11 Ends of Complete Minimal Surfaces

By Osserman's theorem, any complete minimal surface of finite total curvature is an immersion $X: M = S_k - \{p_1, \cdots, p_r\} \hookrightarrow \mathbf{R}^3$, where S_k is a closed Riemann surface of genus k. Consider conformal closed disks $D_i \subset S_k$ such that $p_i \in D_i$ and $p_j \notin D_i$ for $j \neq i$. Denote $D_i^* := D_i - \{p_i\}$. For any such D_i, the restriction $X: D_i^* \hookrightarrow \mathbf{R}^3$ is called a *representative of an end of X at p_i* or simply *an end.* When we say that some property holds at an end of X at p_i, for example embeddedness, we mean that there is a disk like domain D_i such that for any disk like domain $p_i \in U_i \subset D_i$, $X: U_i - \{p_i\}$ satisfies the property. Such a representative $X : U_i - \{p_i\} \to \mathbf{R}^3$ is called a *subend* of the end $X: D_i^* \hookrightarrow \mathbf{R}^3$.

Osserman's theorem says that the Gauss map g extends to p_i and the extended g is a meromorphic function. Since $N = \tau^{-1} \circ g$ we have a well defined normal vector $N(p_i)$ at p_i, which we call the *limit normal* at p_i. This also defines a *limit tangent plane* at the end E_i corresponding to p_i.

Intuitively, and we will prove it later (see Proposition 11.5), $E_i = X(D_i^*) \subset \mathbf{R}^3$ is an unbounded set. Moreover, since $M - \bigcup_{i=1}^r D_i^*$ is precompact, $X(M) - \bigcup_{i=1}^r E_i$ is bounded. Thus if X is an embedding, an end E_i is just a connected component of $X(M) - B$, where B is any sufficiently large ball in $\mathbf{R}^3$ centred at 0.

In this section, all ends considered are ends of some complete minimal surface of finite total curvature.

Now consider the Enneper-Weierstrass representation of the complete minimal surface $X : M \hookrightarrow \mathbf{R}^3$. By (6.20)

$$\Lambda^2 = \frac{1}{2}\left(|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2\right).$$

Now let $r : [0, 1) \to D_i^*$ be a regular curve such that $|r'(t)| = 1$ and $\lim_{t\to 1} r(t) = p_i$. By completeness,

$$\int_0^1 \Lambda(r(t))\, |r'(t)|\, dt = \infty.$$

This implies that $\Lambda(q) \to \infty$ as $q \to p$. Since ϕ_i's are meromorphic, one of them must have a pole at p. Hence let z be the local coordinate of D_i such that $z(p_i) = 0$, we must have

$$\Lambda^2 = \frac{1}{2}\left(|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2\right) \sim \frac{c}{|z|^{2m}}, \tag{11.46}$$

where $c > 0$ and $m \geq 1$ is an integer.

Definition 11.1 If $\Lambda^2 \sim c/|z|^{2m}$ at an end, we say that Λ has order m at that end.

Remark 11.2 Since Λ^2 is the pull back metric of $X: M \to \mathbf{R}^3$, we see that the order of Λ is invariant under an isometry in $\mathbf{R}^3$. Precisely, if A is an isometry of $\mathbf{R}^3$ then AX and X has the same pull back metric Λ^2. Thus the order of Λ at an end is invariant.

X being complete requires that the order of Λ at an end is at least one. In fact, we can prove that the order of Λ at an end is at least 2.

Lemma 11.3 *Let $X : M = S_k - \{p_1, \cdots, p_r\} \hookrightarrow \mathbf{R}^3$ be a complete minimal immersion with finite total curvature, and $(\omega_1, \omega_2, \omega_3)$ its Enneper-Weierstrass representation. Then at each p_j, at least one of ω_1, ω_2, ω_3 has a pole of order at least 2.*

Proof. Let (D_j, z) be a coordinate neighbourhood such that $z(p_j) = 0$ and on D_j^*, $(\omega_1, \omega_2, \omega_3) = (\phi_1, \phi_2, \phi_3)dz$.

We have shown that at least one of ϕ_1, ϕ_2, ϕ_3 has a pole at p_j. So $m \geq 1$. If $m = 1$, there are complex constants c_1, c_2 and c_3, not all zero, such that $f_i := \phi_i - c_i/z$ is holomorphic in D_j. Now

$$\Re(c_i \log z) = \Re \int (\phi_i - f_i)dz = X_i - \Re \int f_i \, dz, \qquad i = 1,\ 2,\ 3,$$

are well defined harmonic functions on D_i^*. Since

$$\Re(c_i \log z) = (\Re c_i) \log |z| - (\Im c_i) \arg z,$$

c_i must be real. But

$$0 = \phi_1^2 + \phi_2^2 + \phi_3^2 = f_1^2 + f_2^2 + f_3^2 + (c_1^2 + c_2^2 + c_3^2)/z^2 + 2(c_1 f_1 + c_2 f_2 + c_3 f_3)/z.$$

Comparing the terms of the same order, it must be that $c_i = 0$ for $i = 1,\ 2,\ 3$. But then $\phi_i = f_i$ is holomorphic and bounded in D_j, contradicting the fact that X is complete. $\square$

Now recall that by definition $X : S_k - \{p_1, \cdots, p_r\} \hookrightarrow \mathbf{R}^3$ is complete if and only if for any divergent curve γ the arc length of $X \circ \gamma$ is infinity. Thus either $X \circ \gamma$ goes to infinity in $\mathbf{R}^3$ or $X \circ \gamma$ stays in a compact set of $\mathbf{R}^3$ but has infinite arc length. To study these two cases, we introduce the concept of *properness*.

Definition 11.4 A mapping $X : M \to N$ between two topological spaces is *proper* if for any compact set $C \subset N$, $X^{-1}(C)$ is also compact.

Proposition 11.5 (Osserman) *If $X : M \to \mathbf{R}^3$ is a complete minimal surface of finite total curvature then X is proper.*

Proof. We know that $M = S_k - \{p_1, \cdots, p_r\}$ where S_k is a closed Riemann surface of genus k. Let $p \in \{p_1, \cdots, p_r\}$. Since the order of Λ is invariant under isometries of $\mathbf{R}^3$, after a rotation, we may assume that $g(p) = 0$. There is a coordinate disk $U \subset S_k$ at p such that $z(p) = 0$ and $|z| < 1$ on U. So we can write that $g(z) = z^n h(z)$, where $n > 0$ and $h(0) \neq 0$. On $U - \{p\}$, η must have a pole of order $m \geq 2$, hence we can write $\eta = f(z)dz$ where

$$f(z) = \sum_{i=-m}^{\infty} a_i z^i = \frac{1}{z^m} F(z),$$

where F is holomorphic and $a_{-m} = F(0) \neq 0$. We can write

$$f(z)g^2(z) = \sum_{2n-m}^{\infty} b_i z^i.$$

Recall that

$$\phi_1(z) = \frac{1}{2} f(z)\,(1 - g^2(z)), \quad \phi_2(z) = \frac{i}{2} f(z)\,(1 + g^2(z)).$$

Since on the loop $C := \{|z| = \rho < 1\}$,

$$\begin{aligned} 0 &= \Re \int_C \phi_1\, dz - i\,\Re \int_C \phi_2\, dz \\ &= \frac{1}{2}\Re \int_C (a_{-1} - b_{-1}) z^{-1} dz + i\frac{1}{2}\Im \int_C (a_{-1} + b_{-1}) z^{-1} dz \\ &= \pi i(a_{-1} + \overline{b_{-1}}) \quad \text{(by the residue theorem)}, \end{aligned}$$

we have

$$a_{-1} = -\overline{b_{-1}}. \tag{11.47}$$

Let $X(z) = (X^1, X^2, X^3)(z)$, then

$$\begin{aligned} (X^1 - iX^2)(z) &= \Re \int_{z_0}^{z} \phi_1(\zeta) d\zeta - i\Re \int_{z_0}^{z} \phi_2(\zeta) d\zeta + (X^1 - iX^2)(z_0) \\ &= \Re \int_{z_0}^{z} \frac{1}{2} f(\zeta)(1 - g^2(\zeta)) d\zeta + i\Im \int_{z_0}^{z} \frac{1}{2} f(\zeta)(1 + g^2(\zeta)) d\zeta + (X^1 - iX^2)(z_0) \\ &= \frac{1}{2} \int_{z_0}^{z} f(\zeta) d\zeta - \frac{1}{2} \overline{\int_{z_0}^{z} f(\zeta) g^2(\zeta) d\zeta} + (X_1 - iX_2)(z_0) \\ &= \frac{1}{2} \sum_{\substack{i=-m \\ i \neq -1}}^{\infty} \frac{a_i}{1+i} z^{i+1} - \frac{1}{2} \overline{\sum_{\substack{i=2n-m \\ i \neq -1}}^{\infty} \frac{b_i}{1+i} z^{i+1}} + \frac{1}{2}(a_{-1} - \overline{b_{-1}}) \log |z| \\ &= \frac{1}{2} \frac{a_{-m}}{1-m} z^{1-m} + \frac{1}{2}(a_{-1} - \overline{b_{-1}}) \log |z| + O(|z|^{2-m}). \end{aligned} \tag{11.48}$$

Since $a_{-m} \neq 0$ and $m \geq 2$, (11.48) shows that $|X|^2 \to \infty$ as $z \to 0$. Thus for any compact set $B \subset \mathbf{R}^3$, there are open disks $p_i \in D_i \subset S_k$ such that $X^{-1}(B) \subset S_k - \bigcup_{i=1}^{r} D_i$ is compact. $\square$

We want to know how to determine whether an end is embedded by looking at the Enneper-Weierstrass representation.

Lemma 11.6 *If the order of Λ at an end is $m = 2$, then there is an open conformal disk D such that $X : D - \{p\} \hookrightarrow \mathbf{R}^3$ is an embedding, where p is the puncture corresponding to the end.*

Proof. In the proof of Proposition 11.5, since $n \geq 1$ and $m = 2$ we see that $b_{-1} = 0$ and hence $a_{-1} = 0$ by (11.47). Now by the same calculation which led to (11.48),

$$(X^1 - iX^2)(z) = -\frac{1}{2}\frac{a_{-2}}{z} + O(|z|). \tag{11.49}$$

Obviously for some $0 < \rho < 1$ small enough, $X_1 - iX_2 : D - \{p\} := \{z \in U | 0 < |z| < \rho\} \to \mathbf{C}$ is one to one and $\lim_{|z|\to 0} |X_i - iX_2|(z) = \infty$. Hence $X\Big|_{D-\{p\}}$ is an embedding. □

When Λ has order 2 at an end, we can get more information about the behaviour of X at that end; in fact this end can be expressed as a minimal graph with a very nice growth property. To prove this, we first show:

Lemma 11.7 *Let $p \in \{p_1, \cdots, p_r\}$ and Λ have order 2 at p. Then there are $R > 0$ and $\rho > 0$ such that the mapping $X^1 - iX^2 : D - \{p\} \to \mathbf{C}$ defined in Lemma 11.6 is onto $\{\xi \in \mathbf{C} \,|\, |\xi| > R\}$.*

Proof. We have seen in Lemma 11.6 that for some $0 < \rho < 1$, $X^1 - iX^2 : D - \{p\} = \{0 < |z| \leq \rho\} \to \mathbf{C}$ is one to one and $\lim_{|z|\to 0} |X^1 - iX^2|(z) = \infty$. Let $R = \max_{|z|=\rho}\{|X^1 - iX^2|(z)\}$. Note that $\alpha := (X^1 - iX^2)(\{|z| = \rho\})$ is a Jordan curve in $\mathbf{C}$. If there is a $\xi \in \mathbf{C}$, $|\xi| > R$ and $\xi \notin (X^1 - iX^2)(D - \{p\})$, then there is a $0 < r < \rho$ such that $\min_{|z|=r}\{|X_1 - iX_2|(z)\} > |\xi|$. Let $\beta := (X^1 - iX^2)(\{|z| = r\})$, then β is a Jordan curve in $\mathbf{C}$ and $\alpha \cap \beta = \emptyset$. Let $\Omega := \mathbf{C} - \{0\} - \{\xi\}$, where α and β are not free homotopic to each other in Ω. But clearly $(X^1 - iX^2)(\{r < |z| < \rho\}) \subset \Omega$ and $\phi(\theta, t) := (X^1 - iX^2)[(r + t(\rho - r))e^{i\theta}]$, $0 \leq t \leq 1$, $0 \leq \theta \leq 2\pi$, is a homotopy from β to α in Ω. Thus we get a contradiction. This contradiction proves that $\xi \in (X^1 - iX^2)(D - \{p\})$. The lemma is proved. □

Theorem 11.8 *Let the notation be as in Lemmas 11.6 and 11.7. Then there is an $R > 0$ and an $\epsilon \in (0, 1)$ such that outside the solid cylinder $\{(x_1, x_2, x_3) \in \mathbf{R}^3 \,|\, x_1^2 + x_2^2 \leq R^2\}$, $X(0 < |z| < \epsilon)$ is a graph $(x_1, x_2, u(x_1, x_2))$ over the x_1x_2-plane. Furthermore, asymptotically,*

$$u(x_1, x_2) = \alpha \log r + \beta + r^{-2}(\gamma_1 x_1 + \gamma_2 x_2) + O(r^{-2}), \tag{11.50}$$

where $r = (x_1^2 + x_2^2)^{1/2}$, and α, β, γ_1 and γ_2 are real constants.

Proof. We have proved that there is an $\epsilon \in (0, 1)$ such that the mapping $X^1 - iX^2 : D^* := \{z \,|\, 0 < |z| < \epsilon\} \to \mathbf{C}$ is one to one and onto $|\xi| > R$ for some $R > 0$. Let $\Omega = \{|\xi| > R\}$. For any $(x_1, x_2) \in \Omega$ there is a unique $z \in D^*$ such that $x_1 = X^1(z)$ and $x_2 = X^2(z)$. Define $u(x_1, x_2) = X^3(z)$ on $(X^1 - iX^2)^{-1}(\Omega)$, then u is a well defined function. Now use the data written down in the proof of Proposition 11.5, recalling that $g(z) = z^n h(z)$, $f(z) = a_{-2}z^{-2} + \sum_{i=0}^{\infty} a_i z^i$, and so $\phi_3(z) = a_{-2}h(0)z^{n-2} + a_{-2}h'(0)z^{n-1} + \sum_{i=n}^{\infty} b_i z^i$.

We consider the two cases of $n = 1$ or $n > 1$. If $n = 1$, let $C := \{|z| = \epsilon_1\}$ for some $0 < \epsilon_1 < \epsilon$. Since

$$0 = \Re \int_C \phi_3(z)dz = \Re(a_{-2}h(0)2\pi i),$$

we see that $\alpha := -a_{-2}h(0) \neq 0$ is real. Thus

$$\begin{aligned} u(x_1, x_2) &= X^3(z) = \Re \int_{z_0}^{z} \phi_3(\zeta)d\zeta + X^3(z_0) \\ &= -\alpha \log|z| + \Re(a_{-2}h'(0)z) + O(|z|^2) + X^3(z_0). \end{aligned}$$

By (11.49),

$$r^2 = |x_1 - ix_2|^2 = \frac{|a_{-2}|^2}{4|z|^2} + O(1) = \frac{1}{|z|^2}\left(\frac{|a_{-2}|^2}{4} + O(|z|^2)\right),$$

$$2\log r = -2\log|z| + \log\left(\frac{|a_{-2}|^2}{4} + O(|z|^2)\right) = -2\log|z| + 2\log\frac{|a_{-2}|}{2} + O(|z|^2).$$

Also by (11.49),

$$z = \frac{-a_{-2}}{2(x_1 - ix_2)} + O(r^{-2}) = \frac{-a_{-2}(x_1 + ix_2)}{2r^2} + O(r^{-2}).$$

Thus there are real constants γ_1 and γ_2 such that

$$\Re(a_{-2}h'(0)z) = \frac{\gamma_1 x_1 + \gamma_2 x_2}{r^2}.$$

Setting $\beta = -\alpha \log \frac{|a_{-2}|}{2} + X^3(z_0)$, we have

$$u(x_1, x_2) = \alpha \log r + \beta + r^{-2}(\gamma_1 x_1 + \gamma_2 x_2) + O(r^{-2}).$$

If $n > 1$ then ϕ_3 is bounded in D^*, hence $\alpha = 0$. In this case, the end approximates a plane. $\square$

We have shown that if Λ has order 2 at an end, then that end is embedded and is a minimal graph. Next we will show that if an end is embedded, then Λ must have order 2 at that end.

An outline of the proof is as follows: If $m > 2$ and $g(0) = 0$ then

$$(X^1 - iX^2)(z) = \frac{c}{z^k} + O(|z|^{1-k})$$

with $k > 1$. This shows that $(X^1 - iX^2)$ is not one to one, and $\lim_{|z|\to 0} |X_1 - iX_2|(z) = \infty$. But it is possible that the surface $X = (X^1, X^2, X^3)$ is embedded. However, intuitively we know that X is a graph over $\mathbf{C} - B$, where B is a large disk in $\mathbf{C}$, since our surface has a limit tangent plane corresponding to the puncture. It follows that X is embedded is equivalent to $X^1 - iX^2$ being one to one. The next lemma gives a rigorous proof of this fact.

Lemma 11.9 *Let D and p be as in Proposition 11.5. If $X : D - \{p\}$ is an embedding then there is an $R > 0$ such that X is a graph over $\mathbf{R}^2 - B_R$, where $B_R := \{x \in \mathbf{R}^2 \mid |x| \leq R\}$. In particular, Λ has order 2 at p.*

Proof. We assume that the limit normal to X at p is $(0, 0, -1)$. Let $P(x_1, x_2, x_3) = (x_1, x_2)$ be the perpendicular projection. Let $C_r := \{(x_1, x_2, x_3) \in \mathbf{R}^3 \mid x_1^2 + x_2^2 = r^2\}$, $V_r := \{x_1, x_2, x_3) \in \mathbf{R}^3 \mid x_1^2 + x_2^2 > r^2\}$.

We will prove that there is an $R > 0$ such that $P : X(D - \{p\}) \cap V_R \to \mathbf{R}^2 - B_R$ is one to one and onto $\mathbf{R}^2 - B_R$. Hence X is a graph over $\mathbf{R}^2 - B_R$. Moreover, $\partial[X^{-1}(V_R)]$ is a homotopically non-trivial Jordan curve $J_R \subset D - \{p\}$, hence $X^{-1}(V_R)$ is conformally a punctured disk.

Since the limit normal of X at p is $(0, 0, -1)$, there is an $0 < \rho < 1$ such that $N_3(z) < -1/2$ for any $0 < |z| \leq \rho$. Let $D_\rho^* := \{z \mid 0 < |z| < \rho\}$. Since X is continuous, there is an $R > 0$ such that $|X^1 - iX^2|^2(z) < R^2$ for $|z| = \rho$. For any $r > R$, consider the set $X^{-1}(C_r) \subset D_\rho^*$. Since $N_3(z) < -1/2$ for any $0 < |z| < \rho$, X is transverse to C_r. (i.e., $X(D_\rho^*)$ and C_r have different tangent planes at common points.) This implies that $X^{-1}(C_r)$ is a family of one-dimensional submanifolds in D_ρ^*. From the expression for $X^1 - iX^2$ we know that $|X^1 - iX^2|(z) \to \infty$ when $|z| \to 0$, hence any component J_r of $X^{-1}(C_r)$ is a compact one-dimensional submanifold, i.e., it is a Jordan curve in D_ρ^*. If J_r is homotopically trivial, then it bounds a disk like domain $\Omega \subset D_\rho^*$. We will prove that $|X^1 - iX^2|^2(z) \equiv r^2$ on Ω. In fact, let $z \in \Omega$ be such that $|X^1 - iX^2|^2(z)$ achieves a maximum or minimum other than r^2 on $\overline{\Omega}$. Then z is an interior point of Ω and $D|X^1 - iX^2|^2(z) = (0, 0)$. This says that

$$(X^1, X^2)_x \bullet (X^1, X^2) = 0, \quad (X^1, X^2)_y \bullet (X^1, X^2) = 0. \tag{11.51}$$

Since $(X^1, X^2)(z) \neq (0, 0)$, (11.51) implies that $(X^1, X^2)_x$ and $(X^1, X^2)_y$ are linearly dependent. This then implies that $N_3(z) = 0$, contradicting $N_3(z) < -1/2$. But if $|X^1 - iX^2|^2 \equiv r^2$ on Ω, X maps Ω to C_r, another contradiction to the fact that $N_3(z) < -1/2$ in D_ρ^*. These contradictions prove that J_r is homotopically non-trivial. Now if $X^{-1}(C_r)$ has more than one component, say J_r^1 and J_r^2. The above argument shows that they are both homotopically non-trivial. Thus they are in the same $\mathbf{Z}_2$ homotopy class, and bound a compact doubly-connected domain $\Omega \subset D_\rho^*$. By the same argument we can prove that $X(\Omega) \subset C_r$, which is impossible. Thus we have shown that $J_r := (|X^1 - iX^2|^2)^{-1}(r^2) = X^{-1}(C_r)$ is a homotopically non-trivial Jordan curve in D_ρ^*.

Now $X : D_\rho^* \to \mathbf{R}^3$ is an embedding, so $\alpha := X(J_r)$ is a Jordan curve on C_r. Let $\beta : S^1 \to D_\rho^*$ be a parametrisation of J_r. If $\beta(t_i) = z_i \in J_r$ for $i = 1, 2$ where $z_1 \neq z_2$ and $(X^1, X^2)(z_1) = (X^1, X^2)(z_2)$, then there is a $t \in S^1$ such that $\alpha'(t) = C(0, 0, 1)$ for some non-zero constant C. Since $\alpha'(t)$ is a tangent vector of X, we must have $N_3(\beta(t)) = 0$, a contradiction to $N_3(z) < -1/2$. This shows that $P : X(J_r) \to \partial B_r$ is one to one and onto for any $r > R$; hence (X^1, X^2) is one to one and onto $\mathbf{R}^2 - B_R$. □

Remark 11.10 The fact that X is an embedding is used only when claiming that $\alpha = X(J_r)$ is a Jordan curve. Hence it is true that $(|X^1 - iX^2|^2)^{-1}(r^2) = X^{-1}(C_r)$ is a homotopically non-trivial Jordan curve when X is only an immersion. In general, $P: X(J_r) \to \partial B_r$ is an m to one projection except for a finite number of points in ∂B_r. The number m is the I_i in Theorem 12.1.

An immediate application of Theorem 11.8 and Lemma 11.9 is:

Corollary 11.11 *If $X: S_k - \{p_1, \ldots, p_n\} \hookrightarrow \mathbf{R}^3$ is a complete minimal embedding, then the limit normal must be parallel.*

Definition 11.12 An embedded end of a complete immersed minimal surface in $\mathbf{R}^3$ of finite total curvature is a *flat* (or *planar end*) if $\alpha = 0$ in (11.50), and is a *catenoid end* otherwise.

Remark 11.13 We have proved that X is embedded at an end E if and only if Λ has order 2. Let p be the puncture corresponding to E. From the proof of Theorem 11.8, we know that E is flat if and only if p is a branch point of the Gauss map g.

Finally, we give a description of the image of a flat end at the limit height.

Proposition 11.14 *Let $E = X(D - \{p\})$ be an embedded flat end and g have branch order $k > 0$. Let β be as in Theorem 11.8, and B be a large ball centre at $(0, 0, \beta)$. Then $(E - B) \cap \{(x, y, z) \in \mathbf{R}^3 \mid z = \beta\}$ has $2k$ components.*

Proof. Without loss of generality we may assume that $g(p) = 0$ and $g(z) = z^{k+1}$. Now $\eta = z^{-2}h(z)dz$, $h(0) \neq 0$, so

$$X_3(z) = \beta + \Re\left(\frac{1}{k}h(0)z^k\right) + o(|z|^k).$$

Thus $X_3^{-1}(\beta) \cap (D - \{p\})$ consists of k curves intersecting at $z = 0$. This is equivalent to $(E - B) \cap \{(x, y, z) \mid z = \beta\}$ consisting of $2k$ components. $\square$