SUPPORTS AND LOCALIZATION FOR MULTIPLE FOURIER SERIES

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ABSTRACT

Relationships are described between the supports of functions on the d-dimensional torus and the convergence of their multiple Fourier series both at 0 and in compact neighbourhoods of 0.

1. INTRODUCTION

Let $T^d = \{x=(x_1,\ldots,x_d): -\pi < x_j \le \pi,\ j=1,\ldots,d\}$ denote the d-dimensional torus. For each $f \in L^1 = L^1(T^d)$ and $n = (n_1,\ldots,n_d) \in Z^d$ define

$$\hat{f}(n) = (2\pi)^{-d} \int f(x) e^{-inx} dx,$$

where $nx = n_1x_1 + \ldots + n_dx_d$ and the integral is over T^d. (All integrals and all spaces will be over T^d unles otherwise specified.) The Fourier series of f is

$$f \sim \Sigma_{n\in Z^d} \hat{f}(n) e^{inx}$$

and its partial sums are

$$S_N f(x) = \Sigma_{|m|\le N} \hat{f}(n) e^{inx} ,$$

where $N = (N_1,\dots,N_d)$ and $|m| \le N$ signifies $|m_j| \le N_j$ for $j = 1,\dots,d$.

The pointwise localization principle states that the behaviour of the limit (in a suitable sense) of $S_N f$ at a point x depends only on the values of f in a neighbourhood of x. This principle in clearly valid when $d = 1$, for in this case we have the classical result:

1.1 THEOREM *Let* $d = 1$. *If* $f \in L^1$ *vanishes in a neighbourhood of* x, *then*

$$S_N f(x) \to 0 \quad \text{as} \quad N \to \infty.$$

However, the obvious generalization of this principle does not hold when $d \ge 2$.

1.2 THEOREM *Suppose* $d \ge 2$. *There exists a continuous* f, *vanishing in a neighbourhood of* 0, *such that*

$$\limsup_n S_{(n,\dots,n)} f(0) = \infty.$$

(This was first proved by Igari [3] using the uniform boundedness principle. For example of functions with the properties described in this theorem see Goffman and Liu [4] and Price and Shepp [5].) Our first main result shows how Theorems 1.1 and 1.2 can both be seen as special cases of a more general result. (The result is stated for the point 0 but evidently is equally valid for any point in T^d.) A pointer to the method of reconciliation of these results can be seen in Edwards [2,5.2.1].

1.3 THEOREM *Let* E *be a measurable subset of* T^d. *The following conditions on* E *are equivalent*:

(i) $(x_1 \dots x_d)^{-1} 1_E \in L^1$,

(ii) $S_N f(0) \to 0$ *as* $|N| \to \infty$ *for all continuous* f *supported in* E,

(iii) $S_{(n,\ldots,n)}f(0) \to 0$ *as* $n \to \infty$ *for all continuous* f *supported in* E.

1.4 DISCUSSION (i) Suppose $d = 1$ and $E \subseteq \mathbb{T}$ is a closed connected set. Evidently $x^{-1}1_E$ is integrable if and only if $0 \notin E$. Thus Theorem 1.1 follows (at least for continuous functions).

(ii) For $d \geq 2$, let$E = \mathbb{T}^d \backslash \{x: |x| < 1\}$.
Then $(x_1 \ldots x_d)^{-1} 1_E \notin L^1$ and we achieve Theorem 1.2.

In the latter case it is also possible to deduce another classical result, one involving cross neighbourhoods. (A cross neighbourhood $C_\varepsilon(x)$of a point $x \in \mathbb{T}^d$ is a set of the form

$$C_\varepsilon(x) = \{y \in \mathbb{T}^d : |y_j - x_j| < \varepsilon_j \text{ for at least one } j \in \{1,\ldots,d\}\},$$

where $\varepsilon_1,\ldots,\varepsilon_d > 0$.) Clearly complements E of cross neighbourhoods of 0 satisfy $(x_1 \ldots x_d)^{-1} 1_E \in L^1$ and so we have the following result (Zygmund [7, Chap. XVII, (1.27)]).

1.5 THEOREM *Suppose* f *is a continuous function vanishing on a cross neighbourhood of* 0. *Then*

$$S_N f(0) \to 0 \quad \textit{as} \quad |N| \to \infty.$$

2. UNIFORM LOCALIZATION PRINCIPLE In the 1-dimensional case, there is a stronger result than Theorem 1.1, namely:

2.1 THEOREM *Suppose* $d = 1$ *and let* f *be an integrable function vanishing on an open interval containing* x_0. *Then* $S_N f(x) \to 0$ as $N \to \infty$ *uniformly on any compact subset of that interval.*

For proof see Zygmund [7, Chap. II, (6.3)]. First of all we see that nothing like this is true in higher dimensions for ordinary neighbourhoods.

2.2 THEOREM *Suppose* $d > 1$ *and let* U *be an open neighbourhood of* 0 *and* (x_m) *a countable subset of* U *with each* $x_m \neq 0$. *There exist continuous functions* f *so that*

(i) f *vanishes on* U,

(ii) $S_N f(0) \to 0$ *as* $|N| \to \infty$, *and*

(iii) $\limsup_n S_{(n,\ldots,n)} f(x_m) = \infty$ *for all* m.

Our final result shows that if we want to consider uniform convergence, then we are forced to deal with cross neighbourhoods.

2.3 THEOREM *Let* f *be a continuous function on* $\mathbb{T}^d$. *The following conditions are equivalent:*

(i) f *vanishes on a cross neighbourhood* $C_\varepsilon(0)$ *of* 0,

(ii) $S_N f \to 0$ *uniformly as* $|N| \to \infty$ *on every compact subset of the rectangle*

$$R_\varepsilon(0) = \{x\colon |x_j| < \varepsilon_j \text{ for } j=1,\ldots,d\},$$

(iii) $S_{(n,\ldots,n)} f \to 0$ *uniformly as* $n \to \infty$ *on every compact subset of the rectangle* $R_\varepsilon(0)$.

Moreover, the result remains valid if uniformly is replaced by pointwise in both (ii) *and* (iii).

The above discussion has, in the main, been confined to continuous functions. Analogous results are also possible for L^p functions. The proofs of the above results and their L^p generalizations, along with further discussion, will appear elsewhere. For overviews of the very large subject of multiple Fourier series see Ash [1] or Zhizhiashvili [6].

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