HARMONIC ANALYSIS OF THE QUANTUM NONLINEAR SCHRÖDINGER EQUATION

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ABSTRACT

The Hamiltonian of the quantum nonlinear Schrödinger equation is a selfadjoint operator on Fock space whose eigenstates are given by the Bethe Ansatz. The quantum inverse scattering method of the physics literature introduced two families of operators which have been claimed to satisfy important properties. Operators of one family are diagonal on the Bethe Ansatz eigenstates, while the other family creates the Bethe Ansatz eigenstates. In the present work we examine these operators. We establish some of the claims and show the inconsistency of others.

INTRODUCTION

The quantum nonlinear Schrödinger equation (QNLS)

$$i\Psi_t = -\Psi_{xx} + 2c\Psi(x)^\dagger\Psi(x)^2 \tag{1.1}$$

is an integrable model of quantum field theory in $1+1$ space-time. Eq. (1.1) has been much studied in the literature. Recently, a method of solution of QNLS has been proposed [10], which became known as the quantum inverse scattering method [4] (QISM). The method consists of two parts: the direct and the inverse. The direct part associates with the standard quantum field $\Psi(x)$, $\Psi^\dagger(y)$, two families $A(\lambda)$, $B(\lambda)$ of quantum operators acting on the Fock space $\mathcal{H}$. Taking $\Psi(x)$ for initial condition in eq. (1.1) we obtain the time-dependent QNLS field $\Psi(x,t)$. The corresponding time-dependent families $A(\lambda,t)$, $B(\lambda,t)$ satisfy

$$A(\lambda,t)_t = 0\ ,\ iB(\lambda,t)_t = \lambda^2 B(\lambda,t)\ . \tag{1.2}$$

The inverse part of the method reconstructs the quantum field $\Psi(x,t)$ from $A(\lambda,t)$,

$B(\lambda,t)$. Since equations (1.2) have an explicit solution, QISM provides a way of solving the Cauchy problem for the nonlinear quantum equation (1.1).

The quantum inverse method relies on the relations between the operators $A(\lambda)$, $B(\lambda)$ and the QNLS Hamiltonian

$$(1.3) \qquad H = \int_{-\infty}^{\infty} dx[\Psi_x^\dagger \Psi_x + c\Psi^\dagger(x)^2\Psi(x)^2]$$

as well as on the commutation relations between $A(\lambda)$, $B(\lambda)$ and their Hermitian adjoints $A^\dagger(\mu)$, $B^\dagger(\mu)$. The derivation of these relations in the QISM literature is not satisfactory. It is based on formal manipulations with the formal expansions of $A(\lambda)$, $B(\lambda)$ in the quantum fields $\Psi(x)$, $\Psi^\dagger(y)$. Working with $A(\lambda)$, $B(\lambda)$, "formally-algebraically" [10] is not acceptable since we are dealing with very singular operators here.

A different approach to the QNLS is provided by the method of intertwining operators [3,6,7], which we abbreviate as MIO. The MIO expressions for the operators $A(\lambda)$, $B(\lambda)$ and for the QNLS field $\Psi(x,t)$ [7] are different from the QISM formulas. Ref. [8] contains a critical exposition of QISM and a comparison of MIO and QISM expressions. The conclusion is that despite the formal difference, the two methods define the same operators $A(\lambda)$, $B(\lambda)$. The present paper is a byproduct of the work on [8]. Here we investigate the action of the QISM operators $A(\lambda)$, $B(\lambda)$, $A^\dagger(\mu)$, $B^\dagger(\mu)$ from the point of view of the generalized eigenstates of the QNLS Hamiltonian (1.3). The eigenstate decomposition of a selfadjoint operator is the harmonic analysis of the operator, which explains the title of the paper.

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2. PRELIMINARIES

QNLS (1.1) takes place on the bosonic Fock space

$$\mathcal{H} = \bigoplus_{N=0}^{\infty} \mathcal{H}_N \tag{2.1}$$

of the quantum field theory in $1 + 1$ space-time. For $N > 0$ there is a (noncanonical) isomorphism of the N-particle sector $\mathcal{H}_N$ and the Hilbert space $L_2^{sym}(\mathbf{R}^N)$ of square-integrable functions $f(x_1,\dots,x_N)$, symmetric with respect to permutations of $x_1,\dots,x_N$. We denote by $x \to wx$ the action of the permutation group W_N. The space $\mathcal{H}_0$ is one-dimensional and is spanned by the vacuum vector $|0\rangle$. We say that we have a quantum field when there is a family $\Psi(x)$ of operators on $\mathcal{H}$, and their adjoints $\Psi^\dagger(x)$ satisfying the canonical commutation relations (CCR)

$$[\Psi(x),\Psi(y)] = 0\ , \quad [\Psi(x),\Psi^\dagger(y)] = \delta(x-y)\ . \tag{2.2}$$

The reader can find a mathematical treatment of CCR in [1], for instance. Here we only point out that $\Psi(x)$, $\Psi^\dagger(y)$ are not really operators but the operator-valued distributions (operator densities in physics terminology). We further require that

$$\Psi^\dagger(x) : \mathcal{H}_N \to \mathcal{H}_{N+1}\ , \quad \Psi(x) : \mathcal{H}_N \to \mathcal{H}_{N-1} \tag{2.3}$$

and that the action of $\Psi^\dagger(x)$ on $|0\rangle$ generates $\mathcal{H}$. The densities $\Psi^\dagger(x)$ are called creation operators and $\Psi(y)$ are the annihilation operators. It is known that any representation of CCR satisfying conditions above, is unitarily equivalent to the standard representation $\Psi_0(x)$, $\Psi_0^\dagger(y)$

$$\begin{aligned} (\Psi_0(x)f)(x_1,\dots,x_{N-1}) &= \sqrt{N}\, f(x,x_1,\dots,x_{N-1})\ , \\ (\Psi_0^\dagger(y)f)(x_1,\dots,x_{N+1}) &= (1/\sqrt{N+1}) \sum_{i=1}^{N+1} \delta(x_i-y) f(x_1,\dots,\hat{x}_i,\dots,x_{N+1})\ . \end{aligned} \tag{2.4}$$

A quantum field $\Psi(x)$, $\Psi^\dagger(y)$ can be used to represent operators on $\mathcal{H}$ in the normal ordered form [1]. For instance, eq. (1.3) represents the QNLS Hamiltonian in normal form, where $\Psi(x)$, $\Psi^\dagger(x)$ in (1.3) mean the standard quantum field (2.4).

An expression in $\Psi(x)$, $\Psi^\dagger(y)$ is said to be in normal form if the creation operators are on the left of all annihilation operators.

The Hamiltonian (1.3) is the second quantized form [1] of the bosonic many-body problem (in one dimension) with the delta-function interaction [2,9]. More precisely, the QNLS Hamiltonian preserves the N-particle sectors, and the restriction of H to $\mathcal{H}_N$ is given by

$$H_N = -\sum_{i=1}^{N} \partial^2/\partial x_i^2 + c \sum_{i\neq j} \delta(x_i - x_j) \tag{2.5}$$

which is the N-body Hamiltonian with the potential $v(x) = c\delta(x)$. Physically, (2.5) describes the system of N quantum balls of unit mass on the line, interacting by elastic collisions, and c is the strength of interaction. When $c = 0$, there is no interaction, this is the free case, $c > 0$ is the repulsive case which is simpler than the attractive case, $c < 0$. The many-body problem (2.5) is called in the physics literature the Bose delta-gas.

By discussion above, the eigenstates of the QNLS Hamiltonian H are the eigenstates of H_N for $N \geq 0$. The eigenstates of the delta-gas Hamiltonians have been known in the literature since [2,9], and later were nicknamed the Bethe Ansatz eigenstates (BAE's), see, e.g., [5] and the bibliography there. We denote by $|k_1,\dots,k_N\rangle_c$ the BAE of (2.5) corresponding to N distinct real quantum numbers $k_1 \neq \dots \neq k_N$. Since $|k_1,\dots,k_N\rangle_c$ is a symmetric function of $x_1,\dots,x_N$, it is determined by its values for $x_1 > \dots > x_N$. We have (see, e.g. [8])

$$|k_1,\dots,k_N\rangle_c(x_1 > \dots > x_N) = \tag{2.6}$$

$$(N!)^{-\frac{1}{2}} \sum_{w=(i_1,\dots,i_N)} \prod_{r<s} \frac{c+i(k_{i_r}-k_{i_s})}{i(k_{i_r}-k_{i_s})}\, e^{i(k_{i_1}x_1+\dots+k_{i_N}x_N)}$$

We rewrite eq. (2.6) in shorthand as

(2.7) $$|k_1,\dots,k_N\rangle_c(x_1>\dots>x_N) =$$

$$(N!)^{-\frac{1}{2}}\sum_{w} w\left\{\prod_{i<j}\frac{c+i(k_i-k_j)}{i(k_i-k_j)}\exp(i\langle k|x\rangle)\right\}$$

where the permutations w act on the vector $k=(k_1,\dots,k_N)$.

The BAE's (2.7) belong to the absolutely continuous spectrum of H

(2.8) $$H|k_1,\dots,k_N\rangle_c = (k_1^2+\dots+k_N^2)|k_1,\dots,k_N\rangle_c .$$

They are orthogonal in the generalized sense but not normalized to the δ-function. In fact, they are very singular because of the denominators (k_i-k_j) in (2.7). The BAE's are complete in $\mathcal{H}$ only if $c \geq 0$, which makes the repulsive case easier. The proofs of the assertions above use the fact that the delta-gas Hamiltonian (2.5) is equal to the Laplacean with special boundary conditions on the hyperplanes $\{x_i = x_j\}$, $i<j$ (see, e.g. [6]). Denote by $(\partial/\partial x_i - \partial/\partial x_j)f|\pm$ the jump of the normal derivative across the hyperplane $x_i = x_j$. The boundary conditions for H_N are

(2.9) $$(\partial/\partial x_i - \partial/\partial x_j)f|\pm = 2cf .$$

Since we are dealing with symmetric functions $f(x_1,\dots,x_N)$, the boundary conditions (2.9) can be translated into conditions on the inbound normal derivatives of f on the walls $\{x_1>\dots>x_i=x_{i+1}>x_{i+2}>\dots>x_N\}$ of the fundamental region $\{x_1\geq\dots\geq x_N\}$. There are $N-1$ walls corresponding to $i=1,\dots,N-1$. The conditions are

(2.10) $$(\partial/\partial x_i - \partial/\partial x_{i+1})f = cf .$$

Solution of the classical nonlinear Schrödinger equation (CNLS) by the classical inverse scattering method [11] associates with CNLS an auxiliary spectral problem, the Zakharov-Shabat problem. The matrix entries $a(\lambda)$, $b(\lambda)$ of the monodromy matrix of Zakharov-Shabat problem are functions of the spectral parameter λ and functionals of the CNLS field $\psi(x,t)$. They satisfy equations [11]

$$a_t(\lambda) = 0\,, \quad ib_t(\lambda) = \lambda^2 b(\lambda) \tag{2.11}$$

which is crucial for the solution of CNLS.

The quantum inverse method associates with QNLS the quantized Zakharov-Shabat problem [10]. The matrix entries of the quantum monodromy matrix are the Fock space operators obtained by quantizing $a(\lambda)$ and $b(\lambda)$. The quantization in question is called normal ordering and, in standard notation, we have

$$A(\lambda) = :a(\lambda):\,, \quad B(\lambda) = :b(\lambda):\,. \tag{2.12}$$

Definition (2.12) corresponds to explicit expansions of $A(\lambda)$, $B(\lambda)$ in terms of the standard quantum field Ψ (we drop the subscript zero from now on)

$$A(\lambda) = \sum_{n=0}^{\infty} c^n \int d\xi^{2n} \theta(\xi_{2n} > \dots > \xi_1)\, e^{i\lambda(\xi_{2n} - + \dots - \xi_1)} \tag{2.13}$$

$$\Psi^\dagger(\xi_{2n}) \dots \Psi^\dagger(\xi_2)\Psi(\xi_{2n-1}) \dots \Psi(\xi_1)\,.$$

$$B(\lambda) = \sum_{n=0}^{\infty} c^n \int d\xi^{2n+1}\, \theta(\xi_{2n+1} > \dots > \xi_1)\, e^{i\lambda(-\xi_{2n+1} + - \dots - \xi_1)} \Psi^\dagger(\xi_{2n}) \dots \tag{2.14}$$

$$\Psi^\dagger(\xi_2)\Psi(\xi_{2n+1}) \dots \Psi(\xi_1)\,.$$

Notation $\theta(\xi_{2n} > \dots > \xi_1)$ in (2.13) means that the integration is over the region $\{\xi_{2n} > \dots > \xi_1\}$, and analogously in (2.14), the integration is over $\{\xi_{2n+1} > \dots > \xi_1\}$. At this point we forget about the quantum inverse method, and investigate the operators given by eqs. (2.13), (2.14) and their adjoints $A^\dagger(\lambda)$, $B^\dagger(\lambda)$.

3. OPERATORS $A(\lambda)$ AND BAE's.

To simplify notation, we take $i\lambda$ for the basic spectral parameter in eqs. (2.13), (2.14), denote it by λ and let λ be arbitrary complex. We want to translate the formal expansion (2.13) into the action of $A(\lambda)$ on the Fock space states $|f\rangle$. In what follows we denote by $|f\rangle$ arbitrary functions of $x_1,\dots,x_N$, invariant under permutations of variables. We denote by $|f\rangle(x_1 > \dots > x_N)$ the value of $|f\rangle$ on $x = (x_1 > \dots > x_N)$. The hat sign over a variable, like $\hat{x}_i$, means that x_i is deleted. The following proposition is proved in [8]. The proof is by straightforward computations using eq. (2.4) for the standard fields.

3.1 PROPOSITION *The action of operators* $A(\lambda)$, $A^\dagger(\lambda)$ *on* N*-particle states is given by*

$$A(\lambda)|f\rangle(x_1 > \dots > x_N) = \sum_{1\le i_1<\dots<i_n\le N} c^n e^{\lambda(x_{i_1}+\dots+x_{i_n})} \int_{x_{i_2}}^{x_{i_1}} d\eta_1 \dots \int_{-\infty}^{x_{i_n}} d\eta_n \, e^{-\lambda(\eta_1+\dots+\eta_n)} f(\eta_1,\dots,\eta_n, x_1,\dots,\hat{x}_{i_1},\dots,\hat{x}_{i_n},\dots,x_N). \tag{3.1}$$

$$A^\dagger(\lambda)|f\rangle(x_1 > \dots > x_N) = \sum_{1\le i_1<\dots<i_n\le N} c^n e^{\lambda(x_{i_1}+\dots+x_{i_n})} \int_{x_{i_1}}^{\infty} d\eta_1 \dots \int_{x_{i_n}}^{x_{i_{n-1}}} d\eta_n \, e^{-\lambda(\eta_1+\dots+\eta_n)} f(\eta_1,\dots,\eta_n, x_1,\dots,\hat{x}_{i_1},\dots,\hat{x}_{i_n},\dots,x_N). \tag{3.2}$$

A few comments on eqs. (3.1), (3.2). The summation is over all n-tuples $1 \le i_1 < \dots < i_n \le N$, where n runs from 0 to N. The term with $n = 0$ is the identity operator. For each N, we interpret $A(\lambda)$, $A^\dagger(\lambda)$ as integral operators on symmetric functions $f(x_1,\dots,x_N)$ defined on all $|f\rangle$ for which the integrals in (3.1), (3.2) converge. For instance, $A(\lambda)$ and $A^\dagger(\lambda)$ are certainly defined on continuous functions with compact support. For $\mathrm{Re}\,\lambda < 0$, the operators $A(\lambda)$ are defined on bounded functions. The same holds for $A^\dagger(\lambda)$ if $\mathrm{Re}\,\lambda > 0$.

We are interested in the action of integral operators $A(\lambda)$ on the BAE's. It is convenient to change notation slightly from §2. In this section we denote by

$|k_1,\ldots,k_N\rangle_c$ the symmetric function of N variables determined by

$$(3.3)\qquad |k_1,\ldots,k_N\rangle_c\,(x_1 > \ldots > x_N) = (N!)^{-\frac{1}{2}} \sum_w w\left\{\prod_{i<j} \frac{c+(k_i-k_j)}{(k_i-k_j)} \exp\langle k|x\rangle\right\}$$

The BAE $|k_1,\ldots,k_N\rangle_c$ is defined for any N-tuple of distinct complex numbers $k_1 \neq \ldots \neq k_N$. The BAE's of §2 correspond to purely imaginary $k_1,\ldots,k_N$.

Denote by T_N the space of bounded continuous symmetric functions $f(x_1,\ldots,x_N)$ which are continuously differentiable everywhere except maybe on the hyperplanes $\{x_i = x_j\,,\ i < j\}$, where the derivatives of f have at most jump discontinuities. The boundary conditions (2.10) make sense for f in T_N. By remarks above, $A(\lambda)$ is defined on f in T_N for $\mathrm{Re}\,\lambda < 0$, and $A^{\dagger}(\lambda)$ is defined on f in T_N for $\mathrm{Re}\,\lambda > 0$.

3.2. LEMMA *Let* f *be a function in* T_N *satisfying the boundary conditions* (2.10). *Then* $A(\lambda)f$ *satisfies* (2.10) *for all* λ *such that* $\mathrm{Re}\,\lambda < 0$.

Proof Denote $A(\lambda)f$ by g, and, in view of (3.1), using the self-explanatory notation, we set

$$(3.4)\qquad g = \sum_{1\leq i_1<\ldots<i_n\leq N} c^n g_{i_1\ldots i_n}\,.$$

For $\mathrm{Re}\,\lambda < 0$, the integrals in eq. (2.13) converge, and we can differentiate under the integral sign. We divide the set I of multi-indices $(i_1 < \ldots < i_n)$ into four disjoint groups: I_0 are the multi-indices that contain neither 1 nor 2, I_1 are those that contain 1 but not 2, I_2 are those that contain 2 but not 1, and $I_{1,2}$ contain both.

By symmetry, it suffices to check that g satisfies (2.10) on the wall $\{x_1 = x_2 = x > x_3 > \ldots > x_N\}$. An elementary calculation shows that for any $(i_1 < \ldots < i_N)$ from I_0

(3.5) $$(\partial/\partial x_1 - \partial/\partial x_2) g_{i_1 \ldots i_n}(x, x > x_3 > \ldots > x_N) = c\, g_{i_1 \ldots i_n}(x, x > x_3 > \ldots > x_N)$$

i.e. that $g_{i_1 \ldots i_n}$ satisfies (2.10). A multi-index in I_1 or I_2 is determined by $i_2 < \ldots < i_n$ with $i_2 > 2$. An elementary but long calculation, which we skip, gives

(3.6) $$(\partial/\partial x_1 - \partial/\partial x_2)(g_{1,i_2,\ldots,i_n} + g_{2,i_2,\ldots,i_n}) = 0$$

on the wall $\{x = x > x_3 > \ldots > x_N\}$. Another elementary calculation shows that for a multi-index $(1 < 2 < i_3 < \ldots < i_n)$ in $I_{1,2}$

(3.7) $$(\partial/\partial x_1 - \partial/\partial x_2) g_{1,2,i_3,\ldots,i_n} = g_{1,i_3,\ldots,i_n} + g_{2,i_3,\ldots,i_n}$$

on the wall above. Combining the preceding equations, we obtain that on the wall in question

(3.8) $$(\partial/\partial x_1 - \partial/\partial x_2) g = c\, \Sigma'\, c^n\, g_{i_1 i_2 \ldots i_n}$$

where the summation Σ' is over the multi-indices that don't belong to $I_{1,2}$. Since for any multi-index from $I_{1,2}$

(3.9) $$g_{1,2,i_3,\ldots,i_n}(x, x > x_3 > \ldots > x_N) = 0$$

we can replace Σ' in (3.8) by summation over all multi-indices, which proves the Lemma.

For any $k_1,\ldots,k_N$, we denote by $|k_1,\ldots,k_N\rangle_S$ the "symmetric plane wave" with wave numbers $k_1,\ldots,k_N$, i.e. the unique symmetric function of $x_1,\ldots,x_N$, such that

(3.10) $$|k_1,\ldots,k_N\rangle_S(x_1 \geq \ldots \geq x_N) = \exp[k_1 x_1 + \ldots + k_N x_N] .$$

Let E_N be the space of functions generated by all $|k_1,\ldots,k_N\rangle_S$. The N-particle BAE's in the sense of (3.3) belong to E_N. We want to determine, which functions in E_N satisfy (2.10).

3.3 LEMMA *A function* $f = \sum_k b(k)|k_1,\dots,k_N\rangle_S$ *in* E_N *satisfies* (2.10) *if and only if for any* $i = 1,\dots,N-1$ *and any* $(N-1)$-tuple $\ell = (\ell_1,\dots,\ell_{N-1})$

$$\sum [(k_i-k_{i+1})-c]b(k) = 0 \tag{3.11}$$

where the summation is over all k *such that* $k_1 = \ell_1,\dots, k_i + k_{i+1} = \ell_i$, $k_{i+2} = \ell_{i+1},\dots, k_N = \ell_{N-1}$.

Proof It suffices to prove the assertion for $i = 1$. The restriction of f to the wall $\{x_1 = x_2 = x > x_3 > \dots > x_N\}$ is given by

$$\sum_\ell \Big[\sum_{k_1+k_2=\ell_1} b(k)\Big] e^{\ell_1 x + k_3x_3+\dots+k_Nx_N}. \tag{3.12}$$

Denote by K the set of k such that $b(k) \neq 0$. Then the first summation in (3.12) is over the set L of exponents $\ell = (\ell_1,\dots,\ell_{N-1})$ such that $\ell = (k_1+k_2,k_3,\dots,k_N)$ for $k \in K$.

The inward derivative $(\partial/\partial x_1 - \partial/\partial x_2)f$ on the wall $\{x_1 = x_2 = x > x_3 > \dots > x_N\}$ is given by

$$\sum_{\ell\in L} \Big[\sum_{k_1+k_2=\ell_1} (k_1-k_2)b(k)\Big] e^{\ell_1 x_1 + k_3x_3+\dots+k_Nx_N}. \tag{3.13}$$

By linear independence of exponential functions with different exponents, f satisfies (2.10) on the wall in question if and only if for any $\ell \in L$, $\Sigma[(k_1-k_2-c]b(k) = 0$, where the summation is over the inverse image of ℓ in K, i.e. over k such that $k_1 + k_2 = \ell_1$, $k_3 = \ell_2,\dots,k_N = \ell_{N-1}$. This proves the Lemma. The following is immediate.

COROLLARY *Let* K *be a finite set and let* $f = \sum_{k\in K} b(k)|k_1,\dots,k_N\rangle_S$ *be a function in* E_N *satisfying* (2.10). *If for some index* $1 \le i \le N$ *and some* $(N-1)$-tuple $(z_1,\dots,z_{N-1})$ *there is only one exponent* $k \in K$ *such that* $k_1 = z_1,\dots,k_{i-1} = z_{i-1}$, $k_i + k_{i+1} = z_i$, $k_{i+2} = z_{i+1},\dots,k_N = z_{N-1}$, *then either* $k_i - k_{i+1} = c$ *or* $b(k) = 0$.

For any complex λ and any N-tuple k we define the set $L(k,\lambda)$ of N-tuples ℓ as follows. For any multi-index $1 \le i_1 < j_1 < \dots < i_n < j_n \le N$, we set

$$\ell = \left[k_1,\dots,k_{i_1-1}, \lambda, k_{i_1},\dots,k_{j_1-1} + k_{j_1} - \lambda, k_{j_1+1},\dots,k_N\right]. \tag{3.14}$$

In (3.14) we have λ on places $i_1,\dots,i_n$ and $k_{j_1-1} + k_{j_1} - \lambda,\dots,k_{j_n-1} + k_{j_n} - \lambda$ on places $j_1,\dots,j_n$. The set $L(k,\lambda)$ consists of the exponents (3.14) for $n > 0$. When $n = 0$, eq. (3.14) produces the exponent k.

3.4 LEMMA *Let* λ *and* $k = (k_1,\dots,k_N)$ *be such that* $A(\lambda)|k\rangle_S$ *is defined. Then* $A(\lambda)|k\rangle_S \in E_N$ *and*

$$A(\lambda)|k_1,\dots,k_N\rangle_S = \sum_{\ell} a(\ell)|\ell_1,\dots,\ell_N\rangle_S \tag{3.15}$$

where the summation is over the set $\{k\} \cup L(k,\lambda)$.

Proof By (3.1), we have

$$A(\lambda)|k_1,\dots,k_N\rangle_S\,(x_1 \ge \dots \ge x_N) = \tag{3.16}$$

$$\sum_{1\le i_1<\dots<i_n\le N} c^n \int_{x_{i_2}}^{x_{i_n}} d\eta_1 \cdots \int_{-\infty}^{x_{i_n}} d\eta_n\, e^{\lambda[(x_{i_1}-\eta_1)+\dots+(x_{i_n}-\eta_n)]}$$

$$|k_1,\dots,k_N\rangle_S(x_1,\dots,\eta_1,\dots,\eta_N,\dots,x_N).$$

The domain of integration in (3.16) partitions into subdomains of the form $x_{j_1} > \eta_1 > x_{j_1+1},\dots,x_{j_n} > \eta_n > x_{j_n+1}$ where $1 \le i_1 \le j_1 < \dots < i_n \le j_n \le N$. In each subdomain the integrand in (3.16) is an exponential function, and, since the subdomain is a product of intervals, the integral is a linear combination of exponentials. It is hard to keep track of the coefficients, but the exponents are much easier to calculate. They have the form (3.14) where each subdomain can produce several exponents. □

By Lemma 3.4, $A(\lambda)|k\rangle_S$ is a linear combination of $|k_1,\dots,k_N\rangle_S$ and $|\ell\rangle_S$ with the "bad exponents" $\ell \in L(k,\lambda)$. As our next step, we calculate the coefficient of the good exponent $|k\rangle$.

3.5 PROPOSITION *Let* $A(\lambda)|k\rangle_S$ *be defined and let* $\lambda \neq k_1,\dots,k_N$. *Then*

$$(3.17) \qquad A(\lambda)|k\rangle_S = \left[\sum_{1\le i_1<\dots<i_n\le N} c^n (k_{i_1}-\lambda)^{-1} \dots (k_{i_n}-\lambda)^{-1} \right] |k\rangle_S$$

$$+ \sum_{\ell\in L(k,\lambda)} a(\ell)|\ell\rangle_S .$$

Proof We use notation introduced in the proof of Lemma 3.4. Calculating the integral (3.16) over a subdomain corresponding to $i_1 \le j_1 < \dots < i_n \le j_n$ we obtain the exponent k only if $i_1 = j_1,\dots,i_n = j_n$. The integral in question is

$$(3.18) \qquad \exp[\Sigma\, k_i] \exp[\lambda(k_{i_1}x_{i_1}+\dots+k_{i_n}x_{i_n})]$$

$$\int_{x_{i_1+1}}^{x_{i_1}} d\eta_1\, e^{(k_{i_1}-\lambda)\eta_1} \dots \int_{x_{i_n+1}}^{x_{i_n}} d\eta_n\, e^{(k_{i_n}-\lambda)\eta_n} .$$

where $\Sigma\, k_i x_i$ is the sum over $i \neq i_1,\dots,i_n$. Although the integral (3.18) is a linear combination of 2^n exponentials, only one of them has exponent k , the one which is obtained by taking the upper limits in (3.18). Thus

$$(3.18) = (k_{i_1}-\lambda)^{-1} \dots (k_{i_n}-\lambda)^{-1} e^{k_1x_1+\dots+k_Nx_N} + \text{"bad exponentials"}$$

which proves the Proposition.

COROLLARY *Let* λ *and* $k_1 \neq \dots \neq k_N$, $\lambda \neq k_i$, *be such that* $A(\lambda)|wk\rangle_S$ is defined for any permutation w . Set $L = \bigcup_w L(wk,\lambda)$. Then

$$(3.19) \qquad A(\lambda)|k_1,\dots,k_N\rangle_c = \left[\prod_{i=1}^{N} \left[1 + \frac{c}{k_i-\lambda}\right]\right] |k_1,\dots,k_N\rangle_c + \sum_{\ell\in L} a(\ell)|\ell\rangle_S .$$

Proof Follows immediately from Lemma 3.4, Proposition 3.5 and the identity

$$\prod_{i=1}^{N} \left[1 + \frac{c}{k_i-\lambda}\right] = \sum_{1\le i_1<\dots<i_n\le N} c^n (k_{i_1}-\lambda)^{-1} \dots (k_{i_n}-\lambda)^{-1} .$$

3.6 THEOREM *For any* λ *and any* $k_1 \neq \dots \neq k_N$, *such that* $\lambda \neq k_1,\dots,k_N$ *and* $A(\lambda)|k_1,\dots,k_N\rangle_c$ is defined, we have

$$A(\lambda)|k_1,\dots,k_N\rangle_c = \left[\prod_{i=1}^{N} (1+c/(k_i-\lambda))\right] |k_1,\dots,k_N\rangle_c . \tag{3.20}$$

Proof We need to show that in (3.19), $a(\ell) = 0$ for all $\ell \in L$. Since $|k_1,\dots,k_N\rangle_c$ satisfies (2.10) and, by Lemma 3.2, $A(\lambda)|k_1,\dots,k_N\rangle_c$ satisfies (2.10), the function

$$g = \sum_{\ell \in L} a(\ell)|\ell\rangle_S \tag{3.21}$$

from the RHS of (3.19) satisfies the boundary condition (2.10). We will show that $a(\ell) = 0$ for $\ell \in L$, using Corollary from Lemma 3.3 and a double induction. Recall that the set L of bad exponents is parametrized by pairs $w = (i_1,\dots,i_N)$ and $1 \le p_1 < q_1 < \dots < p_n < q_n \le N$, $n > 0$.

Set $n = 1$, $p_1 = p$, $q_1 = q$. The corresponding exponents ℓ have λ as p-th coordinate and $k_i + k_j - \lambda$ as q-th coordinate. The remaining coordinates are equal to $k_1,\dots,k_N$. Let $q = p+1$. Set $i = p$ in Corollary 3.3 and consider the set M of $(N-1)$-tuples obtained by adding ℓ_p and ℓ_{p+1}. The exponent ℓ goes into

$$m = (k_{i_1},\dots,k_i + k_j,\dots,k_{i_N}) \tag{3.22}$$

and for $\ell \neq \ell'$ we have $m \neq m'$. Hence by Corollary 3.3, either $a(\ell) = 0$ or

$$2\lambda - k_i - k_j = c . \tag{3.23}$$

We can vary k and λ and, by construction, $a(\ell)$ are rational functions of k and λ. Let us fix c. A rational function vanishing on the complement of the hyperplane (3.23) is identically zero. Hence, $a(\ell) = 0$ for the exponents $\ell \in L$ with $n = 1$ and $q_1 - p_1 = 1$.

We fix $n = 1$ and go by induction on $d = q_1 - p_1 = q - p$. The next step is $d = 2$. The coordinates of an exponent ℓ on places p, $p+1$, $p+2$ are λ, k_t, $k_i + k_j - \lambda$. The reduction of Corollary 3.3 sends this triple into the pair, $\lambda + k_t$, $k_i + k_j - \lambda$. The only other exponent ℓ' with the same reduction must have k_t, λ, $k_i + k_j - \lambda$. Hence, $d = 1$ for ℓ', and we have already shown that $a(\ell') = 0$.

Hence, either $\lambda - k_t = c$ or $a(\ell) = 0$. Same argument as before shows that $a(\ell) = 0$ and concludes the second step of induction on d. Leaving $n = 1$ and increasing d, we show by induction on d that $a(\ell) = 0$ for all ℓ with $n = 1$.

Let now $n = 2$ and set $d = \min(q_1-p_1, q_2-p_2)$. For $d = 1$ the exponent ℓ contains a pair λ, $k_i + k_j - \lambda$, which under the reduction of Corollary 3.3 becomes $k_i + k_j$. The only other possible exponents with the same reduction must have k_i, k_j or k_j, k_i instead of λ, $k_i + k_j - \lambda$. These exponents have $n = 1$, and they are out by the previous steps of induction. By now, the reader should see the pattern of the double induction on n and d. This induction proves the theorem.

3.7 THEOREM *Under the assumptions of Theorem* 3.6

$$A^{\dagger}(\lambda)|k_1,\dots,k_N\rangle_c = \left[\prod_{i=1}^{N} (1+c/(\lambda-k_i))\right] |k_1,\dots,k_N\rangle_c \,. \tag{3.24}$$

Proof Follows from Theorem 3.6 using that $A^{\dagger}(-\bar{\lambda})$ is the formal adjoint of $A(\lambda)$.

4. OPERATORS $B(\lambda)$ AND THE BAE's.

We use the same conventions about the spectral parameter λ and the quantum numbers $k_1,\dots,k_N$ as in §3. The following proposition is proved in [8].

4.1 PROPOSITION *The action of* $B(\lambda)$, $B^{\dagger}(\lambda)$ on N-particle states is given by

$$B(\lambda)|f\rangle(x_1>\dots>x_{N-1}) = N^{\frac{1}{2}} \sum_{1\le i_1<\dots<i_N\le N-1} c^n e^{\lambda(x_{i_1}+\dots+x_{i_n})} \tag{4.1}$$

$$\int_{x_{i_1}}^{\infty} d\eta_1 \dots \int_{x_{i_n}}^{x_{i_{n-1}}} d\eta \int_{-\infty}^{x_{i_n}} d\eta_{n+1}\, e^{-\lambda(\eta_1+\dots+\eta_{n+1})}$$

$$f\left[\eta_1,\dots,\eta_{n+1},x_1,\dots,\hat{x}_{i_1},\dots,\hat{x}_{i_n},\dots,x_{N-1}\right] .$$

$$(4.2)\quad B^{\dagger}(\lambda)|f\rangle(x_1>\ldots>x_{N+1}) = (N+1)^{-\frac{1}{2}} \sum_{1\le i_1<\ldots<i_{N+1}\le N+1} c^n e^{\lambda(x_{i_1}+\ldots+x_{i_{n+1}})}$$

$$\int_{x_{i_2}}^{x_{i_1}} d\eta_1 \ldots \int_{x_{i_{n+1}}}^{x_{i_n}} d\eta_n\, e^{-\lambda(\eta_1+\ldots+\eta_n)}$$

$$f\left[\eta_1,\ldots,\eta_n,x_1,\ldots,\hat{x}_{i_1},\ldots,\hat{x}_{i_{n+1}},\ldots,x_{N+1}\right].$$

It is clear from eqs. (4.1), (4.2) that $B(\lambda)$ is never defined on $|k_1,\ldots,k_N\rangle_S$ while $B^{\dagger}(\lambda)$, on the contrary, is defined on all symmetric plane waves. Our goal is to calculate $B^{\dagger}(\lambda)|k_1,\ldots,k_N\rangle_c$.

4.2 LEMMA Operators $B^{\dagger}(\lambda)$ preserve the boundary conditions (2.10).

The proof is analogous to the proof of Lemma 3.2, and we leave it to the reader.

Fix λ and $k=(k_1,\ldots,k_N)$. With any multi-index $1\le i_1<j_1<\ldots<i_n<j_n<i_{n+1}\le N+1$ we associate the $(N+1)$-tuple

$$(4.3)\quad \ell = \left[k_1,\ldots,k_{i_1-1},\lambda,k_{i_1},\ldots,k_{j_1-1}+k_{j_1}-\lambda,k_{j_1+1},\ldots,\lambda,\ldots,k_{j_2-1}+k_{j_2}-\lambda,\ldots,\lambda,\ldots,k_N\right].$$

In eq. (4.3) we have λ on the places $i_1,\ldots,i_{n+1}$. On the places $j_1,\ldots,j_n$ we have $k_{j_1-1}+k_{j_1}-\lambda,\ldots,k_{j_n-1}+k_{j_n}-\lambda$. For $n=0$, we obtain, by (4.3), the $N+1$ exponents $|k_1,\ldots,k_i,\lambda,k_{i+1},\ldots,k_N)$, $i=0,1,\ldots,N$. We denote by $L(k,\lambda)$ the set of exponents (4.3) with $n>0$.

4.3 LEMMA *For any* λ *and* $k=(k_1,\ldots,k_N)$ *with* $k_1\ne\ldots\ne k_N$, *and* $\lambda\ne k_1,\ldots,k_N$, *we have*

$$(4.4)\quad B^{\dagger}(\lambda)|k\rangle_S = (N+1)^{-\frac{1}{2}} \sum_{n=0}^{N} \left[1+\frac{c}{k_1-\lambda}\right]\cdots\left[1+\frac{c}{k_n-\lambda}\right]\left[1+\frac{c}{\lambda-k_{n+1}}\right]\cdots$$

$$\left[1-\frac{c}{\lambda-k_N}\right]|k_1,\ldots,k_n,\lambda,k_{n+1},\ldots,k_N\rangle_S + \sum_{\ell\in L(k,\lambda)} b(\ell)|\ell\rangle_S .$$

Proof The argument is similar to that used in the proof of Proposition 3.5. Using eq. (4.2) we set

$$(4.5) \qquad B^{\dagger}(\lambda)|k\rangle_S = (N+1)^{-\frac{1}{2}} \sum_{1\le i_1<\dots\le i_{n+1}\le N+1} c^n g_{i_1\dots i_{n+1}} .$$

The function $g_{i_1\dots i_{n+1}}(x_1>\dots>x_{N+1})$ is given by an integral over the domain $x_{i_1} > \eta_1 > x_{i_2},\dots,x_{i_n} > \eta_n > x_{i_{n+1}}$, see eq. (4.2). The domain of integration is partitioned into subdomains, like in the proof of Lemma 3.4, where the integrand in each subdomain is an exponential function. For most subdomains, the integral is a linear combination of exponentials with exponents (4.3) with $n > 0$. We look at the subdomains which produce exponents (4.3) with $n = 0$ more in detail. For every $1 \le m \le n$, we consider the subdomain

$$(4.6) \qquad \begin{gathered} x_{i_1} > \eta_1 > x_{i_1+1},\dots,x_{i_{m-1}} > \eta_{m-1} > x_{i_{m-1}+1} ; \\ x_{i_{m+1}-1} > \eta_m > x_{i_{m+1}},\dots,x_{i_{n+1}-1} > \eta_n > x_{i_{n+1}} . \end{gathered}$$

The integrand in the domain (4.6) is the exponential

$$(4.7) \qquad \exp\left[(k_{i_1}-\lambda)\eta_1 + \dots + (k_{i_{m-1}}-\lambda)\eta_{m-1} + (k_{i_{m+1}}-\lambda)\eta_m + \dots + (k_{i_{n+1}}-\lambda)\eta_n\right] .$$

Integrating (4.7) over the subdomain (4.6), we obtain, by choosing the upper or lower limit of integration in each interval in (4.6), a linear combination of 2^n exponentials. The only good exponent is obtained by taking the upper limit for $\eta_1,\dots,\eta_{m-1}$ and the lower limit for $\eta_m,\dots,\eta_n$. The result is the exponential with exponent $(k_1,\dots,k_{i_m-1},\lambda,k_{i_m},\dots,k_N)$ and the coefficient

$$(4.8) \qquad c^n\left[k_{i_1}-\lambda\right]^{-1} \dots \left[k_{i_{m-1}}-\lambda\right]^{-1}\left[\lambda-k_{i_{m+1}-1}\right]^{-1} \dots \left[\lambda-k_{i_{n+1}-1}\right]^{-1} .$$

Collecting the terms (4.8), we obtain the Lemma.

4.4 THEOREM *For any BAE* $|k_1,\dots,k_N\rangle_c$ *and* $\lambda \ne k_1,\dots,k_N$, we have

$$(4.9) \qquad B^{\dagger}(\lambda)|k_1,\dots,k_N\rangle_c = |\lambda,k_1,\dots,k_N\rangle_c .$$

Proof We can rewrite eq. (3.3) as

$$(4.10) \qquad |k_1,\dots,k_N\rangle_c = (N!)^{-\frac{1}{2}} \sum_w b(wk)|wk\rangle_S$$

where

(4.11) $$b(k) = \prod_{i<j} (c+k_i-k_j)(k_i-k_j)^{-1} .$$

These formulas and eq. (4.4) imply, by simple combinatorics

(4.12) $$B^{\dagger}(\lambda)|k_1,\ldots,k_N\rangle_c = |\lambda,k_1,\ldots,k_N\rangle_c + \sum_{\ell\in L} a(\ell)|\ell\rangle_S$$

where $L = \bigcup_w L(wk,\lambda)$. By Lemma 4.2, the function g

(4.13) $$g = \sum_{\ell\in L} a(\ell)|\ell\rangle_S$$

satisfies the boundary condition (2.10). Arguing, like in the proof of Theorem 3.6, we prove, by a double induction, that $a(\ell) = 0$ for all exponents ℓ in (4.13). □

5. PROPERTIES OF OPERATORS $A(\lambda)$, $B(\lambda)$

We return to the notation of §2 and consider the repulsive case, $c > 0$, only. Recall that for real distinct wave numbers $k_1,\ldots,k_N$, the BAE's $|k_1,\ldots,k_N\rangle_c$ form a complete family of generalized N-particle eigenstates of the QNLS Hamiltonian (1.3). We restate the main results of §3, §4 in a more convenient form.

5.1 THEOREM 1. *For any complex* λ *with* $\operatorname{Im}\lambda > 0$, $A(\lambda)|k_1,\ldots,k_N\rangle_c$ *is well defined and*

(5.1) $$A(\lambda)|k_1,\ldots,k_N\rangle_c = \left[\prod_{i=1}^{N} (1 + ic/(\lambda-k_i))\right] |k_1,\ldots,k_N\rangle_c .$$

Analogously, if $\operatorname{Im}\lambda < 0$

(5.2) $$A^{\dagger}(\lambda)|k_1,\ldots,k_N\rangle_c = \left[\prod_{i=1}^{N} (1 - ic/(\lambda-k_i))\right] |k_1,\ldots,k_N\rangle_c .$$

2. For any real λ such that $\lambda \neq k_1,\ldots,k_N$, $B^{\dagger}(\lambda)|k_1,\ldots,k_N\rangle_c$ is well defined and

(5.3) $$B^{\dagger}(\lambda)|k_1,\ldots,k_N\rangle_c = |\lambda,k_1,\ldots,k_N\rangle_c .$$

Proof The BAE's are bounded functions. For λ satisfying the assumptions, integral operators $A(\lambda)$, $A^{\dagger}(\lambda)$ are well defined on bounded functions. Now eqs. (5.1) and (5.2) follow from theorems 3.6, 3.7, by a simple change of notation. Eq. (5.3) follows from theorem 4.4. □

COROLLARY *For any* $k_1 \neq \ldots \neq k_N$

(5.4) $$|k_1,\ldots,k_N\rangle_c = B^{\dagger}(k_1) \ldots B^{\dagger}(k_N)|0\rangle .$$

In view of eq. (5.4), $B^{\dagger}(k)$ are called "creation operators for the BAE's" in the physics literature. The name should be taken with a grain of salt, since the integral operations $B^{\dagger}(k)$ define neither operators nor operator densities on Fock space [8]. On the other hand, $A(\lambda)$, $A^{\dagger}(\mu)$ define reasonable operators on the Fock space [8] satisfying simple commutation relations.

5.2 THEOREM *For* $\mathrm{Im}\,\lambda > 0$, $\mathrm{Im}\,\mu > 0$, *the integral operations* $A(\lambda)$, $A^{\dagger}(\mu)$ *define bounded operators on* $\mathcal{H}_N$ *for every* N. *These operators are uniformly bounded on every halfplane* $\mathrm{Im}\,\lambda > \epsilon > 0$, $\mathrm{Im}\,\mu < -\epsilon < 0$. *For* λ, λ' *and* μ, μ' *as above, they satisfy the commutation relations*

(5.5) $$[A(\lambda),A(\lambda')] = [A^{\dagger}(\mu),A^{\dagger}(\mu')] = [A(\lambda),A^{\dagger}(\mu)] = 0$$

Proof Rewrite eq. (5.1) as

(5.6) $$A(\lambda)|k\rangle_c = a(\lambda,k)|k\rangle_c .$$

By (5.1)

(5.7) $$|a(\lambda,k)| \leq (1 + |c|\ |\mathrm{Im}\,\lambda|^{-1})^N .$$

Since the bound (5.7) does not depend on k and since the BAE's are complete in $\mathcal{H}_N$, $A^{\dagger}(\lambda)$ uniquely extend to bounded operators on $\mathcal{H}_N$. The rest is obvious. □

Eqs. (5.1), (5.2) and (5.3) yield certain relations between $A(\lambda)$, $A^{\dagger}(\lambda)$ on one hand and $B(\mu)$, $B^{\dagger}(\mu)$ on the other hand. For instance, let λ satisfy $\mathrm{Im}\,\lambda > 0$, and let μ be real. By (5.1) and (5.3), for any BAE $|k_1,\dots,k_N\rangle_c$ such that $\mu \neq k_1,\dots,k_N$

$$(5.8) \quad A(\lambda)B^\dagger(\mu)|k_1,\dots,k_N\rangle_c = \left[1 + \frac{ic}{\lambda-\mu}\right]\left[\prod_{i=1}^{N}\left[1 + ic/(\lambda-k_i)\right]\right]|\mu,k_1,\dots,k_N\rangle_c .$$

$$(5.9) \quad B^\dagger(\mu)A(\lambda)|k_1,\dots,k_N\rangle_c = \left[\prod_{i=1}^{N}\left[1 + ic/(\lambda-k_i)\right]\right]|\mu,k_1,\dots,k_N\rangle_c .$$

By (5.8), (5.9)

$$(5.10) \quad \left[A(\lambda)B^\dagger(\mu) - \left[1 + \frac{ic}{\lambda-\mu}\right]B^\dagger(\mu)A(\lambda)\right]|k_1,\dots,k_N\rangle_c = 0 .$$

In the physics literature (see, e.g. [4, 10]) eq. (5.10) is rewritten as

$$(5.11) \quad A(\lambda)B^\dagger(\mu) = \left[1 + \frac{ic}{\lambda-\mu}\right]B^\dagger(\mu)A(\lambda)$$

and called a relation between operators $A(\lambda)$ and $B^\dagger(\mu)$. The problem with eq. (5.11) is that both sides of it are not operators on Fock space. They are integral operations defined, in particular, on the BAE's. By (5.10), they are equal on BAE's, hence on the space E_N spanned by BAE's. Thus, (5.11) should be understood as equality of operators on E_N. Since E_N intersects trivially with $\mathcal{H}_N$, eq. (5.11) can not be considered a relation between operators on Fock space. In fact, it looks like a relation of operator densities, but neither $A(\lambda)$ nor $B^\dagger(\mu)$ are operator densities [8]. We refer the reader to [8] for a further discussion of relations between $A(\lambda)$, $A^\dagger(\lambda')$, $B(\mu)$, $B^\dagger(\mu')$.

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