SOLVABILITY OF SOME ASYMPTOTICALLY HOMOGENEOUS ELLIPTIC PROBLEMS

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In this talk, we discuss some problems on the existence and uniqueness of solutions of some nonlinear boundary value problems. Most of the time we will discuss existence and finally, at the end, we will discuss a non-uniqueness result.

Assume that Ω is a bounded domain in R^n , L is a self-adjoint linear operator on $L^2(\Omega)$ with compact resolvent and that $g : R \to R$ is continuous such that $y^{-1}g(y) \to \mu(\nu)$ as $y \to \infty\ (-\infty)$. We now want to discuss whether the equation

$$(1) \qquad Lu = g(u) - f$$

has a solution u in $L^2(\Omega)$ for every $f \in L^2(\Omega)$. For *simplicity*, we also assume that L is bounded below. As an example of the type of problem to which we wish to apply our results, we could take L to be $-\Delta$ with Dirichlet boundary conditions.

These type of problems have been studied in some detail in recent years. We will discuss a few of the results known and, in more detail, some of the open problems. A more complete bibliography could be obtained from the references in [3] - [6].

Let $y^+ = \sup\{y,0\}$ and $y^- = y - y^+$. Now it is easy to show that $\|u\|^{-1}(g(u)-\mu u^+-\nu u^-) \to 0$ in $L^2(\Omega)$ as $\|u\| \to \infty$. Then one might expect that there is a close relationship between the solvability of (1) for every $f \in L^2(\Omega)$ and the solvability of

(2) $$Lu = \mu u^{+} + \nu u^{-} - f$$

for every $f \in L^2(\Omega)$. Note that (2) is a linear equation if $\mu = \nu$. By analogy with the linear case, one might expect that (2) is better behaved if

(3) $$Lu = \mu u^{+} + \nu u^{-}$$

has no non-trivial solution. (Note that $u = 0$ is always a solution.) It is not quite clear how good this analogy is. For example, we do not know if (2) can be solvable for all $f \in L^2(\Omega)$ when (3) has a non-trivial solution. (A number of partial results are known.) However, it seems unlikely that (1) can be well-behaved for all g if (3) has a non trivial solution.

Let $A_0 = \{(\mu,\nu) \in R^2 : \text{(3) has a non-trivial solution}\}$. By the above comments, A_0 is of some interest. Unfortunately, A_0 is difficult to calculate even in very simple examples. For example, we have very limited knowledge of A_0 in the following cases :

a) Ω is a ball and $L = -\Delta$ with Dirichlet boundary conditions;

b) $\Omega = [0,\pi]$ and $Ly = -y^4$ with boundary conditions $y(0) = y^1(0) = y(\pi) = y^1(\pi) = 0$.

One of the few cases where we know A_0 explicitly is when

c) $\Omega = [0,\pi]$ and $Ly = -y^2$ with boundary conditions $y(0) = y(\pi) = 0$. In this case, A_0 is the curves in Figure 1.

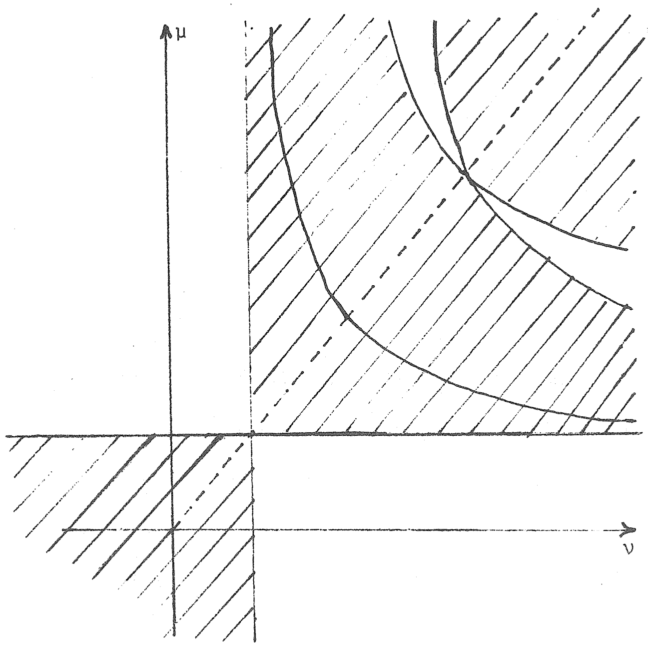

Figure 1

Here we have sketched $A_0 \cap \{(\mu,\nu) : \mu,\nu \le 10\}$. Note that, in general $(\alpha,\alpha) \in A_0$ if and only if α is an eigenvalue of L .

Since it is difficult to calculate A_0 explicitly, it is natural to ask questions about the qualitative behaviour of A_0 . One of the simplest questions one could ask is : *does* A_0 *contain an open set ?* Unfortunately, this is an open problem even for example (a) above. (For example (b) above, A_0 does not contain an open set: in fact, much more can be proved.) It seems likely that A_0 does not contain an open set. Most of the difficulty in attempting to prove this is caused by the non-smoothness of the map $u \to u^+$ (from $D(L)$ with the graph norm to $L^2(\Omega)$.)

It can be shown (cp [4]) that if λ_i is an eigenvalue of L, then the component of A_0 containing (λ_i, λ_i) is unbounded. This gives some information on the structure of A_0. Even the proof of this result needs fairly sophisticated techniques; in particular, it uses Conley's homotopy index [2].

A few other results are known on the structure of A_0. See [3], [4], [7] and [8].

Let us now return to the original problem and assume that $(\mu, \nu) \notin A_0$. An easy geometric type argument shows that, *if there is an* f *in* $L^2(\Omega)$ *for which* (2) *has no solution, then there is an* f_1 *in* $L^2(\Omega)$ *for which* (1) *has no solution.* An important unsolved problem is : does the converse hold? It seems probable that the converse is false in general but it would be interesting to know when it is true. Note that (1) is not always solvable for all $f \in L^2(\Omega)$. For instance, in example (c), if $(\mu, \nu) \notin A_0$ and if $\mu, \nu \leq 10$, then (1) is solvable for all f in $L^2(\Omega)$ precisely when (μ, ν) belongs to the shaded areas in Figure 1.

One can associate with eqn (3) a *finite dimensional* mapping P. If either the degree or the homotopy index determined by P is non-zero, one can prove that (1) is solvable for all $f \in L^2(\Omega)$. However, we do not know whether the homotopy index is non-zero if (2) is solvable for all f in $L^2(\Omega)$. (There is an example where (2) is solvable for all f in $L^2(\Omega)$ but the degree is zero. It would be interesting to find a similar example where L is a differential operator). Unfortunately, P is defined rather implicitly and as a consequence its degree or homotopy index are difficult to calculate. If Ω is a symmetric domain (e.g. a ball) and if L preserves the symmetries, then one can sometimes prove that the degree or homotopy index is non-zero by evaluating the *degree* on a subspace of $L^2(\Omega)$ of symmetric functions. The latter degree is often easier to evaluate. This work is discussed in [5]. Note that, for this method, it is much better to use the homotopy index rather than the degree.

Finally, we want to discuss whether equation (1) has at most one solution for each $f \in L^2(\Omega)$. Let $\lambda_1 < \lambda_2 < \ldots$ denote the distinct eigenvalues of L. If g is differentiable on R and if there is an i such that $\lambda_{i-1} < g'(y) < \lambda_i$ for all y, then it is easy to prove that (2) has at most one solution for all f in $L^2(\Omega)$. (Here $\lambda_0 = -\infty$). On the other hand, assume that g is twice continuously differentiable, that g' is bounded on R, that there is an i and $y_1, y_2 \in R$ with $g'(y_1) < \lambda_i < g'(y_2)$ and that $(L-\alpha I)^{-1} L^\infty(\Omega) \subseteq L^\infty(\Omega)$ for every $\alpha \in L^\infty(\Omega)$. Then there exists f in $L^2(\Omega)$ such that (1) has at least two solutions. This is a special case of a much more general abstract result. Its proof depends upon degree theory and a strengthening of Sard's theorem due to Church [1]. Note that the above uniqueness result does not need the assumptions on g near $\pm\infty$. The boundedness condition on g' can be eliminated for many L's.

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