ON THE EXISTENCE OF POSITIVE ENTIRE SOLUTIONS OF NONLINEAR ELLIPTIC PROBLEMS IN R^N

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Consider the elliptic problem

$$Lu = f(x,u), \quad x \in R^N \tag{1}$$

$$u \in C^2(R^N), \quad \lim_{|x|\to\infty} u(x) = 0, \tag{2}$$

for $N \geq 3$, where

$$Lu = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right);$$

each $a_{ij} \in C^{1+\alpha}_{loc}(R^N)$, $0 < \alpha < 1$, the matrix $(a_{ij}(x))$ is bounded and uniformly positive definite in R^N, and the conditions (A) below hold.

Our main objective is to prove the existence of positive solutions of (1), (2), and obtain asymptotic estimates. A prototype of this class is the equation

$$-\Delta u = p(x)u^{\gamma}, \quad x \in R^N, \tag{3}$$

$1 < \gamma < \frac{N+2}{N-2}$, and $p(x) \not\equiv 0$ is a locally Hölder continuous function in R^N, satisfying $0 \leq p(x) \leq C(1 + |x|^2)^{-b}$ for some constants C, $1 < b < N/2$. The problem has been the subject of intensive investigations in recent years. In particular, there are several results on the existence of positive solutions of equation (1) which are bounded below by positive constants, see, for example, [5], [6], and the references therein. We do not know of any existence result in the literature for the problem in (1), (2). To obtain our main result, we employ a new approach, developed here, based on combining a variational method and the barrier method. i.e. super-and-subsolutions.

Hypotheses (A):

a) $f(x,u)$ is locally Hölder continuous, and there exists an open set $U \subset R^N$ such that $\lim_{u\to\infty} \frac{f(x,u)}{u} = \infty$ uniformly in $x \in U$.

b) $0 \leq f(x,u) \leq C(1 + |x|^2)^{-b} u^{\gamma}$,

$$2b < \inf_{x \in R^N} \Big(\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} (a_{ij} x_j) \Big) \Big(\sum_{i,j=1}^{N} a_{ij} x_i x_j |x|^{-2} \Big)^{-1}$$

$$\leq \sup_{x \in R^N} \Big(\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} (a_{ij} x_j) \Big) \Big(\sum_{i,j=1}^{N} a_{ij} x_i x_j |x|^{-2} \Big)^{-1} = 2a$$

for all $x \in R^N$ and $u \geq 0$, where a, b, γ, C are positive constants with $1 < b < a \leq N/2$, and $1 < \gamma < (N + 2)/(N - 2)$.

c) there is a number $1/2 > \theta > 0$ such that for all $x \in R^N$, $u > 0$, we have

$$\theta u f(x,u) \geq F(x,u) \equiv \int_0^u f(x,t)dt.$$

Notice that (b) is satisfied for (3) with the choice $a = N/2$.

THEOREM. *Let the hypotheses* (A) *hold. Then* (1) *has infinitely many bounded positive entire solutions. There is also at least one positive solution of* (1), (2), *satisfying*

$$C_2(1 + |x|^2)^{1-a-\varepsilon} \leq u(x) \leq C_1(1 + |x|^2)^{1-b}$$

for $x \in R^N$, *where* C_1, C_2 *and* ε *are positive constants.*

To prove this result, we proceed as follows:

1. Using the variational approach in [1], we construct positive solutions u_k of the problems

(4) $$Lu + 1/k\, u = f(x,u), \quad x \in R^N,$$

$k = 1, 2, \ldots$. Each u_k is determined as a critical point, in the Sobolev space $W_0^{1/2}(R^N) \equiv E$, of the functional

$$J_k(u) = \frac{1}{2} \int_{R^N} [\sum_{i,j=1}^{N} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + 1/k\, u^2]dx - \phi(u),$$

where $\phi(u) = \int_{R^N} F(x,u(x))dx.$

To see this, notice first that hypotheses (a) and (b) imply that each J_k, $k = 1, 2, \ldots,$ is well-defined and is in $C^1(E)$. From hypothesis (c) and the embedding theorem 2.2 in [2], it can be shown that the functional ϕ is weakly sequentially compact on E. Thus, the argument in [8] may be adapted to show that the Palais-Smale condition holds, i.e. every sequence $\{\phi_n\} \subset E$ such that $|J_k(\phi_n)| \leqslant M$ and $J_k^1(\phi_n) \to 0$ in norm in E^*(the dual space) as $n \to \infty$, has a convergent subsequence in E. It is not hard to show, see [9], that for some nonincreasing sequences δ_k, α_k, $k = 1, 2, \ldots,$ of positive numbers, $J_k(u) \geqslant \alpha_k$ on $\|u\|_E = \delta_k$, and $J_k(u) > 0$ for $0 < \|u\|_E < \delta_k$.

On the other hand, hypothesis (a) implies that there is an $e \in E$ with $e(x) \geqslant 0$, $\text{supp}\, e \subset U$, $\|e\|_E > \delta_1$, and $J_k(te) < 0$ for all $k,t \geqslant 1$. It thus follows that the conditions of the well-known Mountain-Pass lemma [1], [9] are satisfied, and therefore the numbers

$$\inf_{\gamma\in\Gamma} \max_{u\in\gamma} J_k(u) = c_k, \quad k = 1, 2, \ldots,$$

are positive critical values of J_k, where Γ is the set of all continuous paths in E connecting 0 and e. Let u_k be the corresponding critical points. Thus u_k are positive classical solutions of (4) by standard theorems on elliptic regularity. It is not hard to see from hypothesis (c) and the variational character of u_k, that

$$\| |\nabla u_k| \|_{L^2(R^N)} \leqslant M. \tag{5}$$

Therefore

$$\|u_k\|_{L^{2N/(N-2)}(R^N)} \leqslant M, \tag{6}$$

by the Sobolev embedding theorem, where M is a positive constant independent of k.

2. We use (5), (6), and a device due to H. Brezis and T. Kato [3], to show (Lemma 1 below) that for sufficiently large q, the norms $\|u_k\|_{L^q(R^N)}$ are uniformly bounded in k. Since u_k may be considered as a solution of the linear problem $Lu = \tilde{f}_k(x) = f(x,u_k(x)) - 1/k\ u_k(x)$, lemma 1 and a standard interior Hölder estimate (Theorem 8.24 in [4]) imply that

$$\sup_{x \varepsilon R^N} u_k(x) \leqslant M, \tag{7}$$

for some positive constant M, independent of k. Thus a subsequence, say $\{u_k\}$, converges locally uniformly in $C^2(R^N)$ to a solution $\bar{u} \geqslant 0$, by standard theorems on elliptic regularity. If $\bar{u} \not\equiv 0$, then $\bar{u}$ is a positive solution of (1), (2). If $\bar{u} \equiv 0$, lemma 2 establishes the existence of a supersolution $v(x) = C(1 + |x|^2)^{1-b}$, for some positive constant C, such that $u_k(x) \leqslant v(x)$, $x \varepsilon R^N$, for all sufficiently large k; and the existence of a positive solution of (1), (2) follows from the well-known barrier method [6], [7]. The asymptotic estimate on a positive solution of (1) and (2) may be deduced from the maximum principle using the comparison functions $C|x|^{2-2b-\varepsilon}$, $c|x|^{2-2a}$, $|x| > 1$. Finally, the existence of infinitely many positive solutions of equation (1) which are bounded from above and below by positive constants follows from known results, see [5], [6], [7].

LEMMA 1. *Let* $\{u_k\}$ E *be a sequence of positive classical solutions of* (4) *satisfying* (5). *Then for sufficiently large* q

$$\|u_k\|_{L^q(R^N)} \leqslant M$$

for some constant M, *independent of* k.

Proof. Multiplication of (4) by u_k^p, $p > 1$, integration by parts, using the uniform ellipticity, the hypothesis $u_k(x) \geqslant 0$, and condition (c), we obtain

$$4p(1+p)^{-1} \int_{R^N} |\nabla u_k^{1/2(1+p)}|^2 dx \leqslant M \int_{R^N} (u_k(x))^{p+\gamma} dx,$$

where M is a positive constant, independent of k. Let $\varepsilon > 0$ be arbitrary and let

$$\delta = \varepsilon^{\gamma-(N+2)(N-2)^{-1}},$$

$$c_\varepsilon = \delta^{p+\gamma-2N(N-2)^{-1}}.$$

Then, for $p \geqslant 2N(N-2)^{-1} - \gamma$ we have

$$|t|^{p+\gamma} \leqslant \varepsilon|t|^{p+(N+2)(N-2)^{-1}} + c_\varepsilon|t|^{2N(N-2)^{-1}}.$$

Using (6) we then have

$$\left(\int_{R^N} u_k^q\,dx\right)^{(N-2)N^{-1}} \leqslant M\left(\varepsilon \int_{R^N} u_k^{(N-2)(Nq)^{-1}+4(N-2)^{-1}}\,dx + C_\varepsilon C\right),$$

where $q = N(p + 1)(N - 2)^{-1}$, and M, C are constants, independent of k. Applying Hölder's inequality we obtain

$$\left(\int_{R^N} u_k^q\,dx\right)^{(N-2)N^{-1}} \leqslant C\varepsilon\left(\int_{R^N} u_k^q\,dx\right)^{(N-2)N^{-1}}\left(\int_{R^N} u_k^{2N(N-2)^{-1}}\,dx\right)^{2N^{-1}} + C_\varepsilon.$$

The conclusion of Lemma 1 then follows by choosing ε sufficiently small.

LEMMA 2. *Assume* $\{u_k\}$ *to be a sequence of positive classical solutions of* (4) *satisfying* (7), *and converging locally uniformly in* $C^2(R^N)$ *to* $\bar{u} \equiv 0$. *Then there exists a supersolution* v *of* (1) *and* (2) *such that* $v(x) \geqslant u_k(x)$ *for all* $x \in R^N$, *and for all* k *sufficiently large.*

Proof. Define $v(x) = C(1 + |x|)^2)^{1-b}$. Simple calculations, using hypothesis (b), shows that $Lv(x) \geqslant f(x, v(x))$ for sufficiently small $C > 0$. Now let $\phi_0 = K(1 + |x|)^2)^{1-b}$ with k chosen, using hypothesis (b), such that $L\phi_0 \geqslant \frac{1}{2}K(1 + |x|^2)^{-b}$. Since u_k satisfies $Lu_k \leqslant C(1 + |x|^2)^{-b}$ by (4), (7) and hypothesis (b), we can choose k large enough such that $L\phi_0(x) \geqslant Lu_k(x)$ for all $x \in R^N$. Thus $\phi_0(x) \geqslant u_k(x)$ for all k, and for all $x \in R^N$, by the maximum principle. We then have $Lu_k \leqslant C(1 + |x|)^2)^{-b})(\phi_0(x))^\gamma$. Therefore,

$\varepsilon > 0$ can be chosen such that $b + \varepsilon < a$ and $Lu_k \leq C(1 + |x|)^2)^{-b-\varepsilon}$ for all $k = 1, 2, \ldots,$ and all $x \in R^N$. Now let $\phi_1(x) = K(1 + |x|^2)^{1-b-\varepsilon}$ with K chosen such that $L\phi_1 \geq \frac{1}{2}K(1 + |x|^2)^{-b-\varepsilon}$. Then, as before, $u_k(x) \leq \phi_1(x)$, $x \in R^N$, $k = 1, 2, \ldots,$ by the maximum principle. This implies that we can choose R large enough such that $u_k(x) \leq v(x)$, $k = 1, 2, \ldots,$ $|x| \geq R$.

Finally, we use the uniform convergence hypothesis of $\{u_k\}$ to $\bar{u} \equiv 0$, on $|x| \leq R$, to choose k_0 such that $u_k(x) \leq v(x)$, $x \in R^N$, and $k \geq k_0$.

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