathsf{R}$ and $\forall^0\mathsf{R}$ are also Σ^1_r. The result for Π^1_r follows by dualization. □

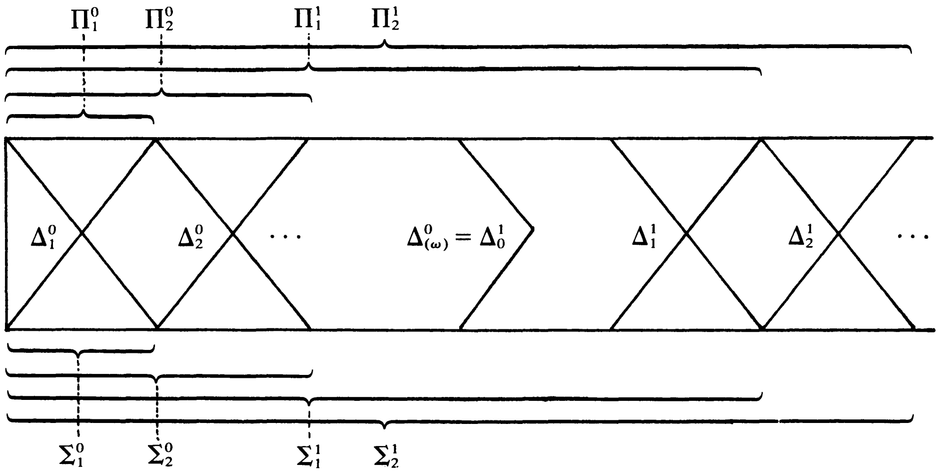
2.6 Corollary. *For all r, $\Sigma^1_r\cup\Pi^1_r\subseteq\Delta^1_{r+1}$.* □

2.7 Corollary. *For any $r>0$, $\mathsf{F}\colon {}^{k,l}\omega\to\omega$, $F\colon {}^k\omega\to\omega$, $\alpha\in{}^\omega\omega$, and $\mathsf{R}\subseteq{}^{k,l}\omega$*

(i) *if $\mathsf{F}\in\Sigma^1_r\cup\Pi^1_r$ and F is total, then also $\mathsf{F}\in\Delta^1_r$;*
(ii) *if $\mathsf{F}\in\Sigma^1_r$ (Π^1_r), then $\mathrm{Dm}\,\mathsf{F}\in\Sigma^1_r$ (Π^1_r);*
(iii) *if $F\in\Sigma^1_r$ (Π^1_r), then $\mathrm{Im}\,F\in\Sigma^1_r$ (Π^1_r);*
(iv) *if $\mathsf{F}\in\Sigma^1_r$, then $\mathrm{Im}\,\mathsf{F}\in\Sigma^1_r$;*
(v) *if $\mathsf{R}\in\Sigma^1_r\cup\Pi^1_r$, then $\mathsf{K}_\mathsf{R}\in\Delta^1_{r+1}$;*
(vi) *Σ^1_r (Π^1_r) is closed under composition with Σ^1_r (Π^1_r) functionals;*
(vii) *$\alpha\in\Delta^1_r$ iff $\{\alpha\}\in\Delta^1_r$ iff $\{\alpha\}\in\Sigma^1_r$.*

Proof. The proofs are in general like those for Corollary 1.8 and we leave them to Exercise 2.21. □

We may picture the arithmetical and analytical hierarchies together as follows:



We want next to show that the indicated inclusions are proper. We postpone to the next section (3.8) the proof that $\Delta_{(\omega)}^0 \neq \Delta_1^1$. For the others we proceed as for the arithmetical hierarchy to define relations universal for Σ_r^1 and Π_r^1. However, we shall need one preliminary result.

2.8 Lemma. $\Sigma_1^1 = \{\exists^1 \mathsf{P} \colon \mathsf{P} \in \Pi_1^0\}$.

Proof. The inclusion ($\supseteq$) is immediate from the definitions. On the other hand, it follows from the last three equivalences used in the proof of Theorem 2.5 that $\{\exists^1 \mathsf{P} \colon \mathsf{P} \in \Pi_1^0\}$ is closed under $\exists^0$, $\forall^0$, and $\exists^1$. Hence this set includes all arithmetical relations and thus also Σ_1^1. $\square$

We now set, for $r \geqslant 1$,

$$\mathsf{U}_1^1(a, \langle \mathbf{m} \rangle, \langle \boldsymbol{\alpha} \rangle) \leftrightarrow \exists \beta \sim \mathsf{U}_1^0(a, \langle \mathbf{m} \rangle, \langle \boldsymbol{\alpha}, \beta \rangle);$$

$$\mathsf{U}_{r+1}^1(a, \langle \mathbf{m} \rangle, \langle \boldsymbol{\alpha} \rangle) \leftrightarrow \exists \beta \sim \mathsf{U}_r^1(a, \langle \mathbf{m} \rangle, \langle \boldsymbol{\alpha}, \beta \rangle);$$

$$U_r^1(a, \langle \mathbf{m} \rangle) \leftrightarrow U_r^1(a, \langle \mathbf{m} \rangle, \langle\ \rangle).$$

2.9 Analytical Indexing Theorem. *For all $r > 0$,*

(i) U_r^1 *is universal for* Σ_r^1;
(ii) $\sim\mathsf{U}_r^1$ *is universal for* Π_r^1;
(iii) U_r^1 *is universal for* $\{R \colon R \in \Sigma_r^1\}$;
(iv) $\sim U_r^1$ *is universal for* $\{R \colon R \in \Pi_r^1\}$.

Proof. We first prove by induction that for all $r > 0$, $\mathsf{U}_r^1 \in \Sigma_r^1$ and $\sim\mathsf{U}_r^1 \in \Pi_r^1$. By the convention discussed following Theorem II.1.8, the definition of U_1^1 given above is an abbreviation for:

$$\mathsf{U}_1^1(a, s, \gamma) \leftrightarrow \exists \beta \sim \mathsf{U}_1^0(a, s, \lambda m\,(\gamma(m) * (\beta(m)))).$$

The part following the quantifier is Π^0_1 by the Arithmetical Substitution Theorem (1.11) and thus $\mathsf{U}^1_1 \in \Sigma^1_1$. Similarly,

$$\mathsf{U}^1_{r+1}(a, s, \gamma) \leftrightarrow \exists \beta \sim \mathsf{U}^1_r(a, s, \lambda m(\gamma(m) * (\beta(m)))).$$

Thus under the induction hypothesis that $\sim \mathsf{U}^1_r \in \Pi^1_r$ we have, using Lemma 2.5, that $\mathsf{U}^1_{r+1} \in \Sigma^1_{r+1}$.

If R is any Σ^1_1 relation, then by Lemma 2.8, $\mathsf{R} = \exists^1 \mathsf{P}$ for some $\mathsf{P} \in \Pi^0_1$. Since $\sim \mathsf{U}^0_1$ is universal for Π^0_1, there is a number a such that

$$\mathsf{P}(\mathbf{m}, \boldsymbol{\alpha}, \beta) \leftrightarrow \sim \mathsf{U}^0_1(a, \langle \mathbf{m} \rangle, \langle \boldsymbol{\alpha}, \beta \rangle).$$

Then

$$\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \mathsf{U}^1_1(a, \langle \mathbf{m} \rangle, \langle \boldsymbol{\alpha} \rangle).$$

The remainder of the proof is by induction on r just as in the arithmetical case. □

2.10 Analytical Hierarchy Theorem. *For all* $r > 0$

(i) $\Sigma^1_r \not\subseteq \Delta^1_r$ *and* $\Pi^1_r \not\subseteq \Delta^1_r$;

(ii) $\Delta^1_{r+1} \not\subseteq \Sigma^1_r \cup \Pi^1_r$.

Proof. Just as for the arithmetical hierarchy. □

From this point on the theory of the analytical hierarchy begins to diverge from that of the arithmetical hierarchy. The question of which of the classes Σ^1_r and Π^1_r have the reduction and separation properties is much more complicated. Results for $r = 1$ and $r = 2$ will be obtained in §§ IV.1 and V.1, respectively, but for $r \geqslant 3$ these questions cannot be decided on the basis of the usual axioms for set theory (cf. end of § V.3).

2.11 Analytical Substitution Theorem. *For any* r, *any* $\mathsf{S} \in \Sigma^1_r$ (Π^1_r), *and any partial functional* $\mathsf{H} \in \Sigma^1_r$ (Π^1_r), *if*

$$\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \mathsf{S}(\mathbf{m}, \boldsymbol{\alpha}, \lambda p \,.\, \mathsf{H}(p, \mathbf{m}, \boldsymbol{\alpha})),$$

then also $\mathsf{R} \in \Sigma^1_r$ (Π^1_r).

Proof. For $r = 0$, suppose S and H are both arithmetical. H may be extended to a total arithmetical functional I:

$$\mathsf{I}(p, \mathbf{m}, \boldsymbol{\alpha}) \simeq n \leftrightarrow \mathsf{H}(p, \mathbf{m}, \boldsymbol{\alpha}) \simeq n \vee (\neg \exists q \,.\, \mathsf{H}(p, \mathbf{m}, \boldsymbol{\alpha}) \simeq q \wedge n = 0).$$

Then

$$\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \forall p\, \exists n\,.\, \mathsf{H}(p, \mathbf{m}, \boldsymbol{\alpha}) \simeq n \wedge \mathsf{S}(\mathbf{m}, \boldsymbol{\alpha}, \lambda p\,.\, \mathsf{I}(p, \mathbf{m}, \boldsymbol{\alpha})),$$

and R is arithmetical by the Arithmetical Substitution Theorem 1.11.

Suppose $r > 0$ and $\mathsf{H}, \mathsf{S} \in \Sigma^1_r$. Then the result follows from the equivalence:

$$\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \exists \beta\, (\forall p\, [\mathsf{H}(p, \mathbf{m}, \boldsymbol{\alpha}) = \beta(p)] \wedge \mathsf{S}(\mathbf{m}, \boldsymbol{\alpha}, \beta)).$$

If $\mathsf{H}, \mathsf{S} \in \Pi^1_r$ we have

$$\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \forall p\, \exists n\, [\mathsf{H}(p, \mathbf{m}, \boldsymbol{\alpha}) \simeq n] \wedge \forall \beta\, (\forall p\, n[\mathsf{H}(p, \mathbf{m}, \boldsymbol{\alpha}) = n \rightarrow \beta(p) = n] \rightarrow \mathsf{S}(\mathbf{m}, \boldsymbol{\alpha}, \beta)). \quad \square$$

The relativized analytical hierarchy may be defined in two equivalent ways:

$\Sigma^1_0[\boldsymbol{\beta}] = \Pi^1_0[\boldsymbol{\beta}] =$ the class of relations arithmetical in $\boldsymbol{\beta}$;

$\Sigma^1_{r+1}[\boldsymbol{\beta}] = \{\exists^1 \mathsf{P}\colon \mathsf{P} \in \Pi^1_r[\boldsymbol{\beta}]\}$;

etc. as in Definition 2.2;

or, for $r > 0$, $\mathsf{R} \in \Sigma^1_r[\boldsymbol{\beta}]$ ($\Pi^1_r[\boldsymbol{\beta}]$) iff $\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \mathsf{S}(\mathbf{m}, \boldsymbol{\alpha}, \boldsymbol{\beta})$ for some $\mathsf{S} \in \Sigma^1_r$ (Π^1_r). The proof that these are equivalent is the same as in the arithmetical case (Exercise 1.24).

Again there is no problem in extending all the results of this section to the relativized hierarchy.

2.12 Corollary. *For all r and β, if $\beta_0, \ldots, \beta_n \in \Delta^1_r$, then*

$$\Sigma^1_r[\boldsymbol{\beta}] = \Sigma^1_r, \quad \Pi^1_r[\boldsymbol{\beta}] = \Pi^1_r, \quad \textit{and} \quad \Delta^1_r[\boldsymbol{\beta}] = \Delta^1_r.$$

Proof. Immediate from Theorem 2.11. $\square$

2.13 Corollary. *For all r, α, β, and γ, if $\alpha \in \Delta^1_r[\beta]$ and $\beta \in \Delta^1_r[\gamma]$, then $\alpha \in \Delta^1_r[\gamma]$.* $\square$

Thus the relation "Δ^1_r in" is transitive and one may consider Δ^1_r degrees analogous to the (ordinary) degrees of § II.5:

$$\Delta^1_r\text{-dg}(\alpha) = \{\beta\colon \alpha \in \Delta^1_r[\beta] \text{ and } \beta \in \Delta^1_r[\alpha]\}.$$

The Δ^1_1 degrees are called *hyperdegrees* and are considered further in § IV.2.

There are several ways one might hope to extend Post's Theorem to the analytical hierarchy. The most direct generalization is

(1) $R \in \Delta^1_{r+1} \overset{?}{\leftrightarrow} R$ recursive in some set $A \in \Sigma^1_r$.

It could be argued that a relationship stronger than "recursive in" is needed here. Since "recursive in" coincides with "Δ^0_1 in", a natural choice is "Δ^1_1 in" or even "Δ^1_r in":

(2) $R \in \Delta^1_{r+1} \overset{?}{\leftrightarrow} R \in \Delta^1_1[A]$ for some $A \in \Sigma^1_r$;

(3) $R \in \Delta^1_{r+1} \overset{?}{\leftrightarrow} R \in \Delta^1_r[A]$ for some $A \in \Sigma^1_r$.

The implication ($\leftarrow$) of (3), and hence of (1) and (2) is immediate from Corollary 2.12: if $A \in \Sigma^1_r$, $\mathsf{K}_A \in \Delta^1_{r+1}$ and $R \in \Delta^1_r[A] \subseteq \Delta^1_{r+1}[A]$, so $R \in \Delta^1_{r+1}$. However the implication ($\rightarrow$) of (3), and hence of (1) and (2), is false for all $r > 0$ (also for $r = 0$ but for a different reason — see Corollary 3.8 below). By the relativized analytical hierarchy theorem there exists a relation $R \in \Sigma^1_r[U^1_r] \sim \Delta^1_r[U^1_r]$. Since every $A \in \Sigma^1_r$ is recursive in U^1_r, $R \notin \Delta^1_r[A]$ for any $A \in \Sigma^1_r$. However, $\Sigma^1_r[U^1_r] \subseteq \Delta^1_{r+1}[U^1_r]$ and $U^1_r \in \Sigma^1_r \subseteq \Delta^1_{r+1}$, so by Corollary 2.12, $R \in \Delta^1_{r+1}$. Thus there is no analogue of Post's Theorem for the analytical hierarchy.

2.14 Definition. For all $r > 0$,

(i) $\mathbf{\Sigma}^1_r = \bigcup \{\Sigma^1_r[\beta]: \beta \in {}^\omega\omega\}$;

(ii) $\mathbf{\Pi}^1_r = \bigcup \{\Pi^1_r[\beta]: \beta \in {}^\omega\omega\}$;

(iii) $\mathbf{\Delta}^1_r = \mathbf{\Sigma}^1_r \cap \mathbf{\Pi}^1_r$.

The classes $\mathbf{\Sigma}^1_r$ and $\mathbf{\Pi}^1_r$ comprise what is known as the *projective hierarchy*, and were known and studied long before the invention of recursion theory. It follows easily from Lemma 2.8 that $\mathbf{\Sigma}^1_1 = \{\exists^1\mathsf{P}: \mathsf{P}$ is closed$\}$. That is, the $\mathbf{\Sigma}^1_1$ relations are exactly the *projections* (with respect to a function coordinate) of closed relations. Similarly the $\mathbf{\Sigma}^1_{r+1}$ relations are exactly the projections of $\mathbf{\Pi}^1_r$ relations. The class of *projective relations* ($= \mathbf{\Delta}^1_{(\omega)}$) is the smallest class containing the closed relations and closed under projection ($\exists^1$) and complementation.

Contrasting with Theorem 1.16 we have

2.15 Theorem. *For all $r > 0$, $\mathbf{\Sigma}^1_r$ and $\mathbf{\Pi}^1_r$ are closed under countable unions and intersections of relations of the same rank.*

Proof. Suppose $\mathsf{P}_p \in \mathbf{\Sigma}^1_r$ for all $p \in \omega$ and let

$$\mathsf{Q}(p, \mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \mathsf{P}_p(\mathbf{m}, \boldsymbol{\alpha}).$$

Then as in the proof of Theorem 1.16, $\mathsf{Q} \in \boldsymbol{\Sigma}_r^1$. Since

$$\bigcup \{\mathsf{P}_p : p \in \omega\} = \exists^0\mathsf{Q} \quad \text{and} \quad \bigcap \{\mathsf{P}_p : p \in \omega\} = \forall^0\mathsf{Q},$$

these are both in $\boldsymbol{\Sigma}_r^1$ by the relativized version of Theorem 2.6. □

The class Bo of *Borel relations* is the smallest class containing the open relations and closed under countable unions and intersections of relations of the same rank.

2.16 Corollary. $\mathrm{Bo} \subseteq \boldsymbol{\Delta}_1^1$. □

We will prove in § IV.3 that in fact $\mathrm{Bo} = \boldsymbol{\Delta}_1^1$. This classical result of descriptive set theory is paradigmatic for many of the other results of later chapters.

2.17–2.23 Exercises

2.17. Show that $\{\alpha : \alpha \in \Delta^0_{(\omega)}\} \in \Delta_1^1$ (cf. Exercise 1.18 and Corollary 4.21 below).

2.18. We might have defined the analytical hierarchy by quantifying sets rather than functions. Let $\dot{\exists}^1\mathsf{P}$ be the relation $\exists B\ \mathsf{P}(\mathbf{m}, \boldsymbol{\alpha}, B)$ and define analogously $\dot{\forall}^1$, $\dot{\Sigma}_r^1$, etc. Show that

$$\dot{\Sigma}_r^1 = \Sigma_r^1, \text{ but that corresponding to Lemma 2.8 we have}$$
$$\{\dot{\exists}^1\mathsf{P} : \mathsf{P} \in \Pi_1^0\} \subsetneqq \{\dot{\exists}^1\mathsf{P} : \mathsf{P} \in \Pi_2^0\} = \dot{\Sigma}_1^1.$$

2.19. For any countable indexed family of relations $\langle \mathsf{P}_s : s \in \omega\rangle$, let $\mathscr{A}\langle \mathsf{P}_s : s \in \omega\rangle = \bigcup\{\bigcap\{\mathsf{P}_{\bar{\beta}(p)} : p \in \omega\} : \beta \in {}^\omega\omega\}$. Show that for any R, the following are equivalent:

(i) $\mathsf{R} \in \boldsymbol{\Sigma}_1^1$;
(ii) $\mathsf{R} = \mathscr{A}\langle \mathsf{P}_s : s \in \omega\rangle$ for some $\mathsf{P}_s \in \boldsymbol{\Delta}_1^0$;
(iii) $\mathsf{R} = \exists^1\mathsf{P}$ for some $\mathsf{P} \in \boldsymbol{\Delta}_1^1$.

2.20. The operation $\mathscr{A}$ of the preceding exercise may also be regarded as a quantifier:

$$(\mathscr{A}\mathsf{P})(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \exists\beta\ \forall p\ \mathsf{P}(\bar{\beta}(p), \mathbf{m}, \boldsymbol{\alpha}).$$

Show that for all r and P

(i) if $r \geqslant 1$ and $\mathsf{P} \in \Sigma_r^1$ $(\boldsymbol{\Sigma}_r^1)$, then $\mathscr{A}\mathsf{P} \in \Sigma_r^1$ $(\boldsymbol{\Sigma}_r^1)$;
(ii) if $r \geqslant 2$ and $\mathsf{P} \in \Pi_r^1$ $(\boldsymbol{\Pi}_r^1)$, then $\mathscr{A}\mathsf{P} \in \Pi_r^1$ $(\boldsymbol{\Pi}_r^1)$.

2.21. Prove Corollary 2.7.

2.22. Prove that the class of Borel relations is closed under complementation.

2.23. Prove the following *Strong Hierarchy Theorem*: for all $r>0$, $\boldsymbol{\Sigma}_r^1 \not\subseteq \boldsymbol{\Delta}_r^1$.

2.24 Notes. The arithmetical hierarchy was developed independently by Kleene [1943] and Mostowski [1946] (because of the Second World War Mostowski in Warsaw did not see Kleene's paper until his manuscript was finished). The analytical hierarchy was first studied in Kleene [1955b]. Kleene's approach was motivated by Gödel's results on incompleteness, whereas Mostowski saw the arithmetical hierarchy as analogous to the projective hierarchy, with "recursive" corresponding to "Borel" and existential number quantification ($\exists^0$) corresponding to projection ($\exists^1$). When Kleene [1950] showed that Σ_1^0 does not have the separation property, Mostowski pointed out that this is a flaw in the analogy, since $\boldsymbol{\Sigma}_1^1$ does have the separation property. Addison, in his thesis (announced in Addison [1955]) proposed that a better correspondence is that between "recursive" and "closed-open" with $\exists^0$ corresponding to countable union. The unified approach to these hierarchies grew out of this analogy and is also largely due to Addison.

The technique of proof of Theorem 2.11 is due to Shoenfield [1962].

3. Inductive Definability

To this point we have used inductive definitions mainly as a tool. We begin now to consider their use as a measure of complexity. We shall be interested here in the relationship between the complexity of Γ and that of $\bar{\Gamma}$ as measured by their classifications in the arithmetical and analytical hierarchies.

For each $\alpha \in {}^{\omega}\omega$ we put $Z_\alpha = \{p\colon \alpha(p)=0\}$. Then for any inductive operator Γ over ω we define a relation P_Γ by

$$\mathsf{P}_\Gamma(m,\alpha) \leftrightarrow m \in \Gamma(Z_\alpha).$$

We write $\Gamma \in \Sigma_r^i$, etc. to mean $\mathsf{P}_\Gamma \in \Sigma_r^i$, etc. We showed in Example 2.3 that for monotone $\Gamma \in \Delta_{(\omega)}^0$, $\bar{\Gamma} \in \Pi_1^1$. The same proof establishes

3.1 Theorem. *For any $r>0$ and any monotone operator Γ over ω,*

$$\Gamma \in \Pi_r^1 \to \bar{\Gamma} \in \Pi_r^1.$$

Proof. If Γ is monotone, we have

$$m \in \bar{\Gamma} \leftrightarrow \forall B\,[\Gamma(B) \subseteq B \rightarrow m \in B]$$
$$\leftrightarrow \forall \alpha\,[\forall p\,(\mathsf{P}_\Gamma(p, \alpha) \rightarrow \alpha(p) = 0) \rightarrow \alpha(m) = 0].$$

An easy calculation based on the techniques of the preceding section shows that if $\mathsf{P}_\Gamma \in \Pi^1_r$, so is $\bar{\Gamma}$. $\square$

Much of the rest of this section concerns the question of when a strengthening of the hypothesis of Theorem 3.1 allows a strengthening of the conclusion. We begin with the case $r = 1$ and find that limiting Γ to lie in Π^0_1 permits no stronger conclusion, whereas if $\Gamma \in \Sigma^0_1$ (and Γ is monotone), then $\bar{\Gamma} \in \Sigma^0_1$.

3.2 Theorem. *For every set $A \in \Pi^1_1$, there exists a monotone operator $\Gamma \in \Pi^0_1$ such that A is many-one reducible to $\bar{\Gamma}$.*

Proof. Let A be an arbitrary Π^1_1 set. By Lemma 2.9, $A = \forall^1\mathsf{P}$ for some $\mathsf{P} \in \Sigma^0_1$. Thus by Theorem II.4.12 there exists a recursive relation R such that

$$m \in A \leftrightarrow \forall \beta\, \exists p\, R(m, \bar{\beta}(p)).$$

Let Γ be the operator defined by

$$\langle m, s\rangle \in \Gamma(B) \leftrightarrow R(m, s) \vee \forall n\, (\langle m, s * \langle n\rangle\rangle \in B).$$

Clearly Γ is monotone and $\Gamma \in \Pi^0_1$. We claim that

$$\langle m, s\rangle \in \bar{\Gamma} \leftrightarrow \forall \beta\, \exists p\, R(m, s * \bar{\beta}(p)).$$

Once this is established we have

$$m \in A \leftrightarrow \langle m, \langle\ \rangle\rangle \in \bar{\Gamma}$$

so that A is many-one reducible to $\bar{\Gamma}$.

To establish the claim, let

$$C = \{\langle m, s\rangle : \forall \beta\, \exists p\, R(m, s * \bar{\beta}(p))\}.$$

We must show $C = \bar{\Gamma}$. As usual, to show $\bar{\Gamma} \subseteq C$ it suffices to show $\Gamma(C) \subseteq C$. Suppose $\langle m, s\rangle \in \Gamma(C)$. Then either

(1) $R(m, s)$ or (2) $\forall n . \langle m, s * \langle n\rangle\rangle \in C$.

In case (1), any β satisfies $R(m, s * \bar{\beta}(0))$ so that $\langle m, s\rangle \in C$. In case (2), for all n, $\forall \beta\, \exists p\, R(m, s * \langle n\rangle * \bar{\beta}(p))$, so also $\langle m, s\rangle \in C$.

For the converse inclusion we assume $\langle m, s\rangle \notin \bar{\Gamma}$ and construct a function β such that $\forall p \sim R(m, s * \bar{\beta}(p))$. Let $D = \{t: \langle m, s * t\rangle \notin \bar{\Gamma}\}$. By assumption, $\langle\ \rangle \in D$. Since $\Gamma(\bar{\Gamma}) = \bar{\Gamma}$ we have

$$t \in D \rightarrow \langle m, s * t\rangle \notin \Gamma(\bar{\Gamma}) \rightarrow \exists n\,(t * \langle n\rangle \in D).$$

Hence there is a unique function β such that for all p,

$$\beta(p) = \text{least } n[\bar{\beta}(p) * \langle n\rangle \in D].$$

For this β, $\forall p\,(\langle m, s * \bar{\beta}(p)\rangle \notin \bar{\Gamma})$ so in particular, $\forall p \sim R(m, s * \bar{\beta}(p))$. Hence $\langle m, s\rangle \notin C$. $\square$

Combining 3.1 and 3.2, a set A is Π_1^1 iff A is many-one reducible to $\bar{\Gamma}$ for some monotone operator $\Gamma \in \Pi_1^0$. The same proof works also for relations R on numbers.

Since by the Analytical Hierarchy Theorem there exist sets A wich are Π_1^1 but not Δ_1^1, there exist Π_1^0 monotone operators such that $\bar{\Gamma}$ is Π_1^1 but not Δ_1^1. A natural question to ask in conjunction with this theorem is whether or not every Π_1^1 set is equal to $\bar{\Gamma}$ for some $\Gamma \in \Pi_1^0$. We shall show in § 6 that this is false if Γ is required to be monotone. The answer is unknown if non-monotone Γ are admitted.

We turn now to Σ_1^0 operators, which turn out to be much weaker.

3.3 Lemma. *For any inductive operator $\Gamma \in \Sigma_1^0$, $|\Gamma| \leqslant \omega$.*

Proof. Suppose $\Gamma \in \Sigma_1^0$ and let R be a recursive relation such that $\mathsf{P}_\Gamma(m, \alpha) \leftrightarrow \exists n R(m, \bar{\alpha}(n))$. $|\Gamma|$ is the least ordinal σ such that $\Gamma(\Gamma^{(\sigma)}) \subseteq \Gamma^{(\sigma)}$, so to show $|\Gamma| \leqslant \omega$ it suffices to show $\Gamma(\Gamma^{(\omega)}) \subseteq \Gamma^{(\omega)}$. Suppose $m \in \Gamma(\Gamma^{(\omega)})$, so for some n, $R(m, \bar{\mathsf{K}}_{\Gamma^{(\omega)}}(n))$. For each $i < n$ let

$$s_i = \begin{cases} \text{least } s \,.\, i \in \Gamma^s, & \text{if} \quad i \in \Gamma^{(\omega)}; \\ 0, \quad \text{otherwise}; \end{cases}$$

and

$$t = \max\{s_i : i < n\}.$$

Then $\bar{\mathsf{K}}_{\Gamma^{(\omega)}}(n) = \bar{\mathsf{K}}_{\Gamma^t}(n)$ so $R(m, \bar{\mathsf{K}}_{\Gamma^t}(n))$ and thus $m \in \Gamma(\Gamma^t) = \Gamma^{t+1} \subseteq \Gamma^{(\omega)}$. $\square$

As a special notation for the following lemma and theorem only, we write

$$s < A \leftrightarrow (\forall i < \lg(s))((s)_i \leqslant 1 \wedge [(s)_i = 0 \rightarrow i \in A]).$$

Thus if $s = \bar{\mathsf{K}}_B(n)$, then $s < A \leftrightarrow B \cap n \subseteq A \cap n$.

The import of the following lemma is that for monotone Σ_1^0 operators Γ, the relation $m \in \Gamma(A)$ depends only on *positive* information about A — that is, information concerning membership in A but not concerning *non*-membership in A.

3.4 Lemma. *For any monotone operator $\Gamma \in \Sigma_1^0$, there exists a recursive relation R such that for all m and A,*

$$m \in \Gamma(A) \leftrightarrow \exists s\,[s < A \wedge R(m,s)].$$

Proof. If $\Gamma \in \Sigma_1^0$ there exists by definition a recursive R such that

$$\begin{aligned} m \in \Gamma(A) &\leftrightarrow \exists n\, R(m, \bar{\mathsf{K}}_A(n)) \\ &\leftrightarrow \exists s\, \exists n\, [s = \bar{\mathsf{K}}_A(n) \wedge R(m,s)]. \end{aligned}$$

Thus for this R,

$$m \in \Gamma(A) \rightarrow \exists s\,[s < A \wedge R(m,s)].$$

The converse implication holds also. Suppose $s < A$ and $R(m,s)$. Let

$$B = \{i\colon [i < \lg(s) \wedge (s)_i = 0] \vee [i \geq \lg(s) \wedge i \in A]\}.$$

Then $B \subseteq A$ and $s = \bar{\mathsf{K}}_B(\lg(s))$. Hence $m \in \Gamma(B)$ and by the monotonicity of Γ, $\Gamma(B) \subseteq \Gamma(A)$ so also $m \in \Gamma(A)$. $\square$

3.5 Theorem. *For any monotone operator Γ,*

$$\Gamma \in \Sigma_1^0 \rightarrow \bar{\Gamma} \in \Sigma_1^0.$$

Proof. Let Γ be a Σ_1^0 monotone operator. We shall define a primitive recursive function f such that for all r, $f(r+1)$ is a semi-index for the semi-recursive set Γ^r. Once this is done, we have by Lemma 3.3

$$m \in \bar{\Gamma} \leftrightarrow \exists r.\, m \in \Gamma^r \leftrightarrow \exists r\, \exists u\, \mathsf{T}(f(r+1), \langle m \rangle, u, \langle\ \rangle)$$

and thus $\bar{\Gamma} \in \Sigma_1^0$.

We let $f(0) = 0$ and $f(r+1) = \mathrm{Sb}_0(c, f(r))$, where c is an index chosen as follows. Let R be as in Lemma 3.4, so that

$$m \in \Gamma^r \leftrightarrow \exists s\,[s < \Gamma^{(r)} \wedge R(m,s)].$$

Under the induction hypothesis that $\Gamma^{(r)} = \mathrm{Dm}\{f(r)\}$ (which is valid for $r = 0$) we have

$$m \in \Gamma^r \leftrightarrow \exists s\,(\forall i < \lg(s)[(s)_i \leq 1 \wedge ((s)_i = 0 \rightarrow \exists u\, \mathrm{T}(f(r), \langle i\rangle, u, \langle\quad\rangle))] \wedge R(m, s))$$
$$\leftrightarrow \exists p\, S(p, f(r), m)$$

for an appropriate recursive relation S. Choose c such that

$$(n, m) \in \mathrm{Dm}\{c\} \leftrightarrow \exists p\, S(p, n, m).$$

Then

$$m \in \Gamma^r \leftrightarrow (f(r), m) \in \mathrm{Dm}\{c\} \leftrightarrow m \in \mathrm{Dm}\{\mathrm{Sb}_0(c, f(r))\} = \mathrm{Dm}\{f(r+1)\}. \quad \square$$

If Γ is not monotone, then $\bar{\Gamma}$ need not even be arithmetical. Let

$$U^0_{(\omega)} = \{\langle r, a, \mathbf{m}\rangle\colon\ U^0_r(a, \langle \mathbf{m}\rangle)\}.$$

It is clear that for any arithmetical relation R, there exist r and a such that

$$R(\mathbf{m}) \leftrightarrow \langle r, a, \mathbf{m}\rangle \in U^0_{(\omega)}.$$

Hence $U^0_{(\omega)}$ is not itself arithmetical as if it were, say, Σ^0_r, then so would be every arithmetical relation contrary to the Arithmetical Hierarchy Theorem. On the other hand,

3.6 Theorem. *There exists a Σ^0_1 inductive operator Γ such that $\bar{\Gamma} = U^0_{(\omega)}$.*

Proof. Let

$$s \in \Gamma(A) \leftrightarrow \exists a\, \exists \mathbf{m}\,[s = \langle 1, a, \mathbf{m}\rangle \wedge U^0_1(a, \langle \mathbf{m}\rangle)]$$
$$\vee\ \exists r\, \exists a\, \exists \mathbf{m}\,[\exists b\, \exists \mathbf{n}\, \langle r, b, \mathbf{n}\rangle \in A\ \wedge s = \langle r+1, a, \mathbf{m}\rangle \wedge$$
$$\exists p\, (\langle r, a, p, \mathbf{m}\rangle \notin A)] \vee (s \in A).$$

Clearly $\Gamma \in \Sigma^0_1$ and it is easy to show by induction on r that

$$\Gamma^r = \{\langle t, a, \mathbf{m}\rangle\colon\ 0 < t \leq r+1 \wedge U^0_t(a, \langle \mathbf{m}\rangle)\}.$$

Hence $\bar{\Gamma} = U^0_{(\omega)}$. $\square$

Note that Γ is non-monotone because of the condition in its definition that something *not* belong to A. On the positive side,

3.7 Theorem. *For any inductive operator Γ,*

$$\Gamma \in \Sigma_1^0 \rightarrow \bar{\Gamma} \in \Delta_1^1.$$

Proof. Let Γ be a Σ_1^0 inductive operator and let β^* be the function such that for all r and m,

$$\beta^*(\langle r+1, m\rangle) = 0 \leftrightarrow m \in \Gamma^r,$$

and $\beta^*(t) = 1$, otherwise. Consider the following arithmetical relation S:

$$\begin{aligned}\mathsf{S}(\beta) \leftrightarrow \forall t\,[\beta(t) \leq 1] \wedge \\ \wedge \forall t\,[\sim \mathrm{Sq}(t) \vee \lg(t) \neq 2 \vee (t)_0 = 0 \rightarrow \beta(t) = 1] \\ \wedge \forall r \forall m\,[\beta(\langle r+1, m\rangle) = 0 \leftrightarrow \mathsf{P}_\Gamma(m, \lambda n.\, \beta(\langle r, n\rangle))].\end{aligned}$$

Note first that $\mathsf{S}(\beta^*)$. Furthermore, for any β such that $\mathsf{S}(\beta)$, $\beta(t) = 1 = \beta^*(t)$ for all t not of the form $\langle r+1, m\rangle$ and it is easy to show by induction on r that for all m

$$\beta(\langle r+1, m\rangle) = 0 \leftrightarrow m \in \Gamma^r.$$

Thus β^* is the unique member of S and we have

$$\begin{aligned}m \in \bar{\Gamma} \leftrightarrow \exists r.\, m \in \Gamma^r \leftrightarrow \exists \beta\,[\mathsf{S}(\beta) \wedge \exists r.\, \beta(\langle r+1, m\rangle) = 0] \\ \leftrightarrow \forall \beta\,[\mathsf{S}(\beta) \rightarrow \exists r.\, \beta(\langle r+1, m\rangle) = 0].\end{aligned}$$

Thus $\bar{\Gamma} \in \Delta_1^1$. □

3.8 Corollary. $U^0_{(\omega)} \in \Delta_1^1 \sim \Delta^0_{(\omega)}$.

Proof. By 3.6 and 3.7. □

The proof of 3.7 actually establishes a stronger result:

$$\Gamma \in \Delta_1^1 \wedge |\Gamma| \leq \omega \rightarrow \bar{\Gamma} \in \Delta_1^1.$$

Since by Theorem 3.2 there exist monotone $\Gamma \in \Pi_1^0$ such that $\bar{\Gamma} \notin \Delta_1^1$, it follows that for some such $\Gamma, |\Gamma| > \omega$. It will soon be clear that Π_1^0 operators may have quite large closure ordinal; an upperbound for these is obtained in § IV.2.

For the remainder of this section we exploit the fact (Corollary I.3.2) that all inductive operators over ω have countable closure ordinal. Thus for any such Γ we have

$$m \in \bar{\Gamma} \leftrightarrow (\exists \sigma < \aleph_1) m \in \Gamma^{\sigma}$$
$$\leftrightarrow (\forall \sigma < \aleph_1)[\Gamma^{\sigma} \subseteq \Gamma^{(\sigma)} \rightarrow m \in \Gamma^{\sigma}].$$

Recall from the end of § I.1 that W denotes the set of functions γ such that $\leqslant_\gamma$ is well-ordering of order type $\|\gamma\| < \aleph_1$. Since every countable ordinal is $\|\gamma\|$ for some $\gamma \in \mathsf{W}$, we have

$$m \in \bar{\Gamma} \leftrightarrow (\exists \gamma \in \mathsf{W}) m \in \Gamma^{\|\gamma\|}$$
$$\leftrightarrow (\forall \gamma \in \mathsf{W})[\Gamma^{\|\gamma\|} \subseteq \Gamma^{(\|\gamma\|)} \rightarrow m \in \Gamma^{\|\gamma\|}].$$

Thus we can classify $\bar{\Gamma}$ in the analytical hierarchy if we can classify the relations $m \in \Gamma^{(\|\gamma\|)}$ and $m \in \Gamma^{\|\gamma\|}$.

3.9 Theorem. *For any $r > 0$ and any inductive operator $\Gamma \in \Delta^1_r$, there exist relations $\mathsf{V}^{()}_\Sigma$ and $\mathsf{V}_\Sigma \in \Sigma^1_r$ and $\mathsf{V}^{()}_\Pi$ and $V_\Pi \in \Pi^1_r$ such that for any $\gamma \in \mathsf{W}$ and any m,*

(i) $m \in \Gamma^{(\|\gamma\|)} \leftrightarrow \mathsf{V}^{()}_\Sigma(m, \gamma) \leftrightarrow \mathsf{V}^{()}_\Pi(m, \gamma)$;
(ii) $m \in \Gamma^{\|\gamma\|} \leftrightarrow \mathsf{V}_\Sigma(m, \gamma) \leftrightarrow \mathsf{V}_\Pi(m, \gamma)$.

Proof. The technique is an elaboration of that used for Theorem 3.7 in which we index the stages of Γ by $p \in \mathrm{Fld}(\gamma)$ rather than by $r \in \omega$. Suppose $\Gamma \in \Delta^1_r$ with $r > 0$. For each $\gamma \in \mathsf{W}$ we define α_γ and β_γ by:

$$\alpha_\gamma(\langle p, m\rangle) = 0 \leftrightarrow p \in \mathrm{Fld}(\gamma) \wedge m \in \Gamma^{(|p|_\gamma)};$$
$$\beta_\gamma(\langle p, m\rangle) = 0 \leftrightarrow p \in \mathrm{Fld}(\gamma) \wedge m \in \Gamma^{|p|_\gamma};$$

and $\alpha_\gamma(t) = \beta_\gamma(t) = 1$, otherwise. Let S be the relation defined as follows:

$$\begin{aligned}\mathsf{S}(\alpha, \beta, \gamma) \leftrightarrow {} & \forall t\,[\alpha(t) \leqslant 1 \wedge \beta(t) \leqslant 1] \\ & \wedge \forall t\,[\sim \mathrm{Sq}(t) \vee \mathrm{lg}(t) \neq 2 \vee (t)_0 \notin \mathrm{Fld}(\gamma) \rightarrow \alpha(t) = \beta(t) = 1] \\ & \wedge \forall p\, \forall m\, (p \in \mathrm{Fld}(\gamma) \rightarrow [\alpha(\langle p, m\rangle) = 0 \leftrightarrow (\exists q <_\gamma p)\beta(\langle q, m\rangle) = 0] \\ & \wedge [\beta(\langle p, m\rangle) = 0 \leftrightarrow \mathsf{P}_\Gamma(m, \lambda n\,.\,\alpha(\langle p, n\rangle))]).\end{aligned}$$

As $\mathsf{P}_\Gamma \in \Delta^1_r$ and everything else is arithmetical, $\mathsf{S} \in \Delta^1_r$. It is routine to check that for any $\gamma \in \mathsf{W}$, $S(\alpha_\gamma, \beta_\gamma, \gamma)$. We claim that for any $\gamma \in \mathsf{W}$ and any α and β such that $S(\alpha, \beta, \gamma)$, $\alpha = \alpha_\gamma$ and $\beta = \beta_\gamma$. For any such α, β, and γ, let

$$Z_{\alpha, p} = \{m : \alpha(\langle p, m\rangle) = 0\}$$

and similarly $Z_{\beta, p}$. It will suffice to show that for all $p \in \mathrm{Fld}(\gamma)$,

$$Z_{\alpha,p} = \Gamma^{(|p|_\gamma)} \quad \text{and} \quad Z_{\beta,p} = \Gamma^{|p|_\gamma}.$$

We proceed by induction on $|p|_\gamma$ and assume as induction hypothesis that for all $q <_\gamma p$,

$$Z_{\alpha,q} = \Gamma^{(|q|_\gamma)} \quad \text{and} \quad Z_{\beta,q} = \Gamma^{|q|_\gamma}.$$

Then by the definition of S and formulas (8)–(10) of § I.1,

$$\begin{aligned} m \in Z_{\alpha,p} &\leftrightarrow (\exists q <_\gamma p) m \in Z_{\beta,q} \leftrightarrow (\exists q <_\gamma p) m \in \Gamma^{|q|_\gamma} \\ &\leftrightarrow (\exists \sigma < |p|_\gamma) m \in \Gamma^\sigma \leftrightarrow m \in \Gamma^{(|p|_\gamma)}, \end{aligned}$$

and

$$\begin{aligned} m \in Z_{\beta,p} &\leftrightarrow \mathsf{P}_\Gamma(m, \lambda n\,.\,\alpha(\langle p, n\rangle)) \leftrightarrow m \in \Gamma(Z_{\alpha,p}) \\ &\leftrightarrow m \in \Gamma(\Gamma^{(|p|_\gamma)}) = \Gamma^{|p|_\gamma}. \end{aligned}$$

We now set

$$\mathsf{V}_\Sigma^{()}(m, \gamma) \leftrightarrow \exists \alpha\beta\, [\mathsf{S}(\alpha, \beta, \gamma) \wedge \exists p\, (\beta(\langle p, m\rangle) = 0)];$$
$$\mathsf{V}_\Pi^{()}(m, \gamma) \leftrightarrow \forall \alpha\beta\, [\mathsf{S}(\alpha, \beta, \gamma) \rightarrow \exists p\, (\beta(\langle p, m\rangle) = 0)];$$
$$\mathsf{V}_\Sigma(m, \gamma) \leftrightarrow \exists \alpha\beta\delta\, [\mathsf{S}(\alpha, \beta, \gamma) \wedge \forall n\, (\delta(n) = 0 \leftrightarrow \exists p\, [\beta(\langle p, m\rangle) = 0]) \wedge \mathsf{P}_\Gamma(m, \delta)];$$
$$\mathsf{V}_\Pi(m, \gamma) \leftrightarrow \forall \alpha\beta\delta\, [\mathsf{S}(\alpha, \beta, \gamma) \wedge \forall n\, (\delta(n) = 0 \leftrightarrow \exists p\, [\beta(\langle p, m\rangle) = 0]) \rightarrow \mathsf{P}_\Gamma(m, \delta)].$$

We leave to the reader the straightforward verification that these relations satisfy the conditions of the theorem. □

3.10 Theorem. *For any $r \geqslant 2$ and any inductive operator Γ,*

$$\Gamma \in \Delta_r^1 \rightarrow \bar{\Gamma} \in \Delta_r^1.$$

Proof. With the notation of the preceding theorem and the remarks before it we have

$$\begin{aligned} m \in \bar{\Gamma} &\leftrightarrow \exists \gamma\, [\gamma \in \mathsf{W} \wedge \mathsf{V}_\Sigma(m, \gamma)] \\ &\leftrightarrow \forall \gamma\, [\gamma \in \mathsf{W} \wedge \forall p\, (\mathsf{V}_\Pi(p, \gamma) \rightarrow \mathsf{V}_\Sigma^{()}(p, \gamma)) \rightarrow \mathsf{V}_\Pi(m, \gamma)]. \end{aligned}$$

Since $r \geqslant 2$, $\mathsf{W} \in \Pi_1^1 \subseteq \Delta_r^1$ and an easy computation shows that the first formula gives a Σ_r^1 definition for $\bar{\Gamma}$, the second a Π_r^1 definition. □

By Theorem 3.2, the result fails for $r = 1$, but we can get some information about Δ_1^1 operators.

3.11 Definition. An ordinal σ is *recursive* iff $\sigma = \|\gamma\|$ for some recursive $\gamma \in \mathsf{W}$. The least non-recursive ordinal is denoted by ω_1.

As there are only countably many recursive functions, ω_1 is a countable ordinal. Furthermore it is easy to see that any ordinal less than a recursive ordinal is also recursive so that σ is recursive iff $\sigma < \omega_1$. Other properties of ω_1 are indicated in Exercise 3.27.

3.12 Corollary. *For any inductive operator Γ,*

$$\Gamma \in \Delta_1^1 \wedge |\Gamma| < \omega_1 \rightarrow \bar{\Gamma} \in \Delta_1^1.$$

Proof. If $|\Gamma| < \omega_1$, then there exists a recursive function $\gamma \in \mathsf{W}$ such that $\|\gamma\| = |\Gamma|$. Then by Theorem 3.9,

$$m \in \bar{\Gamma} \leftrightarrow m \in \Gamma^{\|\gamma\|} \leftrightarrow \mathsf{V}_\Sigma(m, \gamma) \leftrightarrow \mathsf{V}_\Pi(m, \gamma).$$

The conclusion follows from Theorem 2.5. □

The proof of Corollary 3.12 establishes an apparently stronger result. We say an ordinal σ is Δ_r^1 iff $\sigma = \|\gamma\|$ for some $\gamma \in \Delta_r^1 \cap \mathsf{W}$; δ_r^1 denotes the least non-Δ_r^1 ordinal. Then using the Analytical Substitution Theorem (2.11) instead of Theorem 2.5 we have by the same argument,

$$\Gamma \in \Delta_1^1 \wedge |\Gamma| < \delta_1^1 \rightarrow \bar{\Gamma} \in \Delta_1^1.$$

However, we shall see in § IV.2 that $\delta_1^1 = \omega_1$ so that this is no improvement. We can also now extend the reasoning following Corollary 3.8 to conclude that for some Π_1^0 monotone operators Γ we have $|\Gamma| \geqslant \omega_1$. In §IV.2 we shall also prove that for any Π_1^0 operator $\Gamma, |\Gamma| \leqslant \omega_1$ and $\bar{\Gamma} \in \Pi_1^1 \sim \Delta_1^1$ just in case $|\Gamma| = \omega_1$.

For arbitrary inductive operators $\Gamma \in \Sigma_r^1$, the best possible classification of $\bar{\Gamma}$ in the analytical hierarchy is that given by Theorem 3.10: $\bar{\Gamma} \in \Delta_{r+1}^1$. For monotone Γ, however, a refinement of the proof of Theorem 3.10 yields a result parallel to Theorem 3.1.

3.13 Theorem. *For any $r > 0$ and any monotone operator $\Gamma \in \Sigma_r^1$ (Π_r^1), there exist relations $\mathsf{V}^{()}$ and $\mathsf{V} \in \Sigma_r^1$ (Π_r^1) such that for any $\gamma \in \mathsf{W}$ and any m,*

(i) $m \in \Gamma^{(\|\gamma\|)} \leftrightarrow \mathsf{V}^{()}(m, \gamma)$;

(ii) $m \in \Gamma^{\|\gamma\|} \leftrightarrow \mathsf{V}(m, \gamma)$.

Proof. We follow closely the proof of Theorem 3.9 and only indicate the necessary modifications. Suppose first that $\Gamma \in \Sigma_r^1$ and is monotone. Define α_γ and β_γ for $\gamma \in \mathsf{W}$ as before and let S' be the relation defined as S except that the clause

$$[\beta(\langle p, m\rangle) = 0 \leftrightarrow \mathsf{P}_\Gamma(m, \lambda n.\, \alpha(\langle p, n\rangle))]$$

is replaced by

$$[\beta(\langle p, m\rangle) = 0 \rightarrow \mathsf{P}_\Gamma(m, \lambda n.\, \alpha(\langle p, n\rangle))].$$

Clearly $\mathsf{S}' \in \Sigma^1_r$ and for any $\gamma \in \mathsf{W}$, $\mathsf{S}'(\alpha_\gamma, \beta_\gamma, \gamma)$, but α_γ and β_γ are no longer the only functions for which this is true. Rather we prove by induction on $|p|_\gamma$ that for all $p \in \mathrm{Fld}(\gamma)$, if $\mathsf{S}'(\alpha, \beta, \gamma)$, then

$$Z_{\alpha, p} \subseteq \Gamma^{(|p|_\gamma)} \quad \text{and} \quad Z_{\beta, p} \subseteq \Gamma^{|p|_\gamma}.$$

Assuming as induction hypothesis that this holds for all $q <_\gamma p$, we have

$$\begin{aligned} m \in Z_{\alpha, p} &\leftrightarrow (\exists q <_\gamma p) m \in Z_{\beta, q} \rightarrow (\exists q <_\gamma p) m \in \Gamma^{|q|_\gamma} \\ &\leftrightarrow (\exists \sigma < |p|_\gamma) m \in \Gamma^\sigma \leftrightarrow m \in \Gamma^{(|p|_\gamma)}, \end{aligned}$$

and

$$\begin{aligned} m \in Z_{\beta, p} &\rightarrow \mathsf{P}_\Gamma(m, \lambda n.\, \alpha(\langle p, n\rangle)) \leftrightarrow m \in \Gamma(Z_{\alpha, p}) \\ &\rightarrow m \in \Gamma(\Gamma^{(|p|_\gamma)}) = \Gamma^{|p|_\gamma}. \end{aligned}$$

The last implication is the only place where the monotonicity of Γ is used.

Finally, define $\mathsf{V}^{(\,)}$ and V from S' just as $\mathsf{V}^{(\,)}_\Sigma$ and V_Σ are defined from S. Clearly $\mathsf{V}^{(\,)}$ and V are Σ^1_r. If $\gamma \in \mathsf{W}$ and $m \in \Gamma^{(\|\gamma\|)}$, then for some $p \in \mathrm{Fld}(\gamma)$, $m \in \Gamma^{|p|_\gamma}$ and thus $\beta_\gamma(\langle p, m\rangle) = 0$. Since $\mathsf{S}'(\alpha_\gamma, \beta_\gamma, \gamma)$, also $\mathsf{V}^{(\,)}(m, \gamma)$. Conversely, suppose $\mathsf{V}^{(\,)}(m, \gamma)$ holds, say $\mathsf{S}'(\alpha, \beta, \gamma)$ and $\beta(\langle p, m\rangle) = 0$. Then $p \in \mathrm{Fld}(\gamma)$ and $m \in Z_{\beta, p} \subseteq \Gamma^{|p|_\gamma}$ so $m \in \Gamma^{(\|\gamma\|)}$. We leave the similar verification of (ii) to the reader.

In case Γ is a monotone Π^1_r operator, we define S'' by replacing the last clause in the definition of S by

$$[\mathsf{P}_\Gamma(m, \lambda n.\, \alpha(\langle p, n\rangle)) \rightarrow \beta(\langle p, m\rangle) = 0].$$

Then $\mathsf{S}'' \in \Sigma^1_r$ (!), $\mathsf{S}''(\alpha_\gamma, \beta_\gamma, \gamma)$ holds for any $\gamma \in \mathsf{W}$, and for all $\gamma \in \mathsf{W}$, α and β such that $\mathsf{S}''(\alpha, \beta, \gamma)$ and $p \in \mathrm{Fld}(\gamma)$:

$$\Gamma^{(|p|_\gamma)} \subseteq Z_{\alpha, p} \quad \text{and} \quad \Gamma^{|p|_\gamma} \subseteq Z_{\beta, p}.$$

Then if $\mathsf{V}^{(\,)}$ and V are defined from S'' as $\mathsf{V}^{(\,)}_\Pi$ and V_Π are defined from S in 3.9, we have $\mathsf{V}^{(\,)}$ and V in Π^1_r and an easy computation verifies (i) and (ii). □

3.14 Theorem. *For any $r \geqslant 2$ and any monotone operator Γ,*

$$\Gamma \in \Sigma^1_r \to \bar{\Gamma} \in \Sigma^1_r.$$

Proof. With the notation of the preceding theorem, we have

$$m \in \bar{\Gamma} \leftrightarrow \exists \gamma\, [\gamma \in \mathsf{W} \wedge m \in \Gamma^{\|\gamma\|}] \leftrightarrow \exists \gamma\, [\gamma \in \mathsf{W} \wedge \mathsf{V}(m, \gamma)]. \quad \square$$

At this point the strongest result we have for $\Gamma \in \Sigma^1_1$ is that $\bar{\Gamma} \in \Delta^1_2$. Although this is clearly the best estimate possible in terms of the analytical hierarchy, we shall return to this question in § VI.6 and find better bounds in terms of other measures of complexity.

So far in this section we have treated only inductive operators over ω. There is little difficulty in extending our analysis to operators over ${}^k\omega$. For any α let

$$Z^k_\alpha = \{\mathbf{p}\colon \mathbf{p} \in {}^k\omega \wedge \alpha(\langle \mathbf{p} \rangle) = 0\},$$

and for any operator Γ over ${}^k\omega$ set

$$\mathsf{P}_\Gamma(\mathbf{m}, \alpha) \leftrightarrow \mathbf{m} \in \Gamma(Z^k_\alpha).$$

Then we classify Γ as Σ^i_r, etc. ($i = 0, 1$), according as $\mathsf{P}_\Gamma \in \Sigma^i_r$, etc. and all the results of this section hold with only minor changes in the proofs.

If we attempt to do the same for operators over ${}^{k,l}\omega$ or even over ${}^\omega\omega$, however, we encounter an immediate difficulty. In place of Z^k_α we should have to use

$$\mathsf{Z}^{k,l}_\mathsf{F} = \{(\mathbf{p}, \boldsymbol{\beta})\colon (\mathbf{p}, \boldsymbol{\beta}) \in {}^{k,l}\omega \wedge \mathsf{F}(\mathbf{p}, \boldsymbol{\beta}) = 0\}$$

and attempt to classify Γ by means of the relation

$$\mathbb{P}_\Gamma(\mathbf{m}, \boldsymbol{\alpha}, \mathsf{F}) \leftrightarrow (\mathbf{m}, \boldsymbol{\alpha}) \in \Gamma(\mathsf{Z}^{k,l}_\mathsf{F}).$$

Relations with functionals as arguments are not included in our present system so we have no way to assess the complexity of such a $\mathbb{P}_\Gamma$. We shall develop such means in Chapter VI and return to this question in § VI.7.

We can, however, with our current machinery, treat inductive operators over ${}^{k,l}\omega$ which are decomposable in the sense of Definition I.3.6. Recall that such an operator Γ is defined by a family $\langle \Gamma_{\boldsymbol{\alpha}}\colon \boldsymbol{\alpha} \in {}^l({}^\omega\omega)\rangle$ of operators over ${}^k\omega$ by

$$(\mathbf{m}, \boldsymbol{\alpha}) \in \Gamma(\mathsf{R}) \leftrightarrow \mathbf{m} \in \Gamma_{\boldsymbol{\alpha}}(\mathsf{R}_{\boldsymbol{\alpha}})$$

where $\mathsf{R}_{\boldsymbol{\alpha}}(\mathbf{m}) \leftrightarrow \mathsf{R}(\mathbf{m}, \boldsymbol{\alpha})$. For such an operator we define

$$\mathsf{P}_\Gamma(\mathbf{m}, \boldsymbol{\alpha}, \delta) \leftrightarrow \mathbf{m} \in \Gamma_{\boldsymbol{\alpha}}(Z^k_\delta)$$

and say that Γ is Σ_r^i, etc. according as $\mathsf{P}_\Gamma \in \Sigma_r^i$, etc. Note that with these conventions,

$$(\mathbf{m}, \boldsymbol{\alpha}) \in \Gamma(\mathsf{R}) \leftrightarrow \mathsf{P}_\Gamma(\mathbf{m}, \boldsymbol{\alpha}, \lambda p\,.\,\mathsf{K}_\mathsf{R}((p)_0, \ldots, (p)_{k-1}, \boldsymbol{\alpha})).$$

The crucial restriction is that the arguments $\boldsymbol{\alpha}$ enter only as *parameters*: whether or not $(\mathbf{m}, \boldsymbol{\alpha}) \in \Gamma(\mathsf{R})$ depends only on the membership or non-membership of other sequences $(\mathbf{p}, \boldsymbol{\alpha})$ in R not on that of sequences $(\mathbf{p}, \boldsymbol{\beta})$ for any $\boldsymbol{\beta} \neq \boldsymbol{\alpha}$. One sometimes says that the operators $\Gamma_{\boldsymbol{\alpha}}$ are Σ_r^i, etc. *uniformly in* $\boldsymbol{\alpha}$.

At first glance, the class of decomposable operators over ${}^{k,l}\omega$ may appear very limited. It turns out, however, that not only can all of the results of this section be extended to this class, but also that these extensions are just what is needed for many applications in later chapters. We shall state some of these but relegate most of the details of the proofs to the exercises.

In what follows we always assume that the decomposable operator Γ is defined by the family $\langle \Gamma_{\boldsymbol{\alpha}} \colon \boldsymbol{\alpha} \in {}^{l}({}^{\omega}\omega)\rangle$. Note that Γ is monotone just in case each $\Gamma_{\boldsymbol{\alpha}}$ is monotone.

3.15 Theorem. *For any $r > 0$ and any decomposable monotone operator Γ over ${}^{k,l}\omega$,*

$$\Gamma \in \Pi_r^1 \rightarrow \bar{\Gamma} \in \Pi_r^1.$$

Proof. As in the proof of Theorem 3.1, by Lemma I.3.7,

$$(\mathbf{m}, \boldsymbol{\alpha}) \in \bar{\Gamma} \leftrightarrow \mathbf{m} \in \bar{\Gamma}_{\boldsymbol{\alpha}} \leftrightarrow \forall \delta\,[\forall \mathbf{p}\,(\mathsf{P}_\Gamma(\mathbf{p}, \boldsymbol{\alpha}, \delta) \rightarrow \delta(\langle \mathbf{p}\rangle) = 0) \rightarrow \delta(\langle \mathbf{m}\rangle) = 0]. \quad \square$$

3.16 Theorem. *For every $\mathsf{R} \in \Pi_1^1$, there exists a decomposable monotone operator $\Gamma \in \Pi_1^0$ and a recursive function f such that*

$$\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow (f(\mathbf{m}), \boldsymbol{\alpha}) \in \bar{\Gamma}.$$

Proof. For any $\mathsf{R} \in \Pi_1^1$ there exists a recursive relation S such that

$$\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \forall \beta\, \exists p\; \mathsf{S}(\bar{\beta}(p), \mathbf{m}, \boldsymbol{\alpha}).$$

Let

$$\langle \mathbf{m}, s\rangle \in \Gamma_{\boldsymbol{\alpha}}(B) \leftrightarrow \mathsf{S}(s, \mathbf{m}, \boldsymbol{\alpha}) \vee \forall n\, (\langle \mathbf{m}, s * \langle n\rangle\rangle \in B).$$

Then $\Gamma \in \Pi_1^0$, Γ is monotone, and as in the proof of Theorem 3.2,

$$\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \langle \mathbf{m}, \langle\ \rangle\rangle \in \bar{\Gamma}_{\boldsymbol{\alpha}} \leftrightarrow (\langle \mathbf{m}, \langle\ \rangle\rangle, \boldsymbol{\alpha}) \in \bar{\Gamma}. \quad \square$$

3.17 Theorem. *For any decomposable operator Γ over ${}^{k,l}\omega$,*

(i) $\Gamma \in \Sigma_1^0 \wedge \Gamma$ *monotone* $\to \bar{\Gamma} \in \Sigma_1^0$;

(ii) $\Gamma \in \Sigma_1^0 \to \bar{\Gamma} \in \Delta_1^1$.

Proof. We leave the proof of (i) to Exercise 3.31. Suppose $\Gamma \in \Sigma_1^0$. For each $\boldsymbol{\alpha}$, let

$$\beta_{\boldsymbol{\alpha}}(\langle r+1, \mathbf{m}\rangle) = 0 \leftrightarrow \mathbf{m} \in \Gamma_{\boldsymbol{\alpha}}^r,$$

and $\beta_{\boldsymbol{\alpha}}(t) = 1$, otherwise. Set

$$\begin{aligned} \mathsf{S}(\boldsymbol{\alpha}, \beta) \leftrightarrow \forall t\,[\beta(t) \leqslant 1] & \\ \wedge \forall t\,[\sim \mathrm{Sq}(t) \vee \mathrm{lg}(t) \neq k+1 \vee (t_0) = 0 \to \beta(t) = 1] & \\ \wedge \forall r\, \forall \mathbf{m}\,[\beta(\langle r+1, \mathbf{m}\rangle) = 0 \leftrightarrow \mathsf{P}_\Gamma(\mathbf{m}, \boldsymbol{\alpha}, \lambda n\,.\,\beta(\langle r, n\rangle))]. & \end{aligned}$$

As in the proof of Theorem 3.7, for each $\boldsymbol{\alpha}$, $\beta_{\boldsymbol{\alpha}}$ is the unique β such that $S(\boldsymbol{\alpha}, \beta)$ and we have

$$\begin{aligned} (\mathbf{m}, \boldsymbol{\alpha}) \in \bar{\Gamma} &\leftrightarrow \exists \beta\,[\mathsf{S}(\boldsymbol{\alpha}, \beta) \wedge \exists r\,.\,\beta(\langle r+1, \mathbf{m}\rangle) = 0] \\ &\leftrightarrow \forall \beta\,[\mathsf{S}(\boldsymbol{\alpha}, \beta) \to \exists r\,.\,\beta(\langle r+1, \mathbf{m}\rangle) = 0]. \quad \square \end{aligned}$$

3.18 Theorem. *For any $r \geqslant 2$ and any decomposable inductive operator Γ over ${}^{k,l}\omega$,*

(i) $\Gamma \in \Delta_r^1 \to \bar{\Gamma} \in \Delta_r^1$;

(ii) Γ *monotone* $\wedge\, \Gamma \in \Sigma_r^1 \to \bar{\Gamma} \in \Sigma_r^1$.

Proof. By modifications of the proof of Theorem 3.9 similar to those in the preceding proof we obtain for $\Gamma \in \Delta_r^1$ relations $\mathsf{V}_\Sigma^{(\ )}$ and $\mathsf{V}_\Sigma \in \Sigma_r^1$ and $\mathsf{V}_\Pi^{(\ )}$ and $\mathsf{V}_\Pi \in \Pi_r^1$ such that for all $\gamma \in \mathsf{W}$,

$$\mathbf{m} \in \Gamma_{\boldsymbol{\alpha}}^{(\|\gamma\|)} \leftrightarrow \mathsf{V}_\Sigma^{(\ )}(\mathbf{m}, \boldsymbol{\alpha}, \gamma) \leftrightarrow \mathsf{V}_\Pi^{(\ )}(\mathbf{m}, \boldsymbol{\alpha}, \gamma)$$

and

$$\mathbf{m} \in \Gamma_{\boldsymbol{\alpha}}^{\|\gamma\|} \leftrightarrow \mathsf{V}_\Sigma(\mathbf{m}, \boldsymbol{\alpha}, \gamma) \leftrightarrow \mathsf{V}_\Pi(\mathbf{m}, \boldsymbol{\alpha}, \gamma).$$

Then the proof may be completed as above. The construction for (ii) is similar. $\square$

As usual the results of this section may be relativized to any $\beta \in {}^\omega\omega$. In particular we shall later have occasion to use the "boldface" versions of Theorems 3.15 and 3.18: for any $r > 0$ and any decomposable operator Γ over ${}^{k,l}\omega$,

$$\Gamma \text{ monotone} \wedge \Gamma \in \mathbf{\Pi}^1_r \to \bar{\Gamma} \in \mathbf{\Pi}^1_r;$$

and for $r \geq 2$,

$$\Gamma \in \mathbf{\Delta}^1_r \to \bar{\Gamma} \in \mathbf{\Delta}^1_r$$

and

$$\Gamma \text{ monotone} \wedge \Gamma \in \mathbf{\Sigma}^1_r \to \bar{\Gamma} \in \mathbf{\Sigma}^1_r.$$

For reference, we summarize in a table the results concerning the relative complexity of Γ and $\bar{\Gamma}$:

Operator Γ	Σ^0_1 monotone	Σ^0_1	Π^0_1 monotone	Arithmetical monotone	Π^1_r monotone ($r \geq 1$)	Σ^1_r monotone ($r \geq 2$)	Δ^1_r ($r \geq 2$)
Closure $\bar{\Gamma}$	Σ^0_1	Δ^1_1	Π^1_1	Π^1_1	Π^1_r	Σ^1_r	Δ^1_r

3.19–3.34 Exercises

3.19. Let D be the smallest subset of ω such that $0 \in D$ and for all a, if $\forall p.\{a\}(p) \in D$, then $a \in D$. Show that $D \in \Pi^1_1$ and every Π^1_1 relation on numbers is reducible to D (D is Π^1_1-*complete*).

3.20. Let $\mathscr{L}$ be a countable first-order language and $\mathscr{T}$ a theory of $\mathscr{L}$ which under some standard Gödel numbering is recursively axiomatizable. Sketch a proof that both $\mathrm{Fm} = \{n: n \text{ is the Gödel number of some formula of } \mathscr{L}\}$ and $\mathrm{Th} = \{n: n \text{ is the Gödel number of a theorem of } \mathscr{T}\}$ are both of the form $\bar{\Gamma}$ for monotone Σ^0_1 operators Γ. What difference in these inductive definitions accounts for the fact that Fm is recursive, whereas in general Th is only Σ^0_1?

3.21. Show that for any monotone Σ^0_1 operator Γ, there exists a partial recursive function f such that

(i) for all e and m,

$$f(e, m) \simeq \begin{cases} 0, & \text{if } m \in \Gamma(\{p: \{e\}(p) \simeq 0\}); \\ \text{undefined, otherwise;} \end{cases}$$

(ii) there exists an index $\bar{e}$ such that for all m,

$$f(\bar{e}, m) \simeq \{\bar{e}\}(m) \quad \text{and} \quad \{\bar{e}\}(m) \simeq 0 \leftrightarrow m \in \bar{\Gamma}.$$

3.22 (Moschovakis [1972]). The Π^1_1 relations are exactly those expressible in the form $\forall\beta\,\exists n\,\mathsf{P}(\bar{\beta}(n), \mathbf{m}, \boldsymbol{\alpha})$ with P recursive. The function quantifier may be thought of as an infinite string of number quantifiers thus:

$$\forall p_0 \forall p_1 \forall p_2 \ldots \exists n\, \mathsf{P}(\langle p_0, \ldots, p_{n-1}\rangle, \mathbf{m}, \boldsymbol{\alpha}).$$

The purpose of this exercise is to establish that the class of relations expressible in the form

$$(*) \qquad \forall p_0 \exists p_1 \forall p_2 \exists p_3 \forall p_4 \ldots \exists n\, \mathsf{P}(\langle p_0, \ldots, p_{n-1}\rangle, \mathbf{m}, \boldsymbol{\alpha})$$

is also exactly Π_1^1. The expression (*) may be interpreted in several equivalent ways. We shall take it here to mean

$$\exists \gamma\, \forall p_0 \forall p_2 \forall p_4 \ldots \exists n\, \mathsf{P}(\langle p_0, \gamma(\langle p_0\rangle), p_2, \gamma(\langle p_0, p_2\rangle), \ldots\rangle, \mathbf{m}, \boldsymbol{\alpha})$$

where the second ... terminates with p_{n-1} if n is odd and with $\gamma(\langle p_0, \ldots, p_{n-2}\rangle)$ if n is even. (The function γ may be thought of either as a Skolem Function or as a strategy for a certain infinite game — cf. V.3 and the discussion preceding Definition V.4.7.)

It is easy to see that every Π_1^1 relation is expressible in the form (*) with P recursive. For the converse, show first that the same class of relations is expressible in the form

$$\exists p_0 \forall p_1 \exists p_2 \forall p_3 \exists p_4 \ldots \exists n\, \mathsf{P}(\langle p_0, \ldots, p_{n-1}\rangle, \mathbf{m}, \boldsymbol{\alpha}),$$

that is,

$$\exists \delta\, \forall p_1 \forall p_3 \forall p_5 \ldots \exists n\, \mathsf{P}(\langle \delta(\langle\ \rangle), p_1, \delta(\langle p_1\rangle), p_3, \ldots\rangle, \mathbf{m}, \boldsymbol{\alpha}).$$

Then imitate the proof of Theorem 3.2 to show that every relation (*) is reducible to the closure of a Δ_2^0 monotone operator.

3.23. Extend the reasoning of the preceding exercise to show that every relation definable in the form

$$\forall \beta_0 \exists p_0 \forall \beta_1 \exists p_1 \ldots \exists n\, \mathsf{P}(\langle \bar{\beta}_0(p_0), \ldots, \bar{\beta}_{n-1}(p_{n-1})\rangle, \mathbf{m}, \boldsymbol{\alpha})$$

with P recursive is Π_1^1.

3.24. Show that every Σ_1^0 set is reducible to $\bar{\Gamma}$ for some $\Gamma \in \Delta_1^0$.

3.25 (Richter). Show that for each $n \geq 1$, there exists a Π_1^0 inductive operator Γ such that $|\Gamma| = \omega^n$. (For any two operators Γ_0 and Γ_1, let

$$[\Gamma_0, \Gamma_1](A) = \begin{cases} \Gamma_0(A), & \text{if } \Gamma_0 \nsubseteq A; \\ \Gamma_1(A), & \text{otherwise.} \end{cases}$$

Choose appropriate Γ_0 and Γ_1 such that $\Gamma = [\Gamma_0, \Gamma_1]$ is Π_1^0 and has closure ordinal ω^2.)

3.26. Show that there exist (non-monotone) inductive operators $\Gamma \in \Pi_2^0$ such that $\bar{\Gamma} \notin \Pi_1^1$. (Choose Γ_0 such that $\bar{\Gamma}_0 \in \Pi_1^1 \sim \Delta_1^1$, set

$$\Gamma_1(A) = \{\langle 0, m\rangle : m \in \Gamma_0(\{p : \langle 0, p\rangle \in A\})\}$$

and

$$\Gamma(A) = \Gamma_1(A) \cup \{\langle 1, m\rangle : \Gamma_1(A) \subseteq A \wedge m \notin A\}.)$$

3.27. Show that the recursive ordinals are closed under ordinal addition and multiplication.

3.28. Show that for X any of Σ_r^i or Π_r^i $(i = 0, 1;\ r > 0)$, there exists an "X-universal" operator Γ_* — that is, $\Gamma_* \in X$ and for any $\Gamma \in X$ there exists an a such that

$$m \in \bar{\Gamma} \leftrightarrow \langle a, m\rangle \in \bar{\Gamma}_*.$$

3.29. Find a monotone operator $\Gamma \in \Sigma_1^1$ such that $\bar{\Gamma} \notin \Sigma_1^1 \cup \Pi_1^1$.

3.30. Show that not every Δ_2^1 subset of ω is reducible to $\bar{\Gamma}$ for some monotone Σ_1^1 operator Γ.

3.31. Complete the proofs of Theorems 3.17 and 3.18.

3.32. Let X be any of the classes Σ_r^i or Π_r^i $(i = 0 \text{ or } 1,\ r \geq 0)$ and $\bar{X}$ the class of relations reducible to some $\bar{\Gamma}$ with $\Gamma \in X$. Show that

(i) $\bar{X}$ is closed under $\exists^0$;

(ii) if X is not Σ_0^0, Π_0^0, or Σ_1^0, then $\bar{X}$ is closed under $\forall^0$ and indeed under the Suslin quantifier $\mathscr{A}$, where

$$(\mathscr{A} R)(\mathbf{m}) \leftrightarrow \forall \beta\, \exists n\, R(\bar{\beta}(n), \mathbf{m}).$$

3.33 (Aczel). For any monotone operator Γ over ω, let

$$|m|_\Gamma = \begin{cases} \text{least } \sigma . m \in \Gamma^\sigma, & \text{if } \quad m \in \bar{\Gamma}; \\ |\Gamma|, \quad \text{otherwise}; \end{cases}$$

$$m \leqslant_\Gamma n \leftrightarrow m \in \bar{\Gamma} \wedge |m|_\Gamma \leqslant |n|_\Gamma;$$

$$m <_\Gamma n \leftrightarrow m \in \bar{\Gamma} \wedge |m|_\Gamma < |n|_\Gamma;$$

$$\Gamma^\circ(A) = \sim \Gamma(\sim A).$$

(i) Show that for any $\tau < |\Gamma|$ and any $m \in \Gamma^\tau$

$$|m|_\Gamma < |n|_\Gamma \leftrightarrow n \in \Gamma^\circ(\{q \colon |m|_\Gamma \leqslant |q|_\Gamma\})$$

and

$$|m|_\Gamma \leqslant |q|_\Gamma \leftrightarrow m \in \Gamma(\{p \colon p \in \Gamma^{(\tau)} \wedge |p|_\Gamma < |q|_\Gamma\}).$$

(ii) Let Δ be the operator defined by:

$$\Delta(A) = \{\langle m, n\rangle \colon n \in \Gamma^\circ(\{q \colon m \in \Gamma(\{p \colon \langle p, q\rangle \in A\})\})\}.$$

Show that for all m and n and all $\tau < |\Gamma|$,

$$m \in \Gamma^\tau \wedge |m|_\Gamma < |n|_\Gamma \leftrightarrow \langle m, n\rangle \in \Delta^\tau,$$

and

$$m <_\Gamma n \leftrightarrow \langle m, n\rangle \in \bar{\Delta}.$$

(iii) Similarly, show

$$n \leqslant_\Gamma m \leftrightarrow \langle m, n\rangle \in \bar{\Delta}^\circ.$$

3.34. Use the results of the preceding exercise together with Theorem 3.2 to show that:

(i) the class of Π^1_1 relations on numbers has the reduction property;

(ii) for any monotone arithmetical operator Γ, $|\Gamma| \leqslant \delta^1_1$, the least ordinal not the order-type of a Δ^1_1 well-ordering of ω; $\bar{\Gamma} \in \Delta^1_1$ iff $|\Gamma| < \delta^1_1$;

(iii) (Cenzer [1974a]) for any $r \geqslant 2$ and any monotone operator $\Gamma \in \Delta^1_r$, $|\Gamma| < \delta^1_r$.

3.35 Notes. Theorem 3.2 is essentially due to Kleene [1955a], but the simple direct proof given here seems to appear first in Lorenzen-Myhill [1959]. Theorem 3.10 is from Putnam [1964], although it is stated there in very different terms.

4. Implicit Definability and Bases

If $\mathsf{A} \subseteq {}^\omega\omega$ is a set of given complexity, what can one say about the complexity of the elements of A? Since ${}^\omega\omega$ is recursive and has elements of all complexities, we shall at best be able to prove that a simple set has some simple elements. This leads to the notion of basis:

4.1 Definition. For any class X of relations and any $\mathsf{B} \subseteq {}^\omega\omega$, B is a *basis* for X iff for all $\mathsf{A} \in X$,

$$\exists\alpha\,.\,\alpha \in \mathsf{A} \rightarrow (\exists\alpha \in \mathsf{B})\,.\,\alpha \in \mathsf{A}.$$

We shall present in this section a number of positive results concerning bases along with counterexamples to indicate that these are optimal in the sense that they fail if B is reduced or X enlarged. Among the classes X to be considered are classes of *singletons*. If $\{\beta\} \in \Delta^i_r$ (etc.) we say that β is *implicitly* Δ^i_r. Note that B is a basis for the class of Δ^i_r singletons iff every implicitly Δ^i_r function belongs to B.

4.2 Theorem. *Δ^0_1 is a basis for Σ^0_1.*

Proof. The set of recursive (Δ^0_1) functions is dense and hence intersects every open (Σ^0_1) set. $\square$

4.3 Theorem. *$\Delta^0_{(\omega)}$ is not a basis for the class of Π^0_2 singletons.*

Proof. Let Γ be the inductive operator of Theorem 3.6 and β^* and S be as in the proof of Theorem 3.7 for this Γ. Since

$$\begin{aligned}\beta^*(\langle r+1, \langle t, a, \mathbf{m}\rangle\rangle) = 0 &\leftrightarrow \langle t, a, \mathbf{m}\rangle \in \Gamma^r \\ &\leftrightarrow 0 < t \leqslant r+1 \wedge U^0_t(a, \langle \mathbf{m}\rangle)\end{aligned}$$

β^* is non-arithmetical. On the other hand, β^* is the unique member of S and it is routine to check that $\mathsf{S} \in \Pi^0_2$. $\square$

4.4 Lemma. *For any β, if β is implicitly arithmetical, then β is recursive in some function γ which is implicitly Π^0_1.*

Proof. We shall prove the conclusion for $\{\beta\} \in \Pi^0_r$ by induction on r. If $r = 1$, there is nothing to prove. Suppose $\{\beta\} \in \Pi^0_{r+1}$ with $r > 0$ and let R be a Δ^0_r relation such that

$$\forall\alpha\,[\alpha = \beta \leftrightarrow \forall m\, \exists n\, \mathsf{R}(m, n, \alpha)].$$

In particular, $\forall m\ \exists n\ \mathsf{R}(m, n, \beta)$ so define

$$\delta(m) = \langle \beta(m), \text{least } n.\, \mathsf{R}(m, n, \beta)\rangle.$$

Since for all m, $\beta(m) = (\delta(m))_0$, β is recursive in δ. We claim that

$$(*) \qquad \forall \alpha\ (\alpha = \delta \leftrightarrow \mathrm{Sq}_1(\alpha) \wedge \lg(\alpha) = 2 \wedge \forall m\ \mathsf{R}(m, (\alpha(m))_1, (\alpha)_0) \\ \wedge \forall m\ \forall p\ [p < (\alpha(m))_1 \to \sim \mathsf{R}(m, p, (\alpha)_0)]),$$

and thus that $\{\delta\} \in \Pi^0_r$. Then by the induction hypothesis, δ is recursive in some implicitly Π^0_1 function γ and thus so is β.

The implication $(\to)$ of $(*)$ is obvious from the definition of δ. Conversely, if α satisfies the right-hand side, then from the third clause follows that $\forall m\ \exists n\ \mathsf{R}(m, n, (\alpha)_0)$ and thus that $(\alpha)_0 = \beta$. Then from the fourth clause it is immediate that $(\alpha(m))_1 = \text{least } n.\, \mathsf{R}(m, n, \beta)$ and thus that $\alpha = \delta$. □

4.5 Corollary. $\Delta^0_{(\omega)}$ *is not a basis for the class of* Π^0_1 *singletons.*

Proof. Immediate from 4.3 and 4.4. □

In the positive direction, we have from the examples of 2.3:

4.6 Lemma. *For all* $r > 0$, Δ^1_r *is a basis for the class of* Σ^1_r *singletons.* □

Hence in particular, Δ^1_1 is a basis for the class of Π^0_1 singletons. To settle the question for arbitrary Π^0_1 sets we shall need a result from the next chapter:

(IV.2.6) $\quad \{\alpha \colon \alpha \in \Delta^1_1\} \in \Pi^1_1$

together with the following Lemma.

4.7 Lemma. *For any* $r > 0$ *and any set* B *which is closed under "recursive in", if* B *is a basis for* Π^0_1 (Π^1_r), *then* B *is also a basis for* Σ^1_1 (Σ^1_{r+1}).

Proof. Suppose $\alpha \in \mathsf{A} \leftrightarrow \exists \beta\ \mathsf{R}(\alpha, \beta)$, $\mathsf{R} \in \Pi^0_1$, and B is a basis for Π^0_1. Let $\mathsf{C} = \{\gamma \colon \mathsf{R}((\gamma)_0, (\gamma)_1)\}$. If $\mathsf{A} \neq \varnothing$, also $\mathsf{C} \neq \varnothing$ so there exists $\gamma \in \mathsf{C} \cap \mathsf{B}$. Then $(\gamma)_0 \in \mathsf{A}$ and as $(\gamma)_0$ is recursive in γ, also $(\gamma)_0 \in \mathsf{B}$, so $\mathsf{A} \cap \mathsf{B} \neq \varnothing$. The proof for $\mathsf{R} \in \Pi^1_r$ is identical. □

4.8 Corollary. Δ^1_1 *is not a basis for* Π^0_1.

Proof. Let $\mathsf{A} = \{\alpha \colon \alpha \notin \Delta^1_1\}$. By IV.2.6, $\mathsf{A} \in \Sigma^1_1$, but clearly A has no Δ^1_1 element. Hence Δ^1_1 is not a basis for Σ^1_1, so by Lemma 4.7 also not for Π^0_1. □

Note that since a function α is Σ_1^1 or Π_1^1 just in case it is Δ_1^1, neither Σ_1^1 nor Π_1^1 is a basis for Π_1^0. However,

4.9 Kleene Basis Theorem. $\{\alpha$: *α is recursive in some* $B \in \Sigma_1^1\}$ *is a basis for* Π_1^0, *hence also for* Σ_1^1.

Proof. Let A be a non-empty Π_1^0 set. Then for some recursive set $A \subseteq \omega$,

$$\alpha \in \mathsf{A} \leftrightarrow \forall m\, [\bar{\alpha}(m) \in A].$$

Let

$$B = \{s: \exists \alpha\, \forall m\, [s * \bar{\alpha}(m) \in A]\}.$$

Then $B \in \Sigma_1^1$ and from the assumption $\mathsf{A} \neq \varnothing$ it follows that $\langle\ \rangle \in B$. Furthermore, $(\forall s \in B) \exists n.\, s * \langle n \rangle \in B$. Thus there is a unique function β such that for all m,

$$\beta(m) = \text{least } n[\bar{\beta}(m) * \langle n \rangle \in B].$$

Clearly β is recursive in B. Since $B \subseteq A$ and for all m, $\bar{\beta}(m) \in B$, also $\beta \in \mathsf{A}$. $\square$

In the remainder of this section we investigate the basis properties of various special arithmetical classes. We have not yet developed techniques to deal successfully with most basis questions in the analytical hierarchy, but for comparison we mention some results from later chapters:

(V.6.1) $\{\alpha$: α is recursive in some $B \in \Sigma_1^1\}$ is not a basis for the class of Π_1^1 singletons;

(IV.7.9) Δ_2^1 is a basis for Π_1^1, hence for Σ_2^1;

(IV.2.20) every $R \in \Delta_1^1$ is recursive in some implicitly Π_1^0 function γ;

but

(III.6.11) there exist $\alpha \in \Delta_1^1$ such that $\{\alpha\} \notin \Delta_{(\omega)}^0$.

The existence of bases for Π_2^1 and higher classes in the analytical hierarchy is independent of the axioms of set theory. We shall show in § V.2 that it is consistent that for all $r \geqslant 2$, Δ_r^1 be a basis for Σ_r^1. However it is also known to be consistent that $\Delta_{(\omega)}^1$ not be a basis for Π_2^1 (Lévy [1965a]).

4.10 Kreisel Basis Theorem. Δ_2^0 *is a basis for* $\{\mathsf{A}: \mathsf{A} \in \Sigma_2^0 \wedge \mathsf{A} \cap {}^{\omega}2 \neq \varnothing\}$.

Proof. Suppose that $\mathsf{A}=\{\alpha: \exists p\,\forall m\, R(p,\bar\alpha(m))\}$ with R recursive and $\mathsf{A}\cap{}^{\omega}2\neq\varnothing$. Then for some p_0 there exists a function $\alpha^*\in{}^{\omega}2$ such that $\forall m\, R(p_0,\bar\alpha^*(m))$. Let

$$B=\{s: (\exists\alpha^*\in{}^{\omega}2)\forall m\, R(p_0, s*\bar\alpha^*(m))\}.$$

As in the proof of Theorem 4.9 there exists a function $\beta\in\mathsf{A}$ which is recursive in B. In this case, however, we shall show $B\in\Pi^0_1$ which by Post's Theorem (1.12) implies $\beta\in\Delta^0_2$.

By Exercise I.2.6,

$$s\in B\leftrightarrow\forall n\,(\exists\alpha^*\in{}^{\omega}2)(\forall m<n)R(p_0,s*\bar\alpha^*(m)).$$

Let δ be the primitive recursive function defined by: $\delta(0)=1$ and $\delta(n+1)=\delta(n)*\langle 1\rangle$. Any code for a sequence of 0's and 1's of length at most n is less than $\delta(n)$. Hence

$$s\in B\leftrightarrow\forall n\,(\exists t\leqslant\delta(n))(\forall m<n)[(t)_m\leqslant 1\wedge R(p_0,s*\langle(t)_0,\dots,(t)_{m-1}\rangle)]$$

which implies $B\in\Pi^0_1$. $\square$

The following shows that this is the best possible result.

4.11 Theorem. (i) *There exists a non-empty* Π^0_1 *set* $A\subseteq{}^{\omega}2$ *which contains no characteristic function of a* Σ^0_1 *or* Π^0_1 *set;*

(ii) *there exists a non-empty* Π^0_2 *set* $\mathsf{B}\subseteq{}^{\omega}2$ *which contains no* Δ^1_1 *function.*

Proof. Corollary II.4.22 asserts that the class of semi-recursive relations does not have the separation property. The proof actually shows that there exist semi-recursive *sets* $A, B\subseteq\omega$ such that no recursive set C separates A and B. For any set D and all $i<2$, let $D_i=\{m:\langle i,m\rangle\in D\}$. Set

$$\mathsf{A}=\{\mathsf{K}_D: A\subseteq D_0\subseteq\sim B\wedge D_0=\sim D_1\}.$$

Since $A, B\in\Sigma^0_1$, it follows directly that $\mathsf{A}\in\Pi^0_1$. Suppose that there is a $D\in\Sigma^0_1$ such that $\mathsf{K}_D\in\mathsf{A}$. Then $D_0, D_1\in\Sigma^0_1$ so D_0 is a recursive set which separates A and B, contrary to assumption.

We established in Corollary 4.8 that there exists a non-empty Π^0_1 set with no Δ^1_1 elements. If C is such a set, it suffices for (ii) to take $\mathsf{B}=\{\mathsf{K}_{\mathrm{Gr}(\alpha)}: \alpha\in\mathsf{C}\}$ (Exercise 4.26). $\square$

Note that without using 4.8 (which depends on IV.2.6) we have already in the proof of Theorem 4.3 a Π^0_2 subset of ${}^{\omega}2$ with no $\Delta^0_{(\omega)}$ element.

The theme of the remainder of this section is that a set of a given complexity which is in some sense "large" is more likely to contain some simple elements than a "small" set of the same complexity. In particular, this is true when we interpret "large" by "non-meager" or "of positive measure".

4.12 Lemma. Δ_1^0 *is a basis for the class of* Π_2^0 *sets which are dense in some interval.*

Proof. Suppose $\mathsf{A} = \{\alpha \colon \forall m\, \exists n\, R(m, \bar{\alpha}(n))\}$ with R recursive, and A is dense in $[s_0]$. Then for any s which extends s_0 and any m, there is a sequence t compatible with s such that $R(m,t)$ holds (cf. proof of II.5.3). Hence if we set

$$s_{m+1} = \text{least } t[\mathrm{Sq}(t) \wedge s_m \subsetneq t \wedge (\exists u \subseteq t) R(m,u)],$$

then s_m is defined for all m and $s_0 \subseteq s_1 \subseteq \cdots \subseteq s_m \subseteq \cdots$. Hence there is a unique function α which is the "limit" of the s_m, given by $\alpha(m) = (s_m)_m$. Clearly α is recursive and belongs to A.

4.13 Corollary. Δ_1^0 *is a basis for the class of non-meager* Σ_3^0 *sets.*

Proof. A Σ_3^0 set is a countable union of Π_2^0 sets. If it is non-meager, one of these must be dense in some interval and hence contain a recursive function. □

An interesting by-product is the classification of the class of recursive functions:

4.14 Corollary. $\{\alpha \colon \alpha \in \Delta_1^0\} \in \Sigma_3^0 \sim \Delta_3^0$.

Proof. It was shown as one of the Examples 1.3 that the set A of recursive functions is Σ_3^0. Suppose also $\mathsf{A} \in \Pi_3^0$ so that $\sim\mathsf{A} \in \Sigma_3^0$. As A is denumerable it is meager, so by the Baire Category Theorem (I.2.2), $\sim\mathsf{A}$ is non-meager. But then by Corollary 4.13, $\sim\mathsf{A}$ contains a recursive element, a contradiction. □

Note that it also follows from this argument that Δ_1^0 is not a basis for the class of non-meager or even co-meager Π_3^0 sets. Also, since the class of primitive recursive functions is Σ_2^0, this class does not form a basis for the class of non-meager Π_2^0 sets.

We shall extend these results to all levels of the arithmetical hierarchy in § 6. If we replace "non-meager" by "of positive measure" the results have a similar flavor but are weaker in the sense that the bases are larger. First, in contrast with 4.13,

4.15 Theorem. Δ_1^0 *is not a basis for either of*

(i) *the class of* Π_1^0 *sets of positive measure, or*

(ii) *the class of* Σ_2^0 *sets of measure* 1.

Proof. For each n and a let

$$A_{n,a} = \{\alpha\colon (\forall m < n + a + 1)\{a\}(m) \simeq \alpha(m)\}.$$

Each $A_{n,a}$ is either empty or is an interval of length $n + a + 1$. In either case, since the measure of an interval of length k is at most 2^{-k}, $\mathrm{mes}(A_{n,a}) \leq 2^{-n} \cdot 2^{-(a+1)}$. Hence for each n, $A_n = \bigcup \{A_{n,a}\colon a \in \omega\}$ has measure at most $\sum_{a=0}^{\infty} 2^{-n} \cdot 2^{-(a+1)} = 2^{-n}$. Furthermore all recursive functions belong to every A_n, so that each $\sim A_n$ is a Π_1^0 set of positive measure with no recursive elements, and $\sim \bigcap \{A_n\colon n \in \omega\}$ is a Σ_2^0 set of measure 1 with no recursive elements. □

On the other hand, we have

4.16 Theorem. *Δ_1^0 is a basis for the class of Π_2^0 sets of measure* 1.

Proof. A Π_2^0 set A of measure 1 is a countable intersection of Σ_1^0 sets A_n each also of measure 1. Clearly each A_n is open and dense so that $\sim A_n$ is nowhere dense. Thus $\sim A = \bigcup \{\sim A_n\colon n \in \omega\}$ is meager and by the Baire Category Theorem, A is non-meager so has a recursive element by Corollary 4.13. □

In the remainder of this section we compute for all r bases for the class of Σ_r^0 sets of positive measure. For the next lemma only, if F is any total functional we set for any R,

$$R_{\mathrm{mes}}^{F}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \mathrm{mes}\{\beta\colon R(\mathbf{m}, \boldsymbol{\alpha}, \beta)\} \leq \frac{1}{F(\mathbf{m}, \boldsymbol{\alpha})}.$$

4.17 Lemma. *For all $r > 0$, all ε, any recursive functional* F *and any* R, *if* $R \in \Sigma_r^0[\varepsilon]$, *then* $R_{\mathrm{mes}}^{F} \in \Pi_r^0[\varepsilon]$.

Proof. We proceed by induction on r and omit reference to ε. Suppose first that $R \in \Sigma_1^0$, say

$$R(\mathbf{m}, \boldsymbol{\alpha}, \beta) \leftrightarrow \exists n\, S(\bar{\beta}(n), \mathbf{m}, \boldsymbol{\alpha})$$

for a recursive S. Let θ be the following function from ${}^{k+1,l}\omega$ into the rational interval $[0, 1]$:

$$\theta(s, \mathbf{m}, \boldsymbol{\alpha}) = \begin{cases} \mathrm{mes}[s], & \text{if } \ s \in \mathrm{Sq} \wedge S(s, \mathbf{m}, \boldsymbol{\alpha}) \wedge (\forall t \subsetneq s){\sim}S(t, \mathbf{m}, \boldsymbol{\alpha}); \\ 0, & \text{otherwise.} \end{cases}$$

Since $\{\beta\colon R(\mathbf{m}, \boldsymbol{\alpha}, \beta)\}$ is the disjoint union of the intervals $[s]$ for which $\theta(s, \mathbf{m}, \boldsymbol{\alpha}) > 0$, we have

$$\begin{aligned} R_{\mathrm{mes}}^{F}(\mathbf{m}, \boldsymbol{\alpha}) &\leftrightarrow \sum_{s=0}^{\infty} \theta(s, \mathbf{m}, \boldsymbol{\alpha}) \leq \frac{1}{F(\mathbf{m}, \boldsymbol{\alpha})} \\ &\leftrightarrow \forall q \left(\sum_{s=0}^{q} \theta(s, \mathbf{m}, \boldsymbol{\alpha}) \leq \frac{1}{F(\mathbf{m}, \boldsymbol{\alpha})} \right). \end{aligned}$$

Since if $s \in \mathrm{Sq}$ and $\lg(s) = k$, $\mathrm{mes}[s] = 2^{-((s)_0 + \cdots + (s)_{k-1} + k)}$ (cf. end of § I.2), a simple calculation shows that there exist recursive H and I such that

$$\sum_{s=0}^{q} \theta(s, \mathbf{m}, \boldsymbol{\alpha}) = \frac{\mathsf{H}(q, \mathbf{m}, \boldsymbol{\alpha})}{\mathsf{I}(q, \mathbf{m}, \boldsymbol{\alpha})}.$$

Thus

$$\mathsf{R}^{\mathsf{F}}_{\mathrm{mes}}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \forall q\, [\mathsf{F}(\mathbf{m}, \boldsymbol{\alpha}) \cdot \mathsf{H}(q, \mathbf{m}, \boldsymbol{\alpha}) \leqslant \mathsf{I}(q, \mathbf{m}, \boldsymbol{\alpha})].$$

Assume now as induction hypothesis that the result holds for $1, \ldots, r$. We consider first $\mathsf{R} \in \Pi^0_r$, say $\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}, \beta) \leftrightarrow \forall p\, \mathsf{S}(p, \mathbf{m}, \boldsymbol{\alpha}, \beta)$, with $\mathsf{S} \in \Sigma^0_{r-1}$. Then

$$\mathsf{R}^{\mathsf{F}}_{\mathrm{mes}}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \forall n\, \exists q \left[\mathrm{mes}\{\beta\colon (\forall p \leqslant q)\mathsf{S}(p, \mathbf{m}, \boldsymbol{\alpha}, \beta)\} \leqslant \frac{1}{\mathsf{F}(\mathbf{m}, \boldsymbol{\alpha})} + \frac{1}{n}\right]$$

so by the induction hypothesis $\mathsf{R}^{\mathsf{F}}_{\mathrm{mes}} \in \Pi^0_{r+1}$.

Now if $\mathsf{R} \in \Sigma^0_{r-1}$, say $\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}, \beta) \leftrightarrow \exists p\, \mathsf{P}(p, \mathbf{m}, \boldsymbol{\alpha}, \beta)$ with $\mathsf{P} \in \Pi^0_r$ we have

$$\mathsf{R}^{\mathsf{F}}_{\mathrm{mes}}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \forall q \left[\mathrm{mes}\{\beta\colon (\exists p \leqslant q)\mathsf{P}(p, \mathbf{m}, \boldsymbol{\alpha}, \beta)\} \leqslant \frac{1}{\mathsf{F}(\mathbf{m}, \boldsymbol{\alpha})}\right]$$

and again $\mathsf{R}^{\mathsf{F}}_{\mathrm{mes}} \in \Pi^0_{r+1}$. $\square$

4.18 Lemma. *For all r and ε, and any $\mathsf{R} \in \Pi^0_r[\varepsilon]$, if* $\mathrm{mes}\{\alpha : \forall p\, \exists q\, \mathsf{R}(p, q, \alpha)\} > 0$, *then there exists a $\beta \in \Delta^0_{r+2}[\varepsilon]$ such that* $\mathrm{mes}\{\alpha : \forall p\, (\exists q \leqslant \beta(p))\mathsf{R}(p, q, \alpha)\} > 0$.

Proof. Let $\mathsf{A} = \{\alpha\colon \forall p\, \exists q\, \mathsf{R}(p, q, \alpha)\}$, $\mathsf{A}_p = \{\alpha\colon \exists q\, \mathsf{R}(p, q, \alpha)\}$ and $\mathsf{A}_{p,n} = \{\alpha\colon (\exists q \leqslant n)\mathsf{R}(p, q, \alpha)\}$. Suppose $\mathrm{mes}\,\mathsf{A} > 1/2^m$ and define β by

$$\beta(p) = \text{least } n \left[\mathrm{mes}(\mathsf{A}_p \sim \mathsf{A}_{p,n}) \leqslant \frac{1}{2^{m+p+2}}\right].$$

As $\mathsf{A}_p \sim \mathsf{A}_{p,n} \in \Sigma^0_{r+1}[\varepsilon]$, it follows from Lemma 4.15 that $\beta \in \Delta^0_{r+2}[\varepsilon]$.

Let $\mathsf{B} = \{\alpha\colon \forall p\, (\exists q \leqslant \beta(p))\mathsf{R}(p, q, \alpha)\}$. Then since

$$\mathsf{A} = \bigcap \{\mathsf{A}_p\colon p \in \omega\} \subseteq B \cup \bigcup \{\mathsf{A}_p \sim \mathsf{A}_{p,\beta(p)}\colon p \in \omega\},$$

we have

$$\frac{1}{2^m} < \mathrm{mes}\,\mathsf{A} \leqslant \mathrm{mes}\,\mathsf{B} + \sum_{p=0}^{\infty} \frac{1}{2^{m+p+2}} = \mathrm{mes}\,\mathsf{B} + \frac{1}{2^{m+1}}$$

which implies $\text{mes}\, \mathsf{B} > 1/2^{m+1} > 0$. □

4.19 Lemma. *For all* r, β, ε, *and all* $\mathsf{R} \in \Pi_r^0[\varepsilon]$, *if* $\mathsf{S}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \forall p\, (\exists q \leq \beta(p))\, \mathsf{R}(p, q, \mathbf{m}, \boldsymbol{\alpha})$, *then* $\mathsf{S} \in \Pi^0_{\max(1,r)}[\beta, \varepsilon]$.

Proof. We leave this straightforward calculation to the reader. □

Now let $f(r)$ denote the greatest integer not exceeding $r^2/4 + 1$.

4.20 Lemma. *For any r and ε, and any* $\mathsf{A} \in \Pi_r^0[\varepsilon]$ *such that* $\text{mes}\, \mathsf{A} > 0$, *there exists* $\gamma \in \Delta^0_{f(r)}[\varepsilon]$ *and* $\mathsf{C} \in \Pi_1^0[\gamma]$ *such that* $\mathsf{C} \subseteq \mathsf{A}$ *and also* $\text{mes}\, \mathsf{C} > 0$.

Proof. We proceed by induction on r. For $r = 0$ or 1 the result is obvious. Suppose $\mathsf{A} \in \Pi^0_{r+2}[\varepsilon]$. By the two preceding lemmas there exist $\mathsf{B} \subseteq \mathsf{A}$ and $\beta \in \Delta^0_{r+2}[\varepsilon]$ such that $\mathsf{B} \in \Pi^0_{\max(1,r)}[\beta, \varepsilon]$ and $\text{mes}\, \mathsf{B} > 0$. By the induction hypothesis there exist $\gamma \in \Delta^0_{f(r)}[\beta, \varepsilon]$ and $\mathsf{C} \in \Pi_1^0[\gamma]$ such that $\mathsf{C} \subseteq \mathsf{B}$ and $\text{mes}\, \mathsf{C} > 0$. Then by the Arithmetical Substitution Theorem, $\gamma \in \Delta^0_{f(r)+r+1}[\varepsilon] = \Delta^0_{f(r+2)}[\varepsilon]$ as required. □

4.21 Theorem. *For all* r, $\Delta^0_{f(r)+2}$ *is a basis for the class of* Π_r^0 *sets of positive measure.*

Proof. Let A be a Π_r^0 set of positive measure and let γ and C be as in Lemma 4.20. Then there exists a unique function δ such that for all m,

$$\delta(m) = \text{least } n\,.\, \text{mes}(\mathsf{C} \cap [\bar{\delta}(m) * \langle n \rangle]) > 0.$$

Since $\mathsf{C} \in \Pi^0_{f(r)}$, by Lemma 4.17 $\delta \in \Delta_{f(r)+2}$. That $\delta \in \mathsf{C}$ is immediate from the fact that C is closed. □

4.22 Corollary. *For all* r, $\Delta^0_{f(r)+2}$ *is a basis for the class of* Σ^0_{r+1} *sets of positive measure.* □

4.23 Corollary. $\{\alpha \colon \alpha \in \Delta^0_{(\omega)}\} \notin \Delta^0_{(\omega)}$.

Proof. Since $\{\alpha \colon \alpha \notin \Delta^0_{(\omega)}\}$ is the complement of a denumerable set it has measure 1. Hence if it were arithmetical it would have an arithmetical element, which is absurd. □

4.24–4.27 Exercises

4.24. Show that for $r \geq 2$, if β is implicitly Π_r^0 and β and γ are each recursive in the other, then also γ is implicitly Π_r^0. (Exercise 6.17 shows that this is false for $r = 1$.)

4.25. Show that if β is implicitly Π_2^0, then so is $\beta^{\circ J}$. (Let F be a recursive functional such that for all β, $\beta = \lambda m\,.\,\mathsf{F}(m, \beta^{\circ J})$. Then $\gamma = \beta^{\circ J} \leftrightarrow \lambda m\,.\,\mathsf{F}(m,\gamma) = \beta$ and $\gamma = (\lambda m\,.\,\mathsf{F}(m,\gamma))^{\circ J}$.)

4.26. Show that the set B in the proof of Theorem 4.11 (ii) is in fact Π_2^0.

4.27. Is $\{\alpha : \alpha$ is primitive recursive$\}$ a basis for the class of non-meager Σ_2^0 sets?

4.28 Notes. Corollary 4.14 is from Shoenfield [1958] and Theorem 4.15 is from Tanaka [1970a]. 4.17–4.22 are due to Sacks [1969] and independently to Tanaka [1967], although neither formulated the sharp versions given here. Corollary 4.23 was originally proved by Addison using forcing methods (cf. Corollary 6.10 and Notes to § 6). Extensions of many of the results of this section and § 6 may be found in Kechris [1973].

5. Definability in Formal Languages for Arithmetic

The reader familiar with formal languages will certainly have noticed a similarity between the arithmetical and analytical hierarchies and classifications of the formulas of a formal language by the complexity of their structure. In this section we show that the classes Σ_r^i and Π_r^i $(i = 0, 1)$ consist exactly of the relations formally definable over the standard model of arithmetic by certain classes $\exists_r^i$ and $\forall_r^i$ $(i = 0, 1)$ of formulas. At the end we sketch briefly how these results may be applied to derive the undecidability and incompleteness theorems for axiomatic theories of arithmetic and discuss how much of the theory of this book could be developed in such systems.

We shall assume in this section some familiarity with formal languages and their interpretations and we shall omit many standard details, all of which may be found in Shoenfield [1967] or Enderton [1972]. Some of our notational conventions are suspended for this section as indicated below.

By the *standard model for arithmetic* we mean the structure

$$\mathfrak{N} = (\omega, {}^{\omega}\omega, <, +, \cdot, ', 0)$$

where $'$ denotes the successor function and $<$, $+$, and $\cdot$ have their usual arithmetical meanings. With $\mathfrak{N}$ we associate the second-order language $\mathscr{L}$ described as follows. The *symbols* of $\mathscr{L}$ are:

$$\neg, \vee, \exists, \ominus, \otimes, \oplus, \odot, {}^{\odot}, \bar{0}, x_0, x_1, \ldots, \phi_0, \phi_1, \ldots.$$

The set of *terms* is defined inductively as the smallest class such that

(i) $\bar{0}, x_0, x_1, \ldots$ are terms;

(ii) if σ and τ are terms, then so are $\sigma^{\oplus}$, $\sigma \oplus \tau$, $\sigma \odot \tau$, and $\phi_j(\sigma)$ for $j = 0, 1, \ldots$.

The *atomic formulas* of $\mathscr{L}$ are the expressions $\sigma \ominus \tau$ and $\sigma \otimes \tau$ for any terms σ and τ. The set of *formulas* is the smallest class such that

(i) atomic formulas are formulas;

(ii) if $\mathfrak{A}$ and $\mathfrak{B}$ are formulas, then so are $\neg\mathfrak{A}$, $\mathfrak{A} \vee \mathfrak{B}$, $\exists x_i \mathfrak{A}$, and $\exists \phi_j \mathfrak{A}$ for $i, j = 0, 1, \ldots$.

We denote by $\mathfrak{A}(\sigma/x_i)$ the result of substituting the term σ for all free occurrences of the variable x_i in the formula $\mathfrak{A}$ and assume that before the substitution all bound variables of $\mathfrak{A}$ which occur in σ are changed to the first unused variables of the proper type. $\mathfrak{A}(\sigma_0, \ldots, \sigma_{k-1})$ is an abbreviation for $\mathfrak{A}(\sigma_0/x_0)\ldots(\sigma_{k-1}/x_{k-1})$. Similarly, $\mathfrak{A}(\phi_j/\phi_i)$ denotes the result of substituting ϕ_j for all free occurrences of ϕ_i in $\mathfrak{A}$. We denote by $\bar{n}$ the term $\bar{0}^{\oplus\cdots\oplus}$ with n successor symbols. The symbols $\wedge, \rightarrow, \leftrightarrow$, and $\forall$ are used as abbreviations in the usual way. We write $(\exists x_i < \sigma)\mathfrak{A}$ for $\exists x_i (x_i < \sigma \wedge \mathfrak{A})$ and $(\forall x_i < \sigma)\mathfrak{A}$ for $\forall x_i (x_i < \sigma \rightarrow \mathfrak{A})$.

5.1 Definition. For any k and l, all $(\mathbf{m}, \boldsymbol{\alpha}) \in {}^{k,l}\omega$, and any term σ whose variables are included among $x_0, \ldots, x_{k-1}, \phi_0, \ldots, \phi_{l-1}$, we define $\sigma[\mathbf{m}, \boldsymbol{\alpha}]$, the *value of* σ *at* $(\mathbf{m}, \boldsymbol{\alpha})$, recursively by:

(i) $\bar{0}[\mathbf{m}, \boldsymbol{\alpha}] = 0$;

(ii) $x_i[\mathbf{m}, \boldsymbol{\alpha}] = m_i$;

(iii) $\sigma^{\oplus}[\mathbf{m}, \boldsymbol{\alpha}] = \sigma[\mathbf{m}, \boldsymbol{\alpha}] + 1$;

(iv) $(\sigma \oplus \tau)[\mathbf{m}, \boldsymbol{\alpha}] = \sigma[\mathbf{m}, \boldsymbol{\alpha}] + \tau[\mathbf{m}, \boldsymbol{\alpha}]$;

(v) $(\sigma \odot \tau)[\mathbf{m}, \boldsymbol{\alpha}] = \sigma[\mathbf{m}, \boldsymbol{\alpha}] \cdot \tau[\mathbf{m}, \boldsymbol{\alpha}]$;

(vi) $\phi_j(\sigma)[\mathbf{m}, \boldsymbol{\alpha}] = \alpha_j(\sigma[\mathbf{m}, \boldsymbol{\alpha}])$.

Note that this definition and the next one rely on Theorem I.3.5 for their justification.

5.2 Definition. For any k and l, all $(\mathbf{m}, \boldsymbol{\alpha}) \in {}^{k,l}\omega$, and any formula $\mathfrak{A}$ whose free variables are included among $x_0, \ldots, x_{k-1}, \phi_0, \ldots, \phi_{l-1}$, we define $\models \mathfrak{A}[\mathbf{m}, \boldsymbol{\alpha}]$, $\mathfrak{A}$ *is true at* $(\mathbf{m}, \boldsymbol{\alpha})$, recursively by:

(i) $\models (\sigma \ominus \tau)[\mathbf{m}, \boldsymbol{\alpha}]$ iff $\sigma[\mathbf{m}, \boldsymbol{\alpha}] = \tau[\mathbf{m}, \boldsymbol{\alpha}]$;

(ii) $\models (\sigma \otimes \tau)[\mathbf{m}, \boldsymbol{\alpha}]$ iff $\sigma[\mathbf{m}, \boldsymbol{\alpha}] < \tau[\mathbf{m}, \boldsymbol{\alpha}]$;

(iii) $\models (\neg\mathfrak{A})[\mathbf{m}, \boldsymbol{\alpha}]$ iff not $\models \mathfrak{A}[\mathbf{m}, \boldsymbol{\alpha}]$;

(iv) $\models (\mathfrak{A} \vee \mathfrak{B})[\mathbf{m}, \boldsymbol{\alpha}]$ iff $\models \mathfrak{A}[\mathbf{m}, \boldsymbol{\alpha}]$ or $\models \mathfrak{B}[\mathbf{m}, \boldsymbol{\alpha}]$;

(v) $\models \exists x_i \mathfrak{A}[\mathbf{m}, \boldsymbol{\alpha}]$ iff $\exists n (\models \mathfrak{A}'(x_k/x_i)[\mathbf{m}, n, \boldsymbol{\alpha}])$;

(vi) $\models \exists \phi_j \mathfrak{A}[\mathbf{m}, \boldsymbol{\alpha}]$ iff $\exists \beta (\models \mathfrak{A}'(\phi_l/\phi_j)[\mathbf{m}, \boldsymbol{\alpha}, \beta])$;

where $\mathfrak{A}'$ is a variant of $\mathfrak{A}$ in which the variables x_k and ϕ_l do not occur.

Note that for any $\mathfrak{A}$,

$$\models \mathfrak{A}[\mathbf{m}, \mathbf{n}, \boldsymbol{\alpha}] \quad \text{iff} \quad \models \mathfrak{A}(\bar{n}_0/x_k)\cdots(\bar{n}_{k'-1}/x_{k+k'-1})[\mathbf{m}, \boldsymbol{\alpha}].$$

5.3 Definition. For all r,

(i) $\exists_0^0 = \forall_0^0 =$ the smallest class of formulas such that

(a) all atomic formulas belong to $\exists_0^0$;

(b) if $\mathfrak{A}$ and $\mathfrak{B}$ belong to $\exists_0^0$, then so do $\neg\mathfrak{A}, \mathfrak{A} \vee \mathfrak{B}$, and $(\exists x_i < \sigma)\mathfrak{A}$ for $i = 0, 1, \ldots$ and σ any term in which x_i does not occur;

(ii) $\exists_{r+1}^0 = \{\exists x_i \mathfrak{A}: \mathfrak{A} \in \forall_r^0 \wedge i \in \omega\}$;

(iii) $\forall_{r+1}^0 = \{\neg\mathfrak{A}: \mathfrak{A} \in \exists_{r+1}^0\}$;

(iv) $\exists_0^1 = \forall_0^1 = \bigcup\{\exists_r^0: r \in \omega\} \cup \bigcup\{\forall_r^0: r \in \omega\}$;

(v) $\exists_{r+1}^1 = \{\exists\phi_j \mathfrak{A}: \mathfrak{A} \in \forall_r^1 \wedge j \in \omega\}$;

(vi) $\forall_{r+1}^1 = \{\neg\mathfrak{A}: \mathfrak{A} \in \exists_{r+1}^1\}$.

5.4 Definition. For any k and l, any $\mathsf{R} \subseteq {}^{k,l}\omega$, any r, and any $i < 2$, R is $\exists_r^i$ ($\forall_r^i$)-*definable* (in the standard model) iff for some formula $\mathfrak{A} \in \exists_r^i$ ($\forall_r^i$) with free variables included among $x_0, \ldots, x_{k-1}, \phi_0, \ldots, \phi_{l-1}$,

$$\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \models \mathfrak{A}[\mathbf{m}, \boldsymbol{\alpha}].$$

A functional is $\exists_r^i$ ($\forall_r^i$)-definable just in case its graph is. We also use $\exists_r^i$ and $\forall_r^i$ to denote the corresponding classes of relations.

5.5 Lemma. *For all r and all $i < 2$, $\exists_r^i \subseteq \Sigma_r^i$ and $\forall_r^i \subseteq \Pi_r^i$.*

Proof. Relations defined by atomic formulas are recursive. Since the class of recursive relations is closed under complementation, union, and bounded quantification, all relations in $\exists_0^0$ are recursive. The remainder of the proof is a straightforward induction based on the fact that $\exists_{r+1}^i = \{\exists^i\mathsf{R}: \mathsf{R} \in \forall_r^i\}$. □

The main result of this section is that for i and r not both 0, the converse inclusions hold also. To this end we first establish some closure properties of $\exists_0^0$ and $\exists_1^0$. First it is clear from the definitions that the class of $\exists_0^0$ relations is a Boolean algebra and is closed under bounded quantification.

5.6 Lemma. *The class of $\exists_1^0$ relations is closed under binary union and intersection, bounded quantification ($\exists_<^0$ and $\forall_<^0$) and existential number quantification ($\exists^0$).*

Proof. For any $\exists_1^0$ relations R and S, there exist $\exists_0^0$ relations P and Q such that $\mathsf{R} = \exists^0\mathsf{P}$ and $\mathsf{S} = \exists^0\mathsf{Q}$. Then (cf. the proof of Theorem 1.6) the lemma follows from the following equivalences:

$$\exists p\, \mathsf{P}(p, \mathbf{m}, \boldsymbol{\alpha}) \vee \exists q\, \mathsf{Q}(q, \mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \exists p\, [\mathsf{P}(p, \mathbf{m}, \boldsymbol{\alpha}) \vee \mathsf{Q}(p, \mathbf{m}, \boldsymbol{\alpha})];$$

$$\exists p\, \mathsf{P}(p, \mathbf{m}, \boldsymbol{\alpha}) \wedge \exists q\, \mathsf{Q}(q, \mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \exists r\, [(\exists p < r)\mathsf{P}(p, \mathbf{m}, \boldsymbol{\alpha}) \wedge (\exists q < r)\mathsf{Q}(q, \mathbf{m}, \boldsymbol{\alpha})];$$

$$(\exists p < r)\exists q\, \mathsf{P}(p, q, r, \mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \exists q\, (\exists p < r)\mathsf{P}(p, q, r, \mathbf{m}, \boldsymbol{\alpha});$$

$$(\forall p < r)\exists q\, \mathsf{P}(p, q, r, \mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \exists s\, (\forall p < r)(\exists q < s)\mathsf{P}(p, q, r, \mathbf{m}, \boldsymbol{\alpha});$$

$$\exists p\, \exists q\, \mathsf{P}(p, q, \mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \exists r\, (\exists p < r)(\exists q < r)\mathsf{P}(p, q, \mathbf{m}, \boldsymbol{\alpha}). \quad \square$$

The key to the fact that all primitive recursive relations are $\exists_1^0$-definable is that there is an $\exists_0^0$-definable function which codes finite sequences.

5.7 Lemma. *There exists an $\exists_0^0$-definable function h such that for any $p, q_0, \ldots, q_{p-1}$, there exist s and t such that for all $i < p$, $h(s, t, i) = q_i$ and $q_i < s$.*

Proof. Let $h(s, t, i)$ be the remainder when s is divided by $1 + t(i + 1)$ — that is,

$$h(s, t, i) = n \leftrightarrow n < 1 + t(i + 1) \wedge (\exists u < s)\,.\, s = u(1 + t(i + 1)) + n.$$

It is obvious that h is $\exists_0^0$-definable. Given p and $q_0, \ldots, q_{p-1}$, let $t = (\max\{p, q_0, \ldots, q_{p-1}\} + 1)!$. It is easy to check that the numbers $1 + t(i + 1)$ for $i < p$ are pairwise relatively prime and greater than q_i. The Chinese Remainder Theorem of number theory asserts that in this situation there exists a number s such that for all $i < p$, $s \equiv q_i \pmod{1 + t(i + 1)}$ as required. $\square$

5.8 Theorem. *All primitive recursive functionals are $\exists_1^0$-definable.*

Proof. We show that the class of $\exists_1^0$-definable functions contains the initial functionals and is closed under composition and primitive recursion. First, we have for the initial functionals:

$$\mathsf{Cs}_p^{k,l}(\mathbf{m}, \boldsymbol{\alpha}) = n \leftrightarrow \models (x_k \circleddash \bar{p})[\mathbf{m}, n, \boldsymbol{\alpha}];$$

$$\mathsf{Pr}_i^{k,l}(\mathbf{m}, \boldsymbol{\alpha}) = n \leftrightarrow \models (x_k \circleddash x_i)[\mathbf{m}, n, \boldsymbol{\alpha}];$$

$$\mathsf{Sc}_i^{k,l}(\mathbf{m}, \boldsymbol{\alpha}) = n \leftrightarrow \models (x_k \circleddash x_i^{\textcircled{1}})[\mathbf{m}, n, \boldsymbol{\alpha}];$$

$$\mathsf{Ap}_{i,j}^{k,l}(\mathbf{m}, \boldsymbol{\alpha}) = n \leftrightarrow \models (x_k \circleddash \phi_j(x_i))[\mathbf{m}, n, \boldsymbol{\alpha}].$$

If $\mathsf{G}, \mathsf{H}_0, \ldots, \mathsf{H}_{k'-1}$ are all $\exists_1^0$-definable and $\mathsf{F} = \mathrm{FCmp}_{k'}^{k,l}(\mathsf{G}, \mathsf{H}_0, \ldots, \mathsf{H}_{k'-1})$, then

$$\mathsf{F}(\mathbf{m}, \boldsymbol{\alpha}) = n \leftrightarrow \exists q_0 \ldots \exists q_{k'-1}[\mathsf{H}_0(\mathbf{m}, \boldsymbol{\alpha}) = q_0 \wedge \ldots$$
$$\wedge\, \mathsf{H}_{k'-1}(\mathbf{m}, \boldsymbol{\alpha}) = q_{k'-1} \wedge \mathsf{G}(q_0, \ldots, q_{k'-1}, \boldsymbol{\alpha}) = n],$$

so F is $\exists_1^0$-definable by Lemma 5.6.

If G and H are $\exists_1^0$-definable and $\mathsf{F} = \mathrm{Rec}^{k+1,l}(\mathsf{G}, \mathsf{H})$, then

$$\mathsf{F}(p, \mathbf{m}, \boldsymbol{\alpha}) = n \leftrightarrow \exists s\, \exists t\, (h(s, t, 0) = \mathsf{G}(\mathbf{m}, \boldsymbol{\alpha}) \wedge h(s, t, p) = n \wedge$$
$$\wedge (\forall i < p)(\exists q < s)[h(s, t, i) = q \wedge h(s, t, i+1) = \mathsf{H}(q, i, \mathbf{m}, \boldsymbol{\alpha})]),$$

where h is the function of Lemma 5.7. Again by Lemma 5.6, F is $\exists_1^0$-definable. □

5.9 Theorem. *For all $r > 0$, $\exists_r^0 = \Sigma_r^0$ and $\forall_r^0 = \Pi_r^0$; for all r, $\exists_r^1 = \Sigma_r^1$ and $\forall_r^1 = \Pi_r^1$.*

Proof. Let R be any Σ_1^0 relation. By Theorem II.4.12, $\mathsf{R} = \exists^0\mathsf{S}$ for some primitive recursive S. By the preceding Theorem, $\mathsf{K_S}$ is definable by some $\exists_1^0$ formula $\mathfrak{A}$. Then

$$\mathsf{S}(p, \mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \models \mathfrak{A}(\bar{0}/x_k)[p, \mathbf{m}, \boldsymbol{\alpha}],$$

so $\mathsf{S} \in \exists_1^0$, and thus also $\mathsf{R} \in \exists_1^0$ by Lemma 5.6. The remainder of the proof is a straightforward induction. □

In the rest of this section we shall give a brief survey of some facts concerning axiomatic theories of arithmetic. We do not intend to give a complete treatment of these topics but only to point out some of the ways they are related to the main themes of this book.

A formal theory $\mathcal{T}$ in the language $\mathcal{L}$ consists of a set of sentences (formulas without free variables) called the *axioms* of $\mathcal{T}$ and some *rules of inference* for deducing theorems from the axioms. We shall always assume that among the axioms and rules of $\mathcal{T}$ are a complete set of logical rules and axioms (for example, as in Shoenfield [1967, §2.6]). We write $\mathcal{T} \vdash \mathfrak{A}$ to mean that $\mathfrak{A}$ is a theorem of $\mathcal{T}$.

We first consider *first-order* theories, that is, theories in which the variables ϕ_i do not occur and the rules of inference are only the usual ones. Let $\mathcal{T}_0$ be the theory with the following nine non-logical axioms. To make the formulas more readable we shall write $x, y, z, \ldots$ instead of $x_0, x_1, x_2, \ldots$.

(1) $\forall x \neg(x^{\circledS} \circleddash \bar{0})$;
(2) $\forall x\, \forall y\, (x^{\circledS} \circleddash y^{\circledS} \rightarrow x \circleddash y)$;
(3) $\forall x\, (x \oplus \bar{0} \circleddash x)$;
(4) $\forall x\, \forall y\, (x \oplus y^{\circledS} \circleddash (x + y)^{\circledS})$;
(5) $\forall x\, (x \odot \bar{0} \circleddash \bar{0})$;
(6) $\forall x\, \forall y\, (x \odot y^{\circledS} \circleddash (x \odot y) \oplus x)$;
(7) $\forall x \neg(x \otimes \bar{0})$;
(8) $\forall x\, \forall y\, (x \otimes y^{\circledS} \rightarrow x \otimes y \vee x \circleddash y)$;
(9) $\forall x\, \forall y\, (x \otimes y \vee x \circleddash y \vee y \otimes x)$.

$\mathcal{T}_0$ is a very weak theory. For example, it is easy to see that the commutativity of addition is not a theorem of $\mathcal{T}_0$ (Exercise 5.18). However it is just strong enough to carry through the undecidability and incompleteness results of Gödel and we sketch these next. Of course, the axioms of $\mathcal{T}_0$ are true and the rules of inference preserve truth, so whenever $\mathcal{T}_0 \vdash \mathfrak{A}$, also $\models \mathfrak{A}$. The converse is false (Corollary 5.15) but at least we have:

5.10 Lemma. *For all $\exists_0^0$-sentences $\mathfrak{A}$, if $\models \mathfrak{A}$, then $\mathcal{T}_0 \vdash \mathfrak{A}$.*

Proof. Consider the set X of all formulas $\mathfrak{A}$ such that for any k such that the free variables of $\mathfrak{A}$ are included among $x_0, \dots, x_{k-1}$ and any $\mathbf{m} \in {}^k\omega$, if $\models \mathfrak{A}(\bar{\mathbf{m}})$, then $\mathcal{T}_0 \vdash \mathfrak{A}(\bar{\mathbf{m}})$. It is straightforward to show that X is closed under the clauses of the inductive definition 5.3(i) of $\exists_0^0$ and hence contains all $\exists_0^0$ formulas. □

5.11 Lemma. *For any two disjoint semi-recursive relations R and $S \subseteq {}^k\omega$, there exists an $\exists_1^0$ formula $\mathfrak{C}$ such that for all $\mathbf{m} \in {}^k\omega$,*

(i) $R(\mathbf{m}) \rightarrow \mathcal{T}_0 \vdash \mathfrak{C}(\bar{\mathbf{m}})$;

(ii) $S(\mathbf{m}) \rightarrow \mathcal{T}_0 \vdash \neg\mathfrak{C}(\bar{\mathbf{m}})$.

Proof. By Theorem 5.9 there exist $\exists_0^0$ formulas $\mathfrak{A}$ and $\mathfrak{B}$ such that for all $\mathbf{m} \in {}^k\omega$,

$$R(\mathbf{m}) \leftrightarrow \models \exists x\, \mathfrak{A}(\bar{\mathbf{m}}) \quad \text{and} \quad S(\mathbf{m}) \leftrightarrow \models \exists y\, \mathfrak{B}(\bar{\mathbf{m}}).$$

Let $\mathfrak{C}$ be the formula

$$\exists x\, (\mathfrak{A} \wedge (\forall y \circledcirc x) \neg \mathfrak{B}).$$

Then (i) and (ii) follow easily by use of Lemma 5.10 and axioms (8) and (9). □

For any theory $\mathcal{T}$, a formula $\mathfrak{A}$ with free variables among $x_0, \dots, x_{k-1}$ *$\mathcal{T}$-represents* a relation R iff for all $\mathbf{m} \in {}^k\omega$,

$$R(\mathbf{m}) \rightarrow \mathcal{T} \vdash \mathfrak{A}(\bar{\mathbf{m}}) \quad \text{and} \quad \sim R(\mathbf{m}) \rightarrow \mathcal{T} \vdash \neg \mathfrak{A}(\bar{\mathbf{m}}).$$

R is *$\mathcal{T}$-representable* iff R is $\mathcal{T}$-represented by some formula. It follows immdiately from Lemma 5.11 that all recursive relations are $\mathcal{T}_0$-representable (take $S = \sim R$).

5.12 Lemma. *For any $\mathcal{T}$ which extends $\mathcal{T}_0$, if $\mathcal{T}$ is consistent, then there exists a formula $\mathfrak{C}$ with only x_0 free such that $\{m : \mathcal{T} \vdash \mathfrak{C}(\bar{m})\}$ is not recursive.*

Proof. By Theorem 1.10(i), there exist disjoint semi-recursive sets A and B such that there is no recursive set C such that $A \subseteq C \subseteq \sim B$. Take A and B for R and S in Lemma 5.11 and let $C = \{m : \mathcal{T} \vdash \mathfrak{C}(\bar{m})\}$. By (i) of 5.11, $A \subseteq C$, and by (ii) together with the consistency of $\mathcal{T}$, $C \subseteq \sim B$. Hence C is not recursive. □

Informally, we say that $\mathcal{T}$ is decidable iff there is an algorithm for deciding among the sentences of $\mathcal{L}$ which are theorems of $\mathcal{T}$ and which are not. Thus on the intuitive level, Lemma 5.12 implies that any consistent $\mathcal{T}$ which extends $\mathcal{T}_0$ is undecidable, as otherwise we could effectively decide for each m whether or not $\mathcal{T} \vdash \mathfrak{C}(\bar{m})$. These intuitions are made precise through the technique of Gödel numbering. To each expression (symbol, term, or formula) z of $\mathcal{L}$ we assign a natural number $\ulcorner z \urcorner$ much as we did to each description of a primitive recursive functional in § II.1. The square quotes are to suggest that the assigned number be thought of as a name for z.

We begin by assigning successive odd numbers $1, 3, \ldots$ to the symbols of $\mathcal{L}$. The terms are (recursively) assigned numbers by: $\ulcorner \sigma^{\textcircled{1}} \urcorner = \langle \ulcorner \textcircled{1} \urcorner, \ulcorner \sigma \urcorner \rangle$, $\ulcorner \sigma \oplus \tau \urcorner = \langle \ulcorner \oplus \urcorner, \ulcorner \sigma \urcorner, \ulcorner \tau \urcorner \rangle$, etc. Similarly for formulas, $\ulcorner \sigma \circleddash \tau \urcorner = \langle \ulcorner \circleddash \urcorner, \ulcorner \sigma \urcorner, \ulcorner \tau \urcorner \rangle$, $\ulcorner \mathfrak{A} \vee \mathfrak{B} \urcorner = \langle \ulcorner \vee \urcorner, \ulcorner \mathfrak{A} \urcorner, \ulcorner \mathfrak{B} \urcorner \rangle$, and so on. It is an elementary exercise to show that the sets of Gödel numbers corresponding to various syntactic classes of expressions are recursive — for example, $\{\ulcorner \sigma \urcorner : \sigma \text{ is a term}\}$, $\{\ulcorner \mathfrak{A} \urcorner : \mathfrak{A} \text{ is a formula}\}$ and $\{\ulcorner \mathfrak{A} \urcorner : \mathfrak{A}$ is an $\exists^0_{29}$-formula with at most x_{12} and ϕ_{47} free$\}$. Similarly, there is a recursive function f such that for all $\mathfrak{A}$ and all $\mathbf{m} \in {}^k\omega$, $f(\ulcorner \mathfrak{A} \urcorner, \mathbf{m}) = \ulcorner \mathfrak{A}(\bar{\mathbf{m}}) \urcorner$. Now we call $\mathcal{T}$ *decidable* iff $\{\ulcorner \mathfrak{A} \urcorner : \mathcal{T} \vdash \mathfrak{A}\}$ is recursive.

5.13 Corollary. *Any consistent theory $\mathcal{T}$ which extends $\mathcal{T}_0$ is undecidable.*

Proof. Immediate from Lemma 5.12 and the above discussion. □

A theory $\mathcal{T}$ is called *(first-order) complete* iff for every (first-order) sentence $\mathfrak{A}$, either $\mathfrak{A}$ or $\neg \mathfrak{A}$ is a theorem of $\mathcal{T}$. $\mathcal{T}$ is *recursively axiomatizable* iff there exists a theory $\mathcal{T}'$ such that $\{\ulcorner \mathfrak{A} \urcorner : \mathfrak{A}$ is an axiom of $\mathcal{T}'\}$ is recursive and for all $\mathfrak{A}$, $\mathcal{T} \vdash \mathfrak{A}$ iff $\mathcal{T}' \vdash \mathfrak{A}$. $\mathcal{T}$ is *semi-decidable* iff $\{\ulcorner \mathfrak{A} \urcorner : \mathcal{T} \vdash \mathfrak{A}\}$ is semi-recursive.

5.14 Lemma. *For any $\mathcal{T}$,*

(i) *if $\mathcal{T}$ is recursively axiomatizable, then $\mathcal{T}$ is semi-decidable;*

(ii) *if $\mathcal{T}$ is recursively axiomatizable and complete, then $\mathcal{T}$ is decidable.*

Proof. Let $\Gamma_{\mathcal{T}}$ be the monotone operator defined by:

$$\Gamma_{\mathcal{T}}(A) = \{m : m \in A \text{ or } m \text{ is the Gödel number of an axiom of } \mathcal{T} \text{ or } m \text{ is the Gödel number of a formula which follows from formulas with Gödel numbers in } A \text{ by a single rule of inference}\}.$$

It is easy to see that $\Gamma_{\mathcal{T}}$ is Σ^0_1 and that $\bar{\Gamma}_{\mathcal{T}} = \{\ulcorner \mathfrak{A} \urcorner : \mathcal{T} \vdash \mathfrak{A}\}$. Hence (i) follows from Theorem 3.5. Suppose now $\mathcal{T}$ is recursively axiomatizable and complete. If $\mathcal{T}$ is inconsistent all formulas are provable so $\mathcal{T}$ is decidable. Otherwise the sets $\{\ulcorner \mathfrak{A} \urcorner : \mathfrak{A}$ is a sentence and $\mathcal{T} \vdash \mathfrak{A}\}$ and $\{m : m$ is not the Gödel number of a sentence or m is the Gödel number of a sentence $\mathfrak{A}$ such that $\mathcal{T} \vdash \neg \mathfrak{A}\}$ are complementary. By (i), both are semi-recursive, hence both are recursive and $\mathcal{T}$ is decidable. □

5.15 Corollary. *Any consistent recursively axiomatizable theory which extends $\mathscr{T}_0$ is incomplete.*

Proof. Immediate from 5.13 and 5.14. □

As a consequence, the complete theory $\mathscr{T}_{\mathfrak{N}}$ whose axioms are all first-order formulas valid in the standard model is not recursively axiomatizable. It also follows from 5.14(i) that for any consistent recursively axiomatizable theory $\mathscr{T}$ which extends $\mathscr{T}_0$, a relation R is $\mathscr{T}$-representable iff R is recursive.

That $\mathscr{T}_{\mathfrak{N}}$ is not recursively axiomatizable can also be seen by a slightly different argument. From Theorem 5.9 and the Arithmetical Hierarchy Theorem it follows that $\{\ulcorner\mathfrak{A}\urcorner : \models \mathfrak{A}\}$ is not arithmetical, in particular not semi-recursive, and thus is not the set of theorems of any recursively axiomatizable theory by 5.14(i).

$\mathscr{T}_{\mathfrak{N}}$ can, however, be recursively axiomatized if the system is expanded by adding a rule of inference to which the proof of 5.14(i) does not apply — that is, one such that the operator $\Gamma_{\mathscr{T}}$ defined there is not Σ_1^0. For any theory $\mathscr{T}$, we write $\mathscr{T} \vdash^{\omega} \mathfrak{A}$ to mean that $\mathfrak{A}$ is derivable from the rules and axioms of $\mathscr{T}$ together with the *ω-rule*: from the (infinitely many) premises $\mathfrak{A}(\bar{0}), \mathfrak{A}(\bar{1}), \ldots,$ infer $\forall x_0 \mathfrak{A}$.

5.16 Theorem. (i) *For all first-order formulas* $\mathfrak{A}$, $\mathscr{T}_0 \vdash^{\omega} \mathfrak{A}$ *iff* $\models \mathfrak{A}$;
(ii) $\{\ulcorner\mathfrak{A}\urcorner : \mathscr{T}_0 \vdash^{\omega} \mathfrak{A}\}$ *is* Π_1^1.

Proof. (i) follows from Lemma 5.10 by an easy induction. (ii) follows from Theorem 3.1 and the observation that the operator $\Gamma_{\mathscr{T},\omega}$ associated with $\mathscr{T}$ and the ω-rule is arithmetical and monotone. □

We turn now to the full (second-order) language $\mathscr{L}$. In the first part of this section we considered only the standard interpretation for $\mathscr{L}$, but now we shall need a more general notion. A (general) *structure* for $\mathscr{L}$ is sequence $\mathfrak{U} = (U, \Phi, <_{\mathfrak{U}}, +_{\mathfrak{U}}, \cdot_{\mathfrak{U}}, '_{\mathfrak{U}}, 0_{\mathfrak{U}})$ such that U is a set, Φ is a set of unary total functions $U \to U$, $<_{\mathfrak{U}}$ is a binary relation on U, $+_{\mathfrak{U}}$ and $\cdot_{\mathfrak{U}}$ are binary total functions on U, $'_{\mathfrak{U}}$ is a unary total function on U and $0_{\mathfrak{U}} \in U$. For $\mathbf{u} \in {}^kU$, $\boldsymbol{\varphi} \in {}^l\Phi$, and σ a term of $\mathscr{L}$ with free variables among $x_0, \ldots, x_{k-1}$, $\phi_0, \ldots, \phi_{l-1}$, we define $\sigma_{\mathfrak{U}}[\mathbf{u}, \boldsymbol{\varphi}]$, the *value of σ in $\mathfrak{U}$ at* $(\mathbf{u}, \boldsymbol{\varphi})$ just as in Definition 5.1 except that 0 is replaced by $0_{\mathfrak{U}}$, $+$ by $+_{\mathfrak{U}}$, etc. Similarly, the relation $\mathfrak{U} \models \mathfrak{A}[\mathbf{u}, \boldsymbol{\varphi}]$ is defined as in Definition 5.2 with $<$ replaced by $<_{\mathfrak{U}}$, $\exists n$ replaced by $(\exists v \in U)$, and $\exists \beta$ replaced by $(\exists \psi \in \Phi)$. $\mathfrak{U}$ is a *model* of a theory $\mathscr{T}$ iff $\mathfrak{U} \models \mathfrak{A}$ for all axioms $\mathfrak{A}$ of $\mathscr{T}$. $\mathfrak{U}$ is called an *ω-structure* iff the values in $\mathfrak{U}$ of all the terms $\bar{n}$ ($n \in \omega$) exhaust U. In this case $\mathfrak{U}$ is isomorphic to a structure $\mathscr{A} = (\omega, \mathsf{A}, <_{\mathscr{A}}, +_{\mathscr{A}}, \cdot_{\mathscr{A}}, ', 0)$, where $'$ and 0 have their usual meanings. If in addition $\mathfrak{U}$ (and hence $\mathscr{A}$) is a model of $\mathscr{T}_0$, then $<_{\mathscr{A}}$, $+_{\mathscr{A}}$, and $\cdot_{\mathscr{A}}$ must also coincide with the usual $<$, $+$, and $\cdot$. Hence such a structure is determined by the set $\mathsf{A} \subseteq {}^{\omega}\omega$ and we write simply $\mathsf{A} \models \mathfrak{A}[\mathbf{m}, \boldsymbol{\alpha}]$. In particular, $\models \mathfrak{A}[\mathbf{m}, \boldsymbol{\alpha}]$ iff ${}^{\omega}\omega \models \mathfrak{A}[\mathbf{m}, \boldsymbol{\alpha}]$.

Let $\mathscr{T}_1$ denote the theory obtained from $\mathscr{T}_0$ by extending the logical axioms and rules of $\mathcal{T}_0$ to all formulas of $\mathcal{L}$ and adjoining the following (infinitely many) axioms: the universal closures of

(10) (*Induction*) $\mathfrak{A}(\bar{0}) \wedge \forall x_0[\mathfrak{A} \rightarrow \mathfrak{A}(x_0^{\circledS})] \rightarrow \forall x_0 \mathfrak{A}$ for all formulas $\mathfrak{A}$;

(11) (*Extensionality*) $\forall \phi_0 \forall \phi_1 [\forall x\, (\phi_0(x) = \phi_1(x)) \rightarrow \phi_0 = \phi_1]$;

(12) (Δ_0^1-*Comprehension*) $\forall x\, \exists!\, y\, \mathfrak{A} \rightarrow \exists \phi_0 \forall x\, \mathfrak{A}(\phi_0(x)/y)$ for all $\exists_0^1$-formulas $\mathfrak{A}$.

The completeness theorem of first-order logic has the following natural extension to $\mathcal{L}$:

5.17 Theorem. *For any theory $\mathcal{T}$ which extends $\mathcal{T}_1$ and any sentence $\mathfrak{A}$ of $\mathcal{L}$*
(i) *$\mathcal{T} \vdash \mathfrak{A}$ iff $\mathfrak{U} \models \mathfrak{A}$ for all models $\mathfrak{U}$ of $\mathcal{T}$;*
(ii) *$\mathcal{T} \vdash^{\omega} \mathfrak{A}$ iff $\mathsf{A} \models \mathfrak{A}$ for all ω-models A of $\mathcal{T}$.*

Proof. See Shoenfield [1967, § 8.5]. □

From this and 5.16(i) it is immediate that for all $\forall_1^1$ sentences $\mathfrak{A}$, $\mathcal{T}_1 \vdash^{\omega} \mathfrak{A}$ iff $\models \mathfrak{A}$. From 5.16(ii) it follows that any relation which is $\mathcal{T}_0$-ω-representable is Δ_1^1. The same proof applies to $\mathcal{T}_1$ and from the preceding it can be derived that the $\mathcal{T}_1$-ω-representable relations are exactly the Δ_1^1-relations.

The Δ_0^1-Comprehension axioms guarantee that the range of the variable ϕ_i is not too small. In particular, if A is an ω-model of Δ_0^1-Comprehension, then not only does A contain all arithmetical functions, but because the formula $\mathfrak{A}$ in the Δ_0^1-Comprehension schema may have free variables, A is also closed under the relation "arithmetical in".

As is sketched in Shoenfield [1967, § 8.5], the notion of truth for first-order formulas in $\mathfrak{N}$ may be formalized in the theory $\mathcal{T}_1$ (although the system S there includes the comprehension axioms for all formulas, only Δ_0^1-Comprehension is needed here; indeed, for a fixed n, the truth of $\exists_n^0$ sentences may be defined without use of the comprehension axioms). By virtue of Theorem 5.9, much of the theory of the arithmetical hierarchy may thus be developed in $\mathcal{T}_1$. Relations are denoted in $\mathcal{T}_1$ by the Gödel numbers of their defining formulas and the results of §1 become first-order arithmetical theorems in $\mathcal{T}_1$. Alternatively, the arithmetical hierarchy of relations on numbers may be developed in terms of characteristic functions.

For each r, the truth of $\exists_r^1$ and $\forall_r^1$ sentences in $\mathfrak{N}$ and the classes of formulas which define Σ_r^1 and Π_r^1 relations can similarly be defined in $\mathcal{T}_1$. However, some of the basic closure properties of Σ_r^1 and Π_r^1 cannot be proved without additional axioms. Furthermore, without additional comprehension axioms the class of functions defined by second-order formulas is not provably larger than the class of first-order definable functions. To guarantee the existence of all characteristic functions of Σ_r^1 relations we need

(13) $(\exists^1_r\text{-}Comprehension)$ $\exists \phi_0 \forall x\, [\phi_0(x) = \bar{0} \leftrightarrow \mathfrak{A}]$ for all $\mathfrak{A} \in \exists^1_r$.

Note that in $\mathscr{T}_1$ this is equivalent to the corresponding $\forall^1_r$-Comprehension schema.

Following the development in § 2, we encounter the first difficulty with the next-to-last formula in the proof of Theorem 2.5. The translation of the implication ($\rightarrow$) into $\mathscr{L}$ need not hold in every model of $\mathscr{T}_1$ (see Exercise 5.20). Hence to prove the closure of Σ^1_r under number quantification we need to add the axioms

(14) $(\exists^1_r\text{-}Choice)$ $\forall x\, \exists \phi_0\, \mathfrak{A} \rightarrow \exists \phi_1 \forall x\, \mathfrak{A}(\phi^x_1/\phi_0)$ for all $\mathfrak{A} \in \exists^1_r$.

Here $\mathfrak{A}(\phi^x_1/\phi_0)$ denotes the formula obtained from $\mathfrak{A}$ by replacing all occurrences of terms $\phi_0(\sigma)$ by $\phi_1(\langle x, \sigma\rangle)$, where $\langle \cdot\,, \cdot\rangle$ is a symbol for a pairing function introduced by definition into $\mathscr{T}_1$.

Then 2.6–2.10 follow easily. It is worth noting that there is a different proof of Lemma 2.8 which does not require the $\exists^1_1$-Choice axioms. This is based on the equivalence

$$\exists \phi_0 \ldots \forall x\, \exists y \ldots \mathfrak{A} \leftrightarrow \exists \phi_0 \ldots \exists \phi_1 \forall x \ldots \mathfrak{A}(\phi_1(x)/y)$$

which is provable in $\mathscr{T}_1$ (by use of the Induction and Δ^1_0-Comprehension axioms).

The Analytical Substitution Theorem (2.11) for r follows from the $\exists^1_r$-Comprehension axioms, but here a smaller collection of comprehension axioms will suffice. The second equivalence in the proof of that theorem depends on the fact that if $\forall p\, \mathsf{H}(p, \mathbf{m}, \boldsymbol{\alpha}) \downarrow$, then $\exists \beta\, \forall p\, [\mathsf{H}(p, \mathbf{m}, \boldsymbol{\alpha}) \simeq \beta(p)]$. This is provable from the schema:

(15) $(\Delta^1_r\text{-}Comprehension)$ $\forall x\, \exists!\, y\, \mathfrak{A} \rightarrow \exists \phi_0 \forall x\, \mathfrak{A}(\phi_0(x)/y)$ for all $\mathfrak{A} \in \exists^1_r$.

Note that Δ^1_r-Comprehension is derivable from $\exists^1_r$-Choice (Kreisel [1962]). If $\forall x\, \exists!\, y\, \mathfrak{A}$, then $\forall x\, \exists \phi_0\, \mathfrak{A}(\phi_0(x)/y)$ so by $\exists^1_r$-Choice, $\exists \phi_1 \forall x\, \mathfrak{A}(\phi^x_1(x)/y)$. Then if $\phi_2(x) = \phi_1(\langle x, x\rangle)$, $\forall x\, \mathfrak{A}(\phi_2(x)/y)$. The existence of ϕ_2 follows from that of ϕ_1 by Δ^1_0-Comprehension (cf. Exercise 5.21).

In the counterexample following Corollary 2.13 an $\exists^1_r$-Comprehension axiom is essential to guarantee the existence of (the characteristic function of) U^1_r. Some results concerning monotone inductive definitions can be established in $\mathscr{T}_1$ together with some of (13)–(15), but those which involve ordinals are in general beyond the scope of these theories. The closure $\bar{\Gamma}$ must be *defined* as the intersection of sets closed under Γ and for $\Gamma \in \Pi^1_r$ the $\exists^1_r$-Comprehension schema is needed to prove the existence of $\bar{\Gamma}$. We leave it to the interested reader to determine which axioms are needed to prove the remaining results of §§ 3 and 4.

5.18–5.21 Exercises

5.18. Show that the commutativity of $\oplus$ is not provable in $\mathcal{T}_0$.

5.19. Let V be the set of all $\langle \ulcorner A \urcorner, \mathbf{m} \rangle$ such that for some k, $\mathfrak{A}$ is a first-order $\mathcal{L}$-formula with free variables among $x_0, \ldots, x_{k-1}$, $\mathbf{m} \in {}^k\omega$, and $\models \mathfrak{A}[\mathbf{m}]$. Show that V is reducible to $\bar{\Gamma}$ for some Σ_1^0 inductive operator Γ. (Use the result of Exercise 3.32(i)).

5.20. Show that $\{\alpha : \alpha \in \Delta^0_{(\omega)}\}$ is not a model of $\exists_1^1$-Choice. In fact, it is not a model of Δ_1^1-Comprehension.

5.21. Show that $\exists_1^1$-Comprehension implies $\exists_1^1$-Choice.

5.22 Notes. The main ideas through Corollary 5.15 are due to Gödel [1931], but the presentation here benefits from many modifications and improvements due to Rosser, Tarski, Kleene, and others.

The difference in the forms of $\exists_r^1$-Comprehension and Δ_r^1-Comprehension is due to the fact that we are working in a language with function variables rather than set variables. A (total) function which is $\exists_r^1$-definable is is also $\forall_r^1$-definable. The import of $\exists_r^1$-Comprehension (for ω-models) is that (the characteristic functions of) *sets* which are $\exists_r^1$-definable belong to the model.

6. *Arithmetical Forcing*

The technique of forcing was first developed by Cohen [1963/64] in the context of set theory. We shall develop here a simpler form of this technique due to Feferman [1964/65] which leads to several interesting results in the arithmetical hierarchy.

As motivation, consider an arbitrary open (Σ_1^0) subset A of ${}^\omega\omega$. There exists a set $A \subseteq \omega$ such that for all α,

$$(*) \qquad \alpha \in \mathsf{A} \leftrightarrow \exists p \,.\, \bar{\alpha}(p) \in A.$$

That a function α belong to A when it does is "forced" by some initial segment $\bar{\alpha}(p)$ and every other function β such that $\bar{\beta}(p) = \bar{\alpha}(p)$ also belongs to A. Of course for A which are not open the situation is different — no finite initial segment s can "force" all $\beta \in [s]$ to belong to $\{\alpha : \forall m \,.\, \alpha(m) = 0\}$, for example.

The key idea of forcing is that even for more complex A, there exists a set A such that (*) holds for "many" α. The success of the applications turns on the interpretation of "many".

It is convenient to use here a formal language $\mathcal{L}'$ slightly different from the language $\mathscr{L}$ of § 5. The *symbols* of $\mathscr{L}'$ are $\neg, \vee, \exists, {}^{\circledcirc}, \bar{0}, x_0, x_1, \ldots, \phi, \bar{R}_a$ ($a \in$ Pri).

Note that we have removed all the relation and function symbols except ${}^{\circledcirc}$ from $\mathscr{L}$ in favor of an infinite collection of relation symbols R_a and left only one function variable. The *terms* of $\mathscr{L}'$ are the expressions obtained from $\bar{0}, x_0, x_1, \ldots$ by application of ${}^{\circledcirc}$ and ϕ. The *atomic formulas* of $\mathscr{L}'$ are all expressions of the form $\bar{R}_a(\sigma_0, \ldots, \sigma_{k-1})$ for $a \in$ Pri and $k = \text{rank}[a]$. The *formulas* of $\mathscr{L}'$ are all those expressions obtained from the atomic formulas by application of $\neg$, $\vee$, and $\exists x_i$ ($i = 0, 1, \ldots$) — the variable ϕ is not quantified.

A term σ with free variables included among $x_0, \ldots, x_{k-1}, \phi$, has *value* $\sigma[\mathbf{m}, \alpha]$ defined recursively by clauses (i)–(iii) and (vi) (without the subscript) of Definition 5.1. If $\mathfrak{A}$ is a formula of $\mathscr{L}'$ with free variables included among $x_0, \ldots, x_{k-1}, \phi$, then $\models \mathfrak{A}[\mathbf{m}, \alpha]$ ($\mathfrak{A}$ is *true at* $(\mathbf{m}, \alpha)$) is defined recursively by clauses (iii)–(v) of Definition 5.2 together with:

$$\models \bar{R}_a(\sigma_0, \ldots, \sigma_{k-1})[\mathbf{m}, \alpha] \quad \text{iff} \quad [a](\sigma_0[\mathbf{m}, \alpha], \ldots, \sigma_{k-1}[\mathbf{m}, \alpha]) = 0.$$

The $\exists_0^0$ and $\forall_0^0$ formulas of $\mathscr{L}'$ are simply the atomic formulas and the classes $\exists_{r+1}^0$ and $\forall_{r+1}^0$ are defined as in 5.3. Relations and functionals of rank $(k, 1)$ are $\exists_r^0$ or $\forall_r^0$ definable as before. Then for $\mathscr{L}'$ as for $\mathscr{L}$ we have:

6.1 Theorem. *For all $r > 0$ and all $\mathsf{R} \subseteq {}^{k,1}\omega$, $\mathsf{R} \in \Sigma_r^0$ iff R is $\exists_r^0$-definable and $\mathsf{R} \in \Pi_r^0$ iff R is $\forall_r^0$-definable.*

Proof. It is immediate from the definition that every primitive recursive relation on numbers is $\exists_0^0$-definable and that the class of $\exists_0^0$-definable relations is a Boolean algebra closed under bounded quantification. Hence the proof of Lemma 5.6 also establishes here that the class of $\exists_1^0$-definable relations is closed under binary union and intersection, bounded quantification, and existential number quantification. The function h of Lemma 5.7 is primitive recursive, hence also $\exists_0^0$-definable here and the result follows as for $\mathscr{L}$. $\square$

For any term σ of $\mathscr{L}'$ the value $\sigma[\mathbf{m}, \alpha]$ clearly depends on only finitely many values of α. Hence we may also define the value of a term relative to a finite sequence s thought of as an initial segemnt of α; if s is too short, this will be undefined.

6.2 Definition. For any k, any $\mathbf{m} \in {}^k\omega$, any $s \in \text{Sq}$, and any term σ of $\mathscr{L}'$ whose variables are included among $x_0, \ldots, x_{k-1}, \phi$, $\sigma[\mathbf{m}, s]$ is defined recursively by:

(i) $\bar{0}[\mathbf{m}, s] \simeq 0$;

(ii) $x_i[\mathbf{m}, s] \simeq m_i$;

(iii) $\sigma^{\circledcirc}[\mathbf{m}, s] \simeq \sigma[\mathbf{m}, s] + 1$;

(iv) if $\sigma[\mathbf{m}, s] < \lg(s)$, $\phi(\sigma)[\mathbf{m}, s] \simeq (s)_{\sigma[\mathbf{m}, s]}$.

It is easy to check that for sufficiently large n, $\sigma[\mathbf{m}, \bar{\alpha}(n)] \simeq \sigma[\mathbf{m}, \alpha]$. Hence, if $\sigma[\mathbf{m}, s] \simeq n$, then $\sigma[\mathbf{m}, \alpha] = n$ for all $\alpha \in [s]$.

6.3 Definition. For any k, any $\mathbf{m} \in {}^k\omega$, and $s \in \mathrm{Sq}$, and any formula $\mathfrak{A}$ of $\mathscr{L}'$ whose free variables are included among $x_0, \ldots, x_{k-1}, \phi$, we define $\Vdash \mathfrak{A}[\mathbf{m}, s]$, $\mathfrak{A}$ is *forced at* $(\mathbf{m}, s)$, recursively by:

(i) $\Vdash \bar{R}_a(\sigma_0, \ldots, \sigma_{k-1})[\mathbf{m}, s]$ iff $[a](\sigma_0[\mathbf{m}, s], \ldots, \sigma_{k-1}[\mathbf{m}, s]) \simeq 0$;

(ii) $\Vdash (\neg \mathfrak{A})[\mathbf{m}, s]$ iff $\sim \exists t\, (t \supseteq s \wedge \Vdash \mathfrak{A}[\mathbf{m}, t])$;

(iii) $\Vdash (\mathfrak{A} \vee \mathfrak{B})[\mathbf{m}, s]$ iff $\Vdash \mathfrak{A}[\mathbf{m}, s]$ or $\Vdash \mathfrak{B}[\mathbf{m}, s]$;

(iv) $\Vdash \exists x_i\, \mathfrak{A}[\mathbf{m}, s]$ iff $\exists n\, (\Vdash \mathfrak{A}'(x_k/x_i))[\mathbf{m}, n, s]$;

where $\mathfrak{A}'$ is a variant of $\mathfrak{A}$ in which the variable x_k does not appear.

Note that except for the clause (ii), this definition is nearly identical with that of $\vDash \mathfrak{A}[\mathbf{m}, \alpha]$. In terms of the discussion at the beginning of the section, clause (ii) says that α is "forced" to belong to $\sim$A by its initial segment s just in case no extension t forces membership in A.

6.4 Lemma. *For all* $\mathbf{m}$, s, *and* $\mathfrak{A}$ *as in Definition* 6.3,

(i) *not both* $\Vdash \mathfrak{A}[\mathbf{m}, s]$ *and* $\Vdash \neg \mathfrak{A}[\mathbf{m}, s]$;

(ii) *for any* $t \supseteq s$, *if* $\Vdash \mathfrak{A}[\mathbf{m}, s]$, *then also* $\Vdash \mathfrak{A}[\mathbf{m}, t]$;

(iii) *for some* $t \supseteq s$, *either* $\Vdash \mathfrak{A}[\mathbf{m}, t]$ *or* $\Vdash \neg \mathfrak{A}[\mathbf{m}, t]$.

Proof. (i) and (iii) are immediate from clause (ii) of the definition. We prove (ii) by induction on formulas. For $\mathfrak{A}$ atomic, the statement follows from the fact that if $\sigma[\mathbf{m}, s] \simeq n$ and $s \subseteq t$, then also $\sigma[\mathbf{m}, t] \simeq n$. If $\Vdash \neg \mathfrak{A}[\mathbf{m}, s]$ and $s \subseteq t$, then for any $u \supseteq t$ also $u \supseteq s$ so *not* $\Vdash A[\mathbf{m}, u]$. Hence $\Vdash \neg \mathfrak{A}[\mathbf{m}, t]$. The other two clauses follow similarly. □

We call a formula of $\mathscr{L}'$ *closed* iff none of the variables x_i occur free in $\mathfrak{A}$ — ϕ may occur.

6.5 Definition. For all r and α, α is *r-generic* iff for all closed formulas $\mathfrak{A}$ in $\bigcup \{\exists_i^0 : i \leqslant r\}$,

$$\exists p\, (\Vdash \mathfrak{A}[\bar{\alpha}(p)] \quad \text{or} \quad \Vdash \neg \mathfrak{A}[\bar{\alpha}(p)]).$$

α is *generic* iff α is r-generic for all $r \in \omega$.

We now obtain the promised generalization of (*). Note that all α are 0-generic.

6.6 Theorem. *For all* r, *all closed* $\mathfrak{A}$ *in* $\exists_{r+1}^0$, *and all r-generic functions* α,

$$\vDash \mathfrak{A}[\alpha] \text{ iff for some } p, \Vdash \mathfrak{A}[\bar{\alpha}(p)].$$

Proof. We proceed by induction on r. Suppose $r = 0$ and $\vDash \bar{R}_a(\sigma_0, \ldots, \sigma_{k-1})[\alpha]$.

Then if p is chosen sufficiently large so that $\sigma_0[\bar{\alpha}(p)], \ldots, \sigma_{k-1}[\bar{\alpha}(p)]$ are all defined, then $\models \bar{R}_a(\sigma_0, \ldots, \sigma_{k-1})[\bar{\alpha}(p)]$, i.e. $\Vdash \bar{R}_a(\sigma_0, \ldots, \sigma_{k-1})[\bar{\alpha}(p)]$. Conversely, if $\Vdash \bar{R}_a(\sigma_0, \ldots, \sigma_{k-1})[\bar{\alpha}(p)]$, then for all $i < k$, $\sigma_i[\alpha] = \sigma_i[\bar{\alpha}(p)]$ so also $\models \bar{R}_a(\sigma_0, \ldots, \sigma_{k-1})[\alpha]$.

Assume the result for r and suppose α is $r+1$-generic and first that $\mathfrak{A} \in \forall^0_{r+1}$. Then $\mathfrak{A} = \neg\mathfrak{B}$ for some $\mathfrak{B} \in \exists^0_{r+1}$ and we have

$$\models \mathfrak{A}[\alpha] \leftrightarrow \text{not} \models \mathfrak{B}[\alpha] \leftrightarrow \text{not}\, \exists p . \Vdash \mathfrak{B}[\bar{\alpha}(p)].$$

Because α is $r+1$-generic, this is equivalent to $\exists p . \Vdash \neg\mathfrak{B}[\bar{\alpha}(p)]$.

Now if $\mathfrak{A} \in \exists^0_{r+2}$, $\mathfrak{A} = \exists x_i\, \mathfrak{C}$ for some $\mathfrak{C} \in \forall^0_{r+1}$ and we have

$$\models \mathfrak{A}[\alpha] \leftrightarrow \exists n . \models \mathfrak{C}(\bar{n}/x_i)[\alpha]$$
$$\leftrightarrow \exists n\, \exists p . \Vdash \mathfrak{C}(\bar{n}/x_i)[\bar{\alpha}(p)] \leftrightarrow \exists p . \Vdash \mathfrak{A}[\bar{\alpha}(p)]. \quad \square$$

For Theorem 6.6 to be useful there must be sufficiently many r-generic functions. First we have

6.7 Lemma. *For any s there are $2^{\aleph_0}$ generic functions in $[s]$.*

Proof. Let $\mathfrak{A}_0, \mathfrak{A}_1, \ldots$ be a list of the denumerably many closed formulas of $\mathscr{L}'$. For any s and any β we define recursively:

$$\gamma(0) = s; \qquad \gamma(2n+1) = \gamma(2n) * \langle \beta(n) \rangle;$$
$$\gamma(2n+2) = \text{least}\ t[t \in \mathrm{Sq} \wedge \gamma(2n+1) \subseteq t \wedge (\Vdash \mathfrak{A}_n[t] \text{ or } \Vdash \neg\mathfrak{A}_n[t])].$$

γ is a well-defined function by Lemma 6.4(iii). As $\gamma(n) \subseteq \gamma(n+1)$ and $\gamma(2n+1)$ is a proper extension of $\gamma(2n)$, there exists a unique limit function β^* such that for all m, $\beta^*(m) = (\gamma(n))_m$ for all sufficiently large n. Clearly $\beta^* \in [s]$ and is generic. Since if $\beta \neq \delta$, also $\beta^* \neq \delta^*$, there are $2^{\aleph_0}$ distinct such β^*. $\square$

Our next aim is to show that there are some relatively simple r-generic and generic functions. For this we shall use a construction similar to that in the preceding proof together with an assessment of the complexity of the relation $\Vdash \mathfrak{A}[s]$. Let $\ulcorner\mathfrak{A}\urcorner$ and $\ulcorner\sigma\urcorner$ denote the Gödel numbers of $\mathfrak{A}$ and σ in an assignment of numbers to the formulas and terms of $\mathscr{L}'$ similar to that described for $\mathscr{L}$ in § 5. By the discussion there it is clear that for each r, $\{\ulcorner\mathfrak{A}\urcorner : \mathfrak{A}$ *is a closed* $\exists^0_r$-formula$\}$ is recursive. Similarly, there are partial recursive functions f_k such that for all terms σ and all m and s, $f_k(\ulcorner\sigma\urcorner, \mathbf{m}, s) \simeq \sigma[\mathbf{m}, s]$. Let

$$\mathrm{Fo}^+_r(\ulcorner\mathfrak{A}\urcorner, s) \leftrightarrow \mathfrak{A} \text{ is a closed } \exists^0_r\text{-formula and } \Vdash \mathfrak{A}[s];$$
$$\mathrm{Fo}^-_r(\ulcorner\mathfrak{A}\urcorner, s) \leftrightarrow \mathfrak{A} \text{ is a closed } \exists^0_r\text{-formula and } \Vdash \neg\mathfrak{A}[s];$$

$$\mathrm{Fo}^+ = \bigcup \{\mathrm{Fo}_r^+ : r \in \omega\};$$

$$\mathrm{Fo}^- = \bigcup \{\mathrm{Fo}_r^- : r \in \omega\}.$$

6.8 Lemma. *For each* $r > 0$, $\mathrm{Fo}_r^+ \in \Sigma_r^0$ *and* $\mathrm{Fo}_r^- \in \Pi_r^0$; Fo^+ *and* Fo^- *are* Δ_1^1.

Proof. If $\mathfrak{A}$ is a closed $\exists_1^0$-formula, then for some $a \in \mathrm{Pri}$ and some terms $\sigma_0, \ldots, \sigma_{k-1}$, $\mathfrak{A} = \exists x_i \bar{R}_a(\sigma_0, \ldots, \sigma_{k-1})$. Then

$$\begin{aligned} \Vdash \mathfrak{A}[s] &\leftrightarrow \exists n . \Vdash \bar{R}_a(\sigma_0, \ldots, \sigma_{k-1})(\bar{n}/x_i)[s] \\ &\leftrightarrow \exists n . [a](\sigma_0[n, s], \ldots, \sigma_{k-1}[n, s]) = 0 \\ &\leftrightarrow \exists n . [a](f_1(\ulcorner \sigma_0 \urcorner, n, s), \ldots, f_1(\ulcorner \sigma_{k-1} \urcorner, n, s)) = 0. \end{aligned}$$

Since the Gödel numbers $\ulcorner \sigma_i \urcorner$ of the terms occuring in $\mathfrak{A}$ can be recursively calculated from $\ulcorner \mathfrak{A} \urcorner$, this shows $\mathrm{Fo}_1^+ \in \Sigma_1^0$. That $\mathrm{Fo}_1^- \in \Pi_1^0$ is immediate from this and the definition of $\Vdash$.

Suppose now that $\mathrm{Fo}_r^+ \in \Sigma_r^0$ and $\mathrm{Fo}_r^- \in \Pi_r^0$. Then for any closed $\exists_{r+1}^0$-formula $\mathfrak{A} = \exists x_i \mathfrak{B}$,

$$\Vdash \mathfrak{A}[s] \leftrightarrow \exists n . \Vdash \mathfrak{B}(\bar{n}/x_i)[s] \leftrightarrow \exists n . \mathrm{Fo}_r^-(g(\ulcorner \mathfrak{A} \urcorner, n), s),$$

where g is a recursive function such that

$$g(\ulcorner \exists x_i \mathfrak{B} \urcorner, n) = \ulcorner \mathfrak{B}(\bar{n}/x_i) \urcorner.$$

Hence $\mathrm{Fo}_{r+1}^+ \in \Sigma_{r+1}^0$. That $\mathrm{Fo}_{r+1}^- \in \Pi_{r+1}^0$ follows immediately.

To evaluate Fo^+ and Fo^-, recall the relation $U_{(\omega)}^0$:

$$U_{(\omega)}^0(r, a, \langle \mathbf{m} \rangle) \leftrightarrow U_r^0(a, \langle \mathbf{m} \rangle).$$

By Theorems 3.6 and 3.7, $U_{(\omega)}^0 \in \Delta_1^1$. The preceding part of the proof may be interpreted as providing instructions for computing recursive functions h^+ and h^- such that for all $r > 0$,

$$\mathrm{Fo}_r^+(m, s) \leftrightarrow U_{(\omega)}^0(r, h^+(r), \langle m, s \rangle), \quad \text{and}$$

$$\mathrm{Fo}_r^-(m, s) \leftrightarrow \sim U_{(\omega)}^0(r, h^-(r), \langle m, s \rangle);$$

that is, $h^+(r)$ is an index of Fo_r^+ as a Σ_r^0 set and correspondingly for h^-. Then

$$\mathrm{Fo}^+(m, s) \leftrightarrow \exists r\, U_{(\omega)}^0(r, h^+(r), \langle m, s \rangle), \quad \text{and}$$

$$\mathrm{Fo}^-(m, s) \leftrightarrow \exists r \sim U_{(\omega)}^0(r, h^-(r), \langle m, s \rangle),$$

so both are Δ_1^1. $\square$

6.9 Theorem. *For all r there exist r-generic functions in* Δ^0_{r+1}. *There exist generic functions in* Δ^1_1.

Proof. Let f be a recursive function which enumerates the Gödel numbers of closed $\exists^0_r$-formulas and let $\mathfrak{A}_n$ denote the formula with Gödel number n. Set $\gamma(0) = \langle \quad \rangle$ and

$$\begin{aligned}\gamma(n+1) &= \text{least } t[t \in \mathrm{Sq} \wedge \gamma(n) \subsetneq t \wedge (\Vdash \mathfrak{A}_n[t] \text{ or } \Vdash \neg \mathfrak{A}_n[t])] \\ &= \text{least } t[t \in \mathrm{Sq} \wedge \gamma(n) \subsetneq t \wedge (\mathrm{Fo}^+_r(f(n), t) \vee \mathrm{Fo}^-_r(f(n), t))].\end{aligned}$$

As in the proof of 6.7, if β the limit of the sequences $\gamma(n)$, β is r-generic. Both γ and β are recursive in the relations Fo^+_r and Fo^-_r, hence are Δ^0_{r+1} by Post's Theorem.

The second part is proved similarly using Fo^+ and Fo^-. □

6.10 Corollary. *For all r,* $\{\alpha: \alpha \in \Delta^0_{r+1}\} \notin \Sigma^0_{r+1}$. $\{\alpha: \alpha \in \Delta^0_{(\omega)}\} \notin \Delta^0_{(\omega)}$ (second proof — cf. Corollary 4.23).

Proof. Suppose, contrary to the first assertion, that there exists an $\exists^0_r$-formula $\mathfrak{A}$ of $\mathscr{L}'$ such that

$$\alpha \in \Delta^0_{r+1} \leftrightarrow \vDash \exists x_0 \neg \mathfrak{A}[\alpha].$$

Then for some n and some r-generic β in Δ^0_{r+1}, $\vDash \neg \mathfrak{A}(\bar{n}/x_0)[\beta]$. It follows from Theorem 6.6 that for some p, $\Vdash \neg \mathfrak{A}(\bar{n}/x_0)[\bar{\beta}(p)]$. By Lemma 6.7 there are $2^{\aleph_0}$ r-generic functions γ such that $\bar{\gamma}(p) = \bar{\beta}(p)$. For each of these $\Vdash \neg \mathfrak{A}(\bar{n}/x_0)[\bar{\gamma}(p)]$, hence $\vDash \neg \mathfrak{A}(n/x_0)[\gamma]$, hence $\vDash \exists x_0 \neg \mathfrak{A}[\gamma]$, hence $\gamma \in \Delta^0_{r+1}$. This contradicts the fact that there are only countably many Δ^0_{r+1} functions.

For the second part, suppose that $\mathsf{A} = \{\alpha: \alpha \in \Delta^0_{(\omega)}\}$ were, say, Σ^0_{r+1}. The foregoing proof shows that no denumerable Σ^0_{r+1} set includes $\{\alpha: \alpha \in \Delta^0_{r+1}\}$, and A would be such a set, a contradiction. □

A strengthening of the first part of 6.10 is proved in 6.15.

In the examples of 2.3 we saw that if $\{\alpha\} \in \Delta^0_{(\omega)}$, then $\alpha \in \Delta^1_1$ and in IV.2.22 we shall show that every $\alpha \in \Delta^1_1$ is recursive in some β with $\{\beta\} \in \Pi^0_1$. The natural conjecture that every Δ^1_1 function is implicitly $\Delta^0_{(\omega)}$ is, however, false:

6.11 Corollary. *There exist* $\alpha \in \Delta^1_1$ *such that* $\{\alpha\} \notin \Delta^0_{(\omega)}$.

Proof. Let α be any Δ^1_1 generic function. Suppose that for some $\mathfrak{A} \in \exists^0_r$, $\mathfrak{A}$ defines $\{\alpha\}$ — that is, for all β,

$$\beta = \alpha \leftrightarrow \vDash \mathfrak{A}[\beta].$$

Since $\models \mathfrak{A}[\alpha]$ and α is generic, $\Vdash \mathfrak{A}[\bar{\alpha}(p)]$ for some p. But then for any generic $\beta \in [\bar{\alpha}(p)]$ also $\Vdash \mathfrak{A}[\bar{\beta}(p)]$, hence $\models \mathfrak{A}[\beta]$. By Lemma 6.7 there are $2^{\aleph_0}$ such β, a contradiction. □

Related to this question is the relationship between Δ_1^1 subsets of ω and sets of the form $\bar{\Gamma}$ for monotone arithmetical operators Γ. It follows from Theorem 3.2 that every Δ_1^1 set is many-one reducible to such a $\bar{\Gamma}$, but the following shows that the reduction cannot in general be omitted.

6.12 Corollary. *There exist $A \in \Delta_1^1$ such that $A \neq \bar{\Gamma}$ for any monotone arithmetical operator Γ.*

Proof. Let α be a Δ_1^1 generic function and $A = \{m : \alpha(m) = 0\} = Z_\alpha$. Suppose that $A = \bar{\Gamma}$ for some monotone arithmetical Γ. Since P_Γ is arithmetical, there exists a formula $\mathfrak{A} \in \exists_0^1$ such that for all β,

$$\forall m\, [\mathsf{P}_\Gamma(m, \beta) \rightarrow \beta(m) = 0] \leftrightarrow \models \mathfrak{A}[\beta].$$

Since Γ is monotone, A is the smallest set whose characteristic function satisfies the left-hand side of this equivalence. Hence $\models \mathfrak{A}[\alpha]$ and for any β, if $\models \mathfrak{A}[\beta]$, then $A \subseteq Z_\beta$. Since α is generic, $\Vdash \mathfrak{A}[\bar{\alpha}(p)]$ for some p. Clearly A is not finite so there exists a $q \geqslant p$ such that $q \in A$. Let β be any generic function in $[\bar{\alpha}(q) * \langle 1 \rangle]$. Then $\Vdash \mathfrak{A}[\bar{\beta}(p)]$ so $\models \mathfrak{A}[\beta]$, but $q \in A \sim Z_\beta$, a contradiction. □

To obtain the promised extensions of 4.12–4.14 we need to measure the size of the set of r-generic functions in yet another way:

6.13 Lemma. *For all r, the set of r-generic functions is comeager.*

Proof. For each closed $\exists_r^0$-formula $\mathfrak{A}$, let

$$\mathsf{A}_{\mathfrak{A}} = \{\alpha : \exists p\, (\Vdash \mathfrak{A}[\bar{\alpha}(p)] \text{ or } \Vdash \neg \mathfrak{A}[\bar{\alpha}(p)])\}.$$

It is immediate from Lemma 6.4 that each $\mathsf{A}_{\mathfrak{A}}$ is open and dense, so that $\sim\mathsf{A}_{\mathfrak{A}}$ is nowhere dense. Since

$$\{\alpha : \alpha \text{ is not } r\text{-generic}\} = \bigcup \{\sim\mathsf{A}_{\mathfrak{A}} : \mathfrak{A} \text{ is a closed } \exists_r^0\text{-formula}\},$$

this set is meager. □

6.14 Theorem. *For all r, Δ_{r+1}^0 is a basis for the class of non-meager Σ_{r+3}^0 sets.*

Proof. Suppose $\mathsf{A} \in \Sigma_{r+3}^0$ is non-meager and let $\mathfrak{B}$ be an $\forall_r^0$-formula such that

$$\alpha \in \mathsf{A} \leftrightarrow \models \exists x_0 \forall x_1 \exists x_2 \mathfrak{B}[\alpha].$$

A is the countable union of sets A_m, where

$$\alpha \in \mathsf{A}_m \leftrightarrow \models \forall x_1 \exists x_2 \mathfrak{B}[m, \alpha].$$

Let Ge_r denote the set of r-generic functions. By the preceding Lemma, $\mathsf{A} \sim \mathrm{Ge}_r$ is meager so $\mathsf{A} \cap \mathrm{Ge}_r$ is non-meager. Hence for some $\bar{m}$ and some $\bar{s}$, $\mathsf{A}_{\bar{m}} \cap \mathrm{Ge}_r$ is dense in $[\bar{s}]$.

We shall contruct a Δ^0_{r+1} function in $\mathsf{A}_{\bar{m}} \cap \mathrm{Ge}_r$. As in the proof of Theorem 6.9, let f be a recursive function which enumerates $\{\ulcorner\mathfrak{A}\urcorner : \mathfrak{A}$ is a closed $\exists^0_r$ formula$\}$, and let $\mathfrak{A}_n$ be the formula with Gödel number $f(n)$. Set

$$\gamma(0) = \bar{s};$$

$$\gamma(2n+1) = \text{least } t[t \in \mathrm{Sq} \wedge \gamma(2n) \subseteq t \wedge (\Vdash \mathfrak{A}_n[t] \text{ or } \Vdash \neg \mathfrak{A}_n[t])];$$

$$\gamma(2n+2) = (\text{least } u[(u)_1 \in \mathrm{Sq} \wedge \gamma(2n+1) \subsetneqq (u)_1 \wedge \Vdash \mathfrak{B}[\bar{m}, n, (u)_0, (u)_1]])_1.$$

To see that γ is well defined, suppose that $\gamma(2n+1)$ is defined. Since

$$\mathsf{A}_{\bar{m}} \cap \mathrm{Ge}_r \subseteq \{\alpha : \models \exists x_2 \mathfrak{B}[\bar{m}, n, \alpha]\} \cap \mathrm{Ge}_r,$$

this latter set is also dense in $[\bar{s}]$ and thus has a non-empty intersection with $[\gamma(2n+1)]$. If δ is a member of this intersection, then for some p, $\models \mathfrak{B}[\bar{m}, n, p, \delta]$, so since δ is r-generic, $\Vdash \mathfrak{B}[\bar{m}, n, p, \bar{\delta}(q)]$ for some q. We may choose q larger than $\lg(\gamma(2n+1))$. Then $u = \langle p, \bar{\delta}(q)\rangle$ satisfies the condition and $\gamma(2n+2)$ is defined.

Now let β be the limit of the sequences $\gamma(n)$. The odd stages of this construction ensure that β is r-generic and the even stages ensure that

$$\forall n \exists p \exists q . \Vdash \mathfrak{B}[\bar{m}, n, p, \bar{\beta}(q)].$$

Hence

$$\forall n \exists p \models \mathfrak{B}[\bar{m}, n, p, \beta],$$

and thus

$$\models \forall x_1 \exists x_2 \mathfrak{B}[\bar{m}, \beta].$$

Thus $\beta \in \mathsf{A}_{\bar{m}} \cap \mathrm{Ge}_r$. That also $\beta \in \Delta^0_{r+1}$ follows from Lemma 6.8 and Post's Theorem. β is the required element of $\mathsf{A} \cap \Delta^0_{r+1}$. □

6.15 Corollary. *For all* $r > 0$, $\{\alpha : \alpha \in \Delta^0_r\} \in \Sigma^0_{r+2} \sim \Delta^0_{r+2}$.

Proof. The positive half follows from the equivalence

$$\alpha \in \Delta_r^0 \leftrightarrow \exists a\, \forall m\, \forall n\, [\alpha(m) = n \leftrightarrow U_r^0(a, \langle m, n\rangle)].$$

Suppose that also $\{\alpha\colon \alpha \in \Delta_r^0\} \in \Pi_{r+2}^0$. Then $\{\alpha\colon \alpha \notin \Delta_r^0\}$ is a comeager, hence non-meager, Σ_{r+2}^0 set, so by the Theorem, it has a Δ_r^0 element, a contradiction. □

6.16–6.17 Exercises

6.16. Show that for all $r \geqslant 1$ there exist functions β which are implicitly Π_{r+1}^0 but not implicitly Π_r^0.

6.17. Use the result of the preceding Exercise to show that Exercise 4.24 is false for $r = 1$.

6.18 Notes. For readers familiar with forcing in the context of set theory as described (say) in Shoenfield [1971], we note that the r-generic functions are those which meet a certain collection of Δ_{r+1}^0 dense sets, namely those of the form

$$\{s\colon \Vdash \mathfrak{A}[s] \text{ or } \Vdash \neg \mathfrak{A}[s]\}$$

for $\exists_r^0$ formulas $\mathfrak{A}$. Similarly, the generic functions are those which meet a certain collection of arithmetical dense sets.

Corollary 6.10 is due to Addison [1965], the proof given here is from Hinman [1969a]. 6.14 and 6.15 are also from Hinman [1969a]. The other results are due to Feferman [1964/65].