FLOW OF HYPERSURFACES BY CURVATURE FUNCTIONS

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This seminar concerns a class of flow equations for immersed hypersurfaces, modelled on the well-known mean curvature flow. The flows in this class share much of the qualitative behaviour of the mean curvature flow, but are in general fully nonlinear; this complicates some parts of their analysis. Other calculations are clarified by the general setting. I will present some results on the behaviour of convex hypersurfaces under these flows, which extend work on specific flows by Huisken (Hu1), Tso (T1) and Chow (C1-2). Also new is a Harnack inequality for solutions of very general flows; this generalises results of Hamilton (Ha1) and Chow (C3). Flows of this kind have some applications in geometry; for such purposes the mean curvature flow is not always the best candidate - I will describe an example which applies to manifolds of non-negative sectional curvature.

In these flows, the evolution of a hypersurface is prescribed in terms of the speed of motion perpendicular to the surface at each point: Suppose the hypersurface is given at time t by the image of an immersion φ_t from a compact n-dimensional manifold M into a Riemannian $(n+1)$-dimensional manifold N. Then the flow is determined by an equation of the form:

$$\frac{\partial}{\partial t}\varphi_t(x) = F(x,t)\,\nu(x,t) \tag{1}$$

which is required to hold at each point x in M and each positive time t; $\nu(x,t)$ is a unit normal to the hypersurface $\varphi_t(M)$ at the point $\varphi_t(x)$, and F is a function which prescribes the normal speed.

In general, some conditions must be imposed on the function F:

Condition 1:

$F(x,t)$ is a symmetric function of the principal curvatures $\lambda_1, \lambda_2, \ldots \lambda_n$ of $\varphi_t(M)$ at $\varphi_t(x)$. This ensures that the speed is given by a second order partial differential operator acting on the surface, and depends only on the local geometry of the hypersurface.

Condition 2:

F is an increasing function of the principal curvatures:

(2) $$\frac{\partial F}{\partial \lambda_i} > 0, \;\; i = 1,2,\ldots,n.$$

This condition ensures that the flow is parabolic.

Some further conditions are required for specific purposes; these will be discussed as they arise.

For compact initial surfaces, the mean curvature flow will always produce a singularity in finite time; it is not known in general what kinds of singularities can appear (see the papers of Huisken and Ecker in this volume). The special case of locally convex initial surfaces (that is, surfaces with second fundamental form definite everywhere) is more completely understood (see example 1). For present purposes I will consider only this case; to begin with I will also restrict attention to the case where N is the $(n+1)$-dimensional Euclidean space, and $n > 1$. To fix definitions, convex surfaces have inward-pointing normal ν, and positive definite second fundamental form.

Several examples of flows of the form (1) have been considered previously:

Example 1: Mean Curvature Flow ($F = H = \sum_{i=1}^{n} \lambda_i$).

The main result for the convex case is that of Huisken (Hu1) which shows that any convex initial hypersurface is contracted to a point in finite time, and that the hypersurfaces become spherical as the final time is approached - in other words, a unique smooth solution to the flow (1) exists for some finite time T; the hypersurfaces converge to a single point as this time is approached, and a suitable rescaling of the hypersurfaces gives convergence to a sphere at the final time.

Example 2: Gauss Curvature Flows ($F = K^{\alpha} = \left(\prod_{i=1}^{n} \lambda_i\right)^{\alpha}$, $\alpha > 0$).

Tso (T1) initially considered the case $\alpha = 1$, and showed that any strictly convex initial surface contracts to a point in finite time. Chow (C1) extended this result to other α, and showed in the particular case $\alpha = \frac{1}{n}$ that the surfaces are asymptotically spherical, as for the mean curvature flow. It remains an open question whether this is true for other powers α.

Example 3: Flow by the Square Root of Scalar Curvature

($F = R^{1/2} = \left(\sum_{i \neq j} \lambda_i \lambda_j\right)^{\frac{1}{2}}$).

Chow (C2) considered this flow, and was able to show that convex surfaces contract to a point and become spherical, provided they satisfy a certain curvature pinching condition.

Example 4: Expansion Flows.

These are flows with rather different behaviour from the other examples discussed above. The speeds F satisfy conditions 1 and 2, and are also required to be homogeneous of degree -1 in the principal curvatures. Such flows have been considered by Urbas (U1) and Huisken (Hu3), who showed (under some simple additional assumptions on the speed F) that arbitrary convex initial surfaces are expanded to infinite radius in infinite time, and become spherical in the process. This conclusion holds also for star-shaped initial surfaces, as has been shown by Gerhardt (G1) and Urbas (U2).

I will describe some results which include examples 1- 3, in a generality similar to the treatment of example 4. For these purposes there are a variety of conditions which will be used to control the speed F:

Condition 3: Homogeneity in the principal curvatures:

(3) $$F(\kappa\lambda_1,\kappa\lambda_2,\ldots,\kappa\lambda_n) = \kappa F(\lambda_1,\lambda_2,\ldots,\lambda_n).$$

Condition 4: Concavity in the principal curvatures:

(4) $$(\mathrm{Hess}(F))(\xi,\xi) \leq 0, \text{ for all vectors } \xi.$$

Note that (3) and (4) are satisfied in examples 1-3; other examples which satisfy these conditions are the symmetric means ($F = H_r = \left(\sum_{i=1}^{n}\lambda_i^r\right)^{1/r}$) for $r \leq 1$, and the m^{th} root of the m^{th} symmetric product ($F = H_{(m)}^{1/m} = \left(\sum_{i_1<i_2<\ldots<i_m}\lambda_{i_1}\lambda_{i_2}\ldots\lambda_{i_m}\right)^{1/m}$) for $m = 1,\ldots,n$.

Condition 5: $\lambda_i = 0 \Rightarrow F = 0,\ i = 1,\ldots,n.$

This is satisfied for example by H_r for $r < 0$ and $K^{1/n}$, but not $H_{(m)}^{1/m}$ for $m < n$.

Condition 6:

$F^*(\lambda_1,\ldots,\lambda_n) = \left(F(\lambda_1^{-1},\ldots,\lambda_n^{-1})\right)^{-1}$ is a concave function of the principal curvatures.

This is satisfied by H_r for $r \geq -1$, and $H_{(m)}^{1/m}$ for $m = 1,\ldots,n$.

Condition 7:

The derivatives of F with respect to the principal curvatures are bounded:

(5) $$\left|\frac{\partial F}{\partial \lambda_i}(\lambda_1,\ldots,\lambda_n)\right| < C_0, \text{ for } i = 1,\ldots,n \text{ and } \lambda_1,\ldots,\lambda_n > 0.$$

This is not satisfied by $K^{1/n}$, for example.

Now I will discuss some general results modelled on the examples above. The proofs of the theorems will be supplied in a forthcoming paper by the author. The partial result of example 3 holds under relatively few assumptions:

Theorem 1:

Suppose F satisfies the conditions 3.1 - 3.2 , and 6.1 - 6.2. Suppose the initial hypersurface M_0 is strictly convex and satisfies the pinching condition

(6) $$\sup_{M_0}\left(\frac{H}{F}\right) < \inf_{\partial\mathfrak{S}}\left(\frac{H}{F}\right)$$

where $\Im = \{(\lambda_1,\ldots,\lambda_n) : \lambda_i > 0,\ i = 1,\ldots,n\}$. *Then there exists a unique smooth solution* $\{M_t\}$ *to the flow (1) for t in some maximal time interval* $[0,T)$; *the surfaces* M_t *converge to a single point as t approaches T; rescaling these surfaces to maintain constant surface area gives convergence to a round sphere as the final time is approached.*

In certain cases this immediately implies a stronger result:

Corollary 2:

If in addition condition 5 holds or $n = 2$, *then equation (6) is trivial, and arbitrary strictly convex initial surfaces contract to points and become spherical in the sense of theorem 1.*

The techniques employed here are similar to those devised by Huisken in (Hu1): the parabolic maximum principle gives some control over the principal curvatures of the surfaces M_t; a more difficult integral estimate is used to show that the principal curvatures must approach each other near any singularity; careful use of curvature gradient estimates gives the final control over the singularity, showing convergence to a sphere for the rescaled surfaces. The regularity theory for fully nonlinear equations developed by Krylov in (K1) is very important here.

Partial results can be obtained in certain other cases - analogous to those in example 2; these use techniques similar to those of Tso in (T1) to control the curvature in terms of the enclosed volume of the surface; control of the curvature gives control over higher derivatives of the surface, again using the regularity of Krylov (K1).

Theorem 3:

Suppose either:

(i). F satisfies conditions 1 - 4 and 6 ,

or *(ii). $F = G^{\alpha}$, where $\alpha > 0$ and G satisfies conditions 1 - 5 .*

Then for any strictly convex initial surface, the flow (1) has a unique smooth solution for a finite time interval, and the surfaces are contracted to a point at the end of the time interval.

As is evident from the results of the previous paragraph, many of the flows of form (1) have similar qualitative behaviour. It is often convenient to perform calculations in this general setting, since the details of particular flows are avoided. An important example is the proof of a parabolic Harnack inequality for these flows. In the context of flows of type (1), the Harnack inequalty was first proved by Hamilton (Ha2), for the mean curvature flow. Similar inequalities were proved earlier for the heat equation by Li and Yau (LY), and for the Ricci flow on a surface by Hamilton (Ha1). The result gives control over the mean curvatures at different times and places under the flow - for points x_1 and x_2 in M and positive times $t_1 < t_2$, the inequality

$$(7) \qquad \frac{H(x_2,t_2)}{H(x_1,t_1)} \geq \left(\frac{t_1}{t_2}\right)^{\frac{1}{2}} \exp\left(-\frac{d_{t_1}^2(x_1,x_2)}{4(t_2-t_1)}\right)$$

is satisfied. Here $d_{t_1}(x_1,x_2)$ is the distance between the points x_1 and x_2 in M at time t_1. This estimate has applications in controlling the types of singularities which can occur under the flow.

Chow (C3) has since proved similar Harnack inequalities for the Gauss curvature flows of example 2.

In both of these cases the Harnack inequality is deduced from a differential inequality: In the case of the mean curvature flow,

(8) $$\frac{\partial H}{\partial t} \geq h^{-1}(\nabla H, \nabla H) - \frac{H}{2t}$$

holds for any convex solution (here h^{-1} is the inverse of the second fundamental form).

A differential inequality of this form can be proved for a wide class of flows - In a forthcoming paper by the author, the following result is proved:

Theorem 4:

Suppose $F = G^{\alpha}$, where $\alpha > 0$ and G satisfies conditions 1 - 3 and 6. Then for any convex solution of (1), the differential inequality

(9) $$\frac{\partial F}{\partial t} \geq h^{-1}(\nabla F, \nabla F) - \frac{\alpha F}{(\alpha + 1)t}$$

holds as long as the solution exists.

This differential inequality can be integrated to give a Harnack inequality of the form (7). The general treatment leads to great simplification of the calculation; this will be discussed elsewhere.

I will now describe the application of the flows in a slightly different setting - the hypersurfaces will now be allowed to reside in any Riemannian $(n+1)$-manifold N with non-negative sectional curvatures, instead of just in Euclidean space. Huisken (Hu2) has considered the mean curvature flow in such situations; he achieves essentially the same results as before - convex surfaces contract to points and become spherical - but requires more than just strict convexity: the surfaces must satisfy a further convexity condition which depends on the gradient of the Riemann curvature of the background space N.

By considering flows other than the mean curvature flow, this result can be improved - the extra convexity condition can be removed:

Theorem 5:

Suppose F satisfies conditions 1 - 5 and 7. Then any strictly convex initial hypersurface in a non-negatively curved space contracts to a point in finite time under the flow (1), and the hypersurfaces become spherical as they approach the final time.

This result has some applications to the geometry of non-negatively curved spaces; for example, it leads to a new proof of the 1/4 - pinching sphere theorem, following methods of Eschenburg and Gromov (E1). This and other applications will be discussed elsewhere.

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