TRANSFERRING FOURIER MULTIPLIERS

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1. FOURIER MULTIPLIERS OF $L^p(G)$

Let G be a compact Lie group, and $\hat{G}$ is dual (a maximal set of irreducible representations of G). The Fourier transform of $f \in L^1(G)$ associates to $\sigma \in \hat{G}$, the $d_\sigma \times d_\sigma$ matrix $\int_G f(x)\, \sigma(x^{-1})\, dx$ (where d_σ is the dimension of the space in which σ acts).

The Fourier multipliers of $L^p(G)$ are sequences (A_σ) of matrices so that if $(\hat{f}(\sigma))$ is the Fourier series of an L^p function, so is $(A_\sigma \hat{f}(\sigma))$.

Example. If $G = SU(2)$, $\hat{G} \equiv \{0,\frac{1}{2},1,\ldots\}$ and if $\ell \in \hat{G}$, σ_ℓ has dimension $2\ell+1$, and we look for sequences $A_0, A_{\frac{1}{2}}, \ldots$, where A_ℓ is a $(2\ell+1) \times (2\ell+1)$ matrix.

2. EXAMPLES OF MULTIPLIERS

(i) Central multipliers. We restrict to $A_\sigma = c_\sigma I$ for $c_\sigma \in \mathbb{C}$. This is the case which has been most studied. For example, Bonami and Clere [1] and Clere [2] have shown that the

$$\text{Poisson kernel} \quad e^{-\sqrt{\frac{\ell}{R}}}\, I_{\sigma_\ell}$$

$$\text{Gauss kernel} \quad e^{-\frac{\ell}{R}}\, I_{\sigma_\ell}$$

$$\text{Riesz kernel} \quad \left(1 - \frac{\ell}{R}\right)^{\delta}_{+} I_{\sigma_\ell} \qquad (\delta > 1)$$

are bounded summability kernels in $L^p(SU(2))$. These results also hold for general G , where ℓ is replaced by μ_σ , the eigenvalue of χ_σ under the biinvariant Laplacian on G .

Coifman and Weiss [4] unified these results; $m_\ell I_{\sigma_\ell}$ is a multiplier of $L^p(SU(2))$ if $\ell \mapsto (2\ell+1)\, m_\ell - (2\ell-1)\, m_{\ell-1}$ is a multiplier of Fourier series. This result also holds in some generality. This criterion can be applied to the above kernels.

The idea is to use the Weyl integration formula to reduce the problem to the maximal torus.

(ii) Noncentral multipliers. Very little has been proved in generality. Coifman and Weiss [3] gave an extremely detailed study of SU(2), and showed that the Riesz kernels are bounded. One defines operators B_1 , $\bar{B}_1$, B_2 , $\bar{B}_2$, as certain differential operators and computes their Fourier transforms to be

$$\hat{B}_1(\sigma_\ell) = \frac{1}{2\ell}\begin{pmatrix} 0 & 0 & \cdots & & 0 \\ \sqrt{2\ell} & 0 & \cdots & & 0 \\ 0 & \sqrt{2\ell(2\ell-1)} & & \cdots & 0 \\ & & & & \vdots \\ 0 & \cdots & & \sqrt{2\ell} & 0 \end{pmatrix} \qquad \hat{\bar{B}}_1 = \hat{B}_1{}^*$$

$$\hat{B}_2(\sigma_\ell) = \frac{1}{2\ell}\begin{pmatrix} 2\ell & 0 & \cdots & 0 \\ 0 & 2\ell-1 & \cdots & 0 \\ & & \ddots & \\ & 0 & & -2\ell \end{pmatrix} \qquad \hat{\bar{B}}_2(\sigma_\ell)\begin{pmatrix} -2\ell & & \\ & \ddots & \\ 0 & & 2\ell \end{pmatrix}$$

shows that, for a given harmonic polynomial which is zero at the origin, $F = f + \sum_{j=1}^{3} \varepsilon_j f_j$ is a generalized analytic function, where ε_j are a basis for the quaternions and f_j are the Riesz transforms

$$\tilde{f}_1 = i(B_1 - \bar{B}_1)\, f , \qquad f_2 = i(B_1 + \bar{B}_1)\, f , \qquad f_3 = -i(\bar{B}_2 - B_2)\, f$$

This generalizes the classical Riesz transform. The critical point is to prove that the B_j are bounded on $L^p(SU(2))$ for then one may attempt an H^p theory.

The proof that the B_j are bounded given by Coifman and Weiss depends on a detailed structural analysis of $SU(2)$, together with some theory of pseudo differential operators. It is hard to estimate the multiplier norm of the B_j.

I wish to propose a new approach, based upon some of my recent work on contractions of Lie groups which gives a simpler proof plus better control over the constants.

3. CONTRACTION OF $\underline{\underline{g}}$ ONTO G

We consider, as have many other authors, the family of maps $\pi_\lambda : \underline{\underline{g}} \to G : X \mapsto \exp \frac{X}{\lambda}$ $(\lambda > 0)$. We use some results of C.S. Herz [6], on periodification of multipliers to "transfer" multipliers of $L^p(\underline{\underline{g}})$ onto multipliers of $L^p(G)$. Specifically, we can prove

Theorem. *There is a canonical norm non-increasing map* $i_\lambda : A_p(G) \to B_p(\underline{\underline{g}})$.

Proof. For X locally compact Hausdorff, Herz defines $V_p(X)$ to be the set of functions of two variables which are pointwise multipliers of $\ell^p \otimes \ell^{p'}(X)$; B_p may be identified as the elements of V_p invariant under right translation in both variables. According to a theorem of Herz, $F \mapsto F \circ \pi_\lambda \times \pi_\lambda$ is a norm non-increasing map $V_p(G) \to V_p(\underline{\underline{g}})$. The map which takes $F \in V_p(\underline{\underline{g}})$ to the invariant mean $\tilde{F}$ of $Z \mapsto F(X+Z, Y+Z)$ gives a projection $V_p(\underline{\underline{g}}) \to B_p(\underline{\underline{g}})$. Combining these maps with the injection $A_p(G) \hookrightarrow V_p(G)$ gives the map i_λ. $\square$

It remains to compute i_λ. The following theorem is proved in [5].

Theorem. *Let* $f \in A_p(G)$.

$$(*) \qquad (i_\lambda f)(X) = \int_{G/T} f\left(g \cdot \exp\left(\frac{g^{-1} \cdot X}{\lambda}\right)_{\underline{\underline{t}}} \cdot g^{-1}\right) d\dot{g}$$

where T *is any maximal torus for* G, $\underline{\underline{t}}$ *is its Lie algebra, and* $(\ )_{\underline{\underline{t}}}$ *denotes projection onto* $\underline{\underline{t}}$.

(*) is a natural generalization of the usual periodification map $A_p(\mathbb{T}) \to B_p(\mathbb{R})$.

We may dualize i_λ, obtaining $i_\lambda^* : B_p^*(\underline{\underline{g}}) \to M_p(G)$. Thus, for any $f \in L^1(\underline{\underline{g}})$, $i_\lambda^* f \in M_p(G)$ and

$$|||\, i_\lambda^* f \,|||_p \leq |||\, f \,|||_p$$

(where $|||\cdot|||_p$ denotes the multiplier norm).

Using (*), we may compute the Fourier transform of $i_\lambda^* f$. For simplicity, we restrict to the case $G = SU(2)$, although the formula holds in generality (when suitably modified).

Theorem. *Let* $\{u_k\}_{k=-\ell}^{\ell}$ *be the usual orthonormal basis for* H_{σ_ℓ}. *Then for each* $\phi \in L^1(\underline{\underline{g}})$,

$$(**) \qquad (i_\lambda^* \phi)\hat{\ }(\sigma_\ell)_{n,m} = \sum_{k=-\ell}^{\ell} \int_{G/T} \hat{\phi}\left(\frac{g \cdot k}{\lambda}\right) t_{n,k}^{\ell}\, \bar{t}_{m,k}^{\ell}(\dot{g})\, d\dot{g}\,.$$

4. APPLICATIONS

(i) Suppose ϕ is a radial multiplier of $L^p(\mathbb{R}^3)$ (alias an $Ad(SU(2))$ invariant multiplier of $L^p(SU(2))$).

Then $\hat{\phi}\left(\frac{g \cdot k}{\lambda}\right)$ is independent of g, so we may use the orthogonality relations on (**) to obtain

$$(i_\lambda{}^*\phi)^\wedge(\sigma_\ell) = \frac{1}{2\ell+1} \sum_{k=-\ell}^{\ell} \hat{\phi}\left(\frac{k}{\lambda}\right) \cdot I_{d_{\sigma_\ell}} \; .$$

Thus $i_\lambda{}^*\phi$ is a central multiplier.

Knowing the L^p boundedness of certain radial multipliers on $\mathbb{R}^3$ now allows us to deduce the boundedness of certain other multipliers on $SU(2)$. In fact, we may solve the equation

$$\psi = i_\lambda{}^*\phi$$

for ϕ, obtaining

$$\hat{\phi}\left(\frac{\ell}{\lambda}\right) = (2\ell+1)\; \hat{\psi}(\sigma_\ell) - (2\ell-1)\; \hat{\psi}(\sigma_{\ell-1})$$

which is effectively the theorem of Coifman and Weiss alluded to above.

(ii) More interesting are the noncentral multipliers. Take

$$\phi(x) = f(|x|)\; Y_{s,q}\left(\frac{x}{|x|}\right)$$

where $Y_{s,q}$ is a spherical harmonic of degree s, and $-s \le q \le s$. In fact, identifying $Y_{s,q}$ with $t^s_{qo}(\dot{g})$, a matrix coefficient of $SU(2)$, we compute the Fourier transform of $(i_\lambda{}^*\phi)$, by using the Bochner-Hecke formula, as

$$(i_\lambda{}^*\phi)^\wedge(\sigma_\ell)_{m,n} = \sum_{k=-\ell}^{\ell} 2\pi i^{-s}\left(\frac{|k|}{\lambda}\right)^{s-\frac12} \hat{f}\left(s+\tfrac12,\ \frac{|k|}{\lambda}\right) \int_{SU(2)} t^s_{qo}\; t_{mk}\; \bar{t}_{nk}(g)\; dg$$

where $\hat{f}(\;,\;)$ is the Bessel transform of f. The integral over $SU(2)$ is just a Clebsch-Gordan coefficient which may be evaluated.

Taking in particular $f(|x|) = |x|^{-3}$, and $Y_{1,0}$, one computes easily that $2^{-3/2}\pi^{-\frac12} i_\lambda{}^*\Phi = I + \overline{B_2}$, and similarly $\overline{|x|^{-3}\; Y_{1,0}(x/|x|)}$ becomes $I + B_2$. One may further check that

$$\frac{Y_{1,1}\left(\frac{x}{|x|}\right)}{|x|^3}$$

has for image $2^{3/2}\,\pi^{\frac12}(-i)\; B_1$, and that

$$\frac{Y_{1,-1}\left(\frac{x}{|x|}\right)}{|x|^3}$$

has as image $2^{3/2}\pi^{\frac{1}{2}}(i)\,\overline{B_1}$.

It follows at once from our approach that

$$|||B_1|||_p \le \frac{1}{\sqrt{8\pi}}|||\phi_1|||_{p,\mathbb{R}^3} = \frac{1}{\sqrt{8\pi}}\frac{p}{p+1}$$

and

$$|||B_2|||_p \le 1 + \frac{1}{\sqrt{8\pi}}|||\phi_2|||_{p,\mathbb{R}^3} = 1 + \frac{1}{\sqrt{8\pi}}\frac{p}{p+1} .$$

These estimates are extremely precise, and avoid completely any mention of pseudo differential operators.

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