

MAPS WHICH HAVE FIXED POINTS FOR ANY BANACH SPACE

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Abstract. We examine a class of maps which have fixed points for all Banach spaces. Included in the class are affine mappings and Banach contractions. The emphasis is on examples, with an eye to applications.

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The aim of this note is to expose the broad class of *almost convex maps*, which have the following useful property.

For any Banach space X and any nonempty weak-compact convex subset C of X , we have that any map $T : C \rightarrow X$ belonging to the class will have a fixed point provided that it ‘tries to have a one’, in the sense that $\inf\{\|x - Tx\| : x \in C\} = 0$.

The theory of such mappings was developed by Enrique Llorens-Fuster, Jesus Garcia-Falset and the author, [2] and extended to the case of multifunctions (set-valued maps $T : C \rightarrow \mathcal{BC}(X)$, where $\mathcal{BC}(X)$ is the class of all nonempty closed bounded subsets of X) by Llorens-Fuster in [4].

Unlike the metric fixed point theory for nonexpansive maps (see, the recent survey by the author, [5], or the book by Goebel and Kirk, [3] for details), fixed point results for almost convex maps make no demand on the underlying space. Further, the class of almost convex maps is invariant under the change to an equivalent norm, as is the necessary condition $\inf\{\|x - Tx\| : x \in C\} = 0$. The results should therefore be widely applicable to areas such as differential equations, dynamical systems theory and control systems. The brief survey which follows is written with such applications in mind. The emphasis is on examples. A stripped down proof of the main existence result is included for the sake of completeness, other proofs are suppressed and the interested reader is directed to the references mentioned above for details.

Classical classes of maps which have fixed points in all Banach spaces include:

- *Banach (strict) contractions*; that is, maps $T : C \rightarrow C$ with $\|Tx - Ty\| \leq K\|x - y\|$ for all $x, y \in C$ and some constant K with $0 \leq K < 1$ and
- *Norm continuous affine maps*; that is, maps $T : C \rightarrow X$ satisfying $T(\lambda x + (1 - \lambda)y) = \lambda Tx + (1 - \lambda)Ty$, for all $x, y \in C$ and all λ between 0 and 1.

The first class have fixed points courtesy of the Banach contraction mapping principle, and the second because they are also weak continuous and so the Schauder fixed point theorem applies provided C is a nonempty weakly compact convex subset of X .

One reason such classes of maps are important lies in the observation that often the type of space is determined by the application, but there is some freedom in constructing the map to be used. For example, this is the case in the Picard existence theorem for ordinary differential equations and in various generalized Newton-Kantorovich methods where the space is dictated by the type of solution sought, though there is choice in the mapping to be iterated.

The case of affine maps provided motivation for the definition of almost convex maps given below. However, as we shall see, it turns out that they also generalize Banach contractions, at least in the Banach space environment.

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Throughout C will always be a nonempty closed and convex subset of the Banach space X .

For $T : C \rightarrow X$ define the associated nonlinear displacement functional to be

$$J_T(x) := \|x - Tx\|.$$

Obviously; $x_0 \in \text{Fix}(T)$, the set of fixed points of T if and only if $J_T(x_0) = 0$. A necessary condition for this is clearly that $\inf J_T(C) = \inf \{\|x - Tx\| : x \in C\} = 0$; that is, T admits an approximate fixed point sequence $(x_n) \subset C$ with $\|x_n - Tx_n\| \rightarrow 0$.

We will see that for the class of continuous almost convex maps this is also a sufficient condition.

Let $\alpha : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a function with $\alpha(0) = 0$ such that $\alpha(t_n) \rightarrow 0$ if and only if $t_n \rightarrow 0$. We say $T : C \rightarrow X$ is α -almost convex if

$$J_T(\lambda x + (1 - \lambda)y) \leq \alpha(\max\{J_T(x), J_T(y)\}),$$

for all $x, y \in C$ and $\lambda \in [0, 1]$.

In this context we will use r to denote the function $t \mapsto rt$. Thus, for $r > 0$, T is r -almost convex means that T is α -almost convex where $\alpha(t) = rt$. In particular T is 1-almost convex if $J_T(\lambda x + (1 - \lambda)y) \leq \max\{J_T(x), J_T(y)\}$. Clearly every affine map is 1-almost convex.

We will refer to T as an *almost convex mapping* if it is α -almost convex for some admissible function α .

Theorem 1. *Let C be a weak-compact convex set (in particular, a closed bounded convex subset of a reflexive space) and let $T : C \rightarrow X$ be an almost convex and norm-continuous map, then $\text{Fix}(T)$ is a nonempty closed convex subset of C if and only if $\inf J_T(C) = 0$.*

Proof. Let α be an admissible function for which T is α -almost convex. Since $\inf J_T(C) = 0$, we can find an approximate fixed point sequence (x_n) for T ; that is, $J_T(x_n) := \|x_n - Tx_n\| \rightarrow 0$. By passing to a subsequence if necessary we can suppose without loss of generality that $x_n \xrightarrow{w} x_0 \in C$. We will show that $J_T(x_0) = 0$.

Since $\alpha(J_T(x_n)) \rightarrow 0$, we can exploit the properties of α to extract a further subsequence so that

$$J_T(x_n) \geq \alpha(J_T(x_{n+1})).$$

By Mazur's theorem there exist convex combinations

$$\sum_{n=k}^{\infty} \lambda_n^{(k)} x_n \xrightarrow{\|\cdot\|} x_0, \quad \text{as } k \rightarrow \infty.$$

Since T and hence J_T is norm-continuous, we therefore have

$$\begin{aligned} J_T(x_0) &\leftarrow \left(\sum_{n=k}^{\infty} \lambda_n^{(k)} x_n \right) \\ &= J_T \left(\lambda_k^{(k)} x_k + (1 - \lambda_k^{(k)}) \sum_{k+1}^{\infty} \frac{\lambda_n^{(k)}}{1 - \lambda_k^{(k)}} x_n \right) \\ &\leq \alpha(\max\{J_T(x_k), J_T(\dots)\}) \\ &\leq \alpha(\max\{J_T(x_k), \alpha(\max\{J_T(x_{k+1}), \dots\})\}) \\ &= \alpha(J_T(x_k)) \rightarrow 0, \end{aligned}$$

establishing that $J_T(x_0) = 0$ and hence that $x_0 \in \text{Fix}(T)$. That $\text{Fix}(T)$ is closed and convex follow readily from the continuity of T and the definition of almost convexity. \square

The remainder of this note is devoted to examples of maps to which the above result might apply. Firstly, examples of non-affine almost convex maps, and then examples of when the necessary condition $\inf J_T(C) = 0$ is satisfied.

Examples of α -almost convex maps

1. To see what such maps might look like we begin with a very simple 1-dimensional example. The map $T : [0, 1] \rightarrow [0, 1] : x \mapsto x(1 - x)$ is

- not affine, but
- $J_T(x) = |x - Tx| = x^2$ is a convex function, and so
- T is 1-almost convex.

In general any map of the form $T = I - V$, where $\|V(x)\|$ is a convex function will be 1-almost convex.

2. Recall, a map $T : C \rightarrow X$ is K -Lipschitz continuous if $\|Tx - Ty\| \leq K\|x - y\|$, for all $x, y \in C$. Any K -Lipschitz continuous map for which there is also a 'lower estimate' of the form

$$\|x - y\| \leq \eta(\max\{J_T(x), J_T(y)\}),$$

where η is a continuous strictly increasing function with $\eta(0) = 0$, is $t + k\eta(t)/2$ -almost convex. This example encompasses:

(2a). *Banach contractions*; that is, K -Lipschitz continuous maps with $K < 1$. Such maps are readily seen to satisfy a lower estimate, and be $1/(1 - K)$ -almost convex.

(2b). Lipschitz continuous *dissipative operators*. Recall that a (nonlinear) operator $A : C \rightarrow X$ is dissipative (or equivalently, $-A$ is accretive) if

$$\langle Ax - Ay, x^* \rangle \leq 0,$$

for all $x, y \in C$ and $x^* \in \mathcal{D}(x - y)$, where \mathcal{D} is the duality map

$$\mathcal{D} : X \setminus \{0\} \rightarrow 2^{X^*} : x \mapsto \{x^* \in X^* : \langle x, x^* \rangle = \|x^*\| \|x\| = \|x\|^2\}.$$

Here, X^* denotes the dual space of X consisting of all continuous linear functionals from X into the scalars. The value of $y^* \in X^*$ at $x \in X$ is denoted by $\langle x, y^* \rangle$.

When X is a Hilbert space this is equivalent to $-A$ being a monotone operator. For bounded linear maps the condition reduces to requiring $\langle Ax, x^* \rangle \leq 0$, for some $x^* \in \mathcal{D}(x)$ and for each $x \in X$, and so for self-adjoint operators on a Hilbert space equates to requiring the spectrum be negative. Of course in the linear case fixed points can also be established via Schauder's theorem using the weak-continuity of the operator.

For example, $A := T - I$ is dissipative (and Lipschitz continuous), whenever T is nonexpansive; that is, $\|Tx - Ty\| \leq \|x - y\|$. More precisely, Kato (1967) shows that A is dissipative if and only if the resolvent $R_\lambda := (I - \lambda A)^{-1}$ is nonexpansive on its domain for all $\lambda \geq 0$, see [3, Ch. 13] for details.

In the context of dynamical systems, dissipativity is often related to stability considerations. To illustrate this, consider the simple feedback control system

$$\dot{x}(t) = Ax(t) + u(t),$$

where A is a dissipative operator. If $x(t)$ and $y(t)$ are solutions corresponding to the same control $u(t)$, but different initial conditions; $x(0) = x_0$ and $y(0) = y_0$, then

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} \|x - y\|_2^2 &= \langle \dot{x} - \dot{y}, x - y \rangle \\ &= \langle Ax - Ay, x - y \rangle \\ &\leq 0. \end{aligned}$$

To see that dissipative operators satisfy a lower estimate we observe that for distinct $x, y \in C$ and $z^* := y^*/\|y^*\|$, where $y^* \in \mathcal{D}(x - y)$ we have

$$\begin{aligned} \|x - y\| &= \langle x - y, z^* \rangle \\ &\leq \langle x - Ax, z^* \rangle - \langle y - Ay, z^* \rangle, \quad \text{as } \langle Ax - Ay, z^* \rangle \leq 0 \\ &\leq \|x - Ax\| + \|y - Ay\| \\ &= J_A(x) + J_A(y) \\ &\leq 2 \max\{J_A(x), J_A(y)\}. \end{aligned}$$

Thus, all Lipschitz continuous dissipative operators are almost convex.

In particular, the Yosida approximants

$$A_\lambda := \frac{(I - \lambda A)^{-1} - I}{\lambda}$$

for any dissipative operator A are themselves dissipative and automatically Lipschitz continuous, and so are always almost convex.

3. Generalized nonexpansive maps $T : C \rightarrow X$ which satisfy

$$\begin{aligned} \|Tx - Ty\| &\leq a\|x - y\| + b(\|x - Tx\| + \|y - Ty\|) \\ &\quad + c(\|x - Ty\| + \|y - Tx\|), \end{aligned}$$

where a, b and c are positive constants with $a + 2b + 2c = 1$, and which also satisfy the Kannan type condition, $b \neq 0$ may be shown to be almost convex.

4. Maps of type Γ , [1]; that is, $T : C \rightarrow X$ for which there exists a continuous convex strictly increasing function $\gamma : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ with $\gamma(0) = 0$ satisfying

$$\gamma(\|\lambda Tx + (1 - \lambda)Ty - T(\lambda x + (1 - \lambda)y)\|) \leq \|\|x - y\| - \|Tx - Ty\|\|$$

are α -almost convex, where $\alpha(t) = t + \gamma^{-1}(2t)$. So, by [1],

(4a). All nonexpansive maps on a closed bounded convex subset of a uniformly convex Banach space, in particular a Hilbert space, are almost convex.

Many more examples are provided by the following simple observation.

Theorem 2 (Alternative Principle). *For C a closed bounded convex set and $T : C \rightarrow C$ at least one of the following holds*

- (i) T is r -almost convex, for some $r > 0$, or
- (ii) $\inf J_T(C) = 0$.

There exist [3] Lipschitz continuous maps T of weak-compact convex sets, indeed of the unit ball of Hilbert space, B_{ℓ_2} , with $\inf J_T(C) > 0$. These then provide examples of r -almost convex maps which, unlike the affine maps they generalize, are

not weak-continuous. Nonetheless, an argument similar to that for the main result establishes the following.

Proposition 1. *If T is 1-almost convex and norm-continuous then $J_T(x)$ is weak lower semi-continuous.*

We do not know if $J_T(x)$ is weak lower semi continuous for arbitrary norm-continuous almost convex maps.

Some examples of maps satisfying the condition $\inf J_T(C) = 0$

1. *Norm continuous affine maps $T : C \rightarrow C$, where C is a closed convex subset of X . Here, the sequence defined by $x_n := \sum_{k=1}^n T^k(x_0)/n$, for $n \in \mathbb{N}$, is readily seen to be an approximate fixed point sequence.*
2. *Banach contractions $T : C \rightarrow C$, where C is a closed subset of X . In this case, for any $x_0 \in C$ the sequence of iterates $x_n := T^n(x_0)$, is well known to be an approximate fixed point sequence for T .*
3. *Nonexpansive maps $T : C \rightarrow C$, where C is a closed bounded convex subset of X . For any particular $x_0 \in C$ and each $n \in \mathbb{N}$ the map defined by $T_n(x) := (1 - 1/n)T(x) + (1/n)x_0$ is a Banach contraction mapping C into C , and so, by the Banach contraction mapping principle, has a unique fixed point $x_n \in C$. It is easily verified that the sequence (x_n) is an approximate fixed point sequence for T .*
4. *Maps $T : C \rightarrow C$ admitting an entropy; that is, a function $\psi : C \rightarrow \mathbb{R}^+$ for which*

$$\|x - Tx\| \leq \psi(x) - \psi(Tx), \quad \text{for all } x \in C.$$

To see that such maps always have an approximate fixed point sequence we proceed as follows. Since ψ takes only positive values,

$$m := \inf\{\psi(x) : x \in C\} \geq 0.$$

Further, for each $n \in \mathbb{N}$ there exists $x_n \in C$ with $m \leq \psi(x_n) < m + 1/n$. Whence,

$$\|x_n - Tx_n\| \leq \psi(x_n) - \psi(Tx_n) < \left(m + \frac{1}{n}\right) - m = \frac{1}{n}.$$

Thus, $\|x_n - Tx_n\| \rightarrow 0$ and so (x_n) is an approximate fixed point sequence for T .

5. *The existence of an entropy is closely related to the idea of passivity (sometimes termed ‘dissipativity’) of a dynamical system.*

NOTE: The key result and most of the examples considered above have been extended to the case of set valued maps in [4].

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