SOME PROPERTIES OF INCREASING CONVEX-ALONG-RAYS FUNCTIONS

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Abstract. In this paper we extend the theory of real-valued increasing convex along rays functions for functions mapping into the semi-extended real line. We give a full description of the Fenchel-Moreau conjugate function to an increasing positively homogeneous of the first degree function.

Key words. Abstract convexity, increasing convex-along-rays functions, normal sets, Fenchel-Moreau conjugate function.

1. Introduction. A function f is called *abstract convex* with respect to a class of *elementary* functions H if f can be represented as the upper envelope of a subset of H. The notion of abstract convexity plays a very important role in the study of various kinds of optimization problems (see for example [5, 1]). This notion is closely related to the Fenchel-Moreau conjugation theory [4, 5, 11] which is a natural extension of classical Fenchel conjugation [7]. From the point of view of this theory, it is quite natural to consider functions mapping into the semi-extended real line $\mathbf{R}_{+\infty} = \mathbf{R} \cup \{+\infty\}$. There are some interesting examples of abstract convex functions (see for instance [4, 3, 6]). One of the most interesting classes of (non-convex) abstract convex functions is generated by the set H of all shifts of the so-called min-type functions defined on the nonnegative orthant $\mathbf{R}^n_+$. It has been shown in [1, 2, 10] that a real-valued function is abstract convex with respect to the set H if and only if this function is increasing and its restriction on each ray starting from the origin is convex. Functions with these properties are called ICAR (increasing convex-along-rays) (see [1, 2, 10]). Real-valued ICAR functions are lower semicontinuous (l.s.c). These functions have interesting applications in global optimization (see, for example, [10, 8]).

In this paper we study H-convex functions mapping into $\mathbf{R}_{+\infty}$ where H is the above mentioned class of shifts of min-type functions. We prove that a function $f : \mathbf{R}^n_+ \to \mathbf{R}_{+\infty}$ is H-convex if and only if this function is l.s.c and ICAR. We show that the class of l.s.c ICAR functions $f : \mathbf{R}^n_+ \to \mathbf{R}_{+\infty}$ is very large. In particular, each l.s.c function defined on the unit simplex $S = \{x \in \mathbf{R}^n_+ : \sum_i x_i = 1\}$ can be extended to an ICAR function. We describe also Fenchel-Moreau conjugate functions with respect to increasing positively homogeneous functions.

2. Preliminaries. Let $\mathbf{R}$ be the set of real numbers and $\mathbf{R}_{+\infty} = \mathbf{R} \cup \{+\infty\}$. In the sequel we shall require the following definitions and elementary results dealing with abstract convexity (see [5, 11]).

DEFINITION 2.1. Let X be an arbitrary set and H be a set of functions $h : X \to \mathbf{R}$. A function $f : X \to \mathbf{R}_{+\infty}$ is called *abstract convex with respect to H* or *H-convex* if there is a set $U \subseteq H$ such that

$$f(x) = \sup\{h(x) : h \in U\} \quad \text{for all} \quad x \in X.$$

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We suppose that the function $-\infty : x \mapsto -\infty$ for all $x \in X$ is also abstract convex.

DEFINITION 2.2.
1) The set

$$s(f,H) = \{h \in H : h(x) \leq f(x) \quad \text{for all} \quad x \in X\}.$$

of H-minorants of a function $f : X \to \mathbf{R}_{+\infty}$ is called the *support set* of f.
2) The set $U \subset H$ is called *abstract convex* with respect to H or *H-convex* if there exists a function $f : X \to \mathbf{R}_{+\infty}$ such that $U = s(f,H)$.

REMARK 2.1. It is easy to check that U is abstract convex if and only if there exists an H-convex function f such that $U = s(f,H)$.

The following assertion directly follows from the definitions.

Proposition 2.1. *A set $U \subset H$ is H-convex if and only if for each $h \in H \setminus U$ there exists a point $x \in X$ such that $h(x) > \sup\{h'(x) : h' \in U\}$.*

Let L be a set of real-valued functions defined on a set X. Shifts of functions $l \in L$, that is functions h of the form $h(x) = l(x) - c$ for all $x \in X$ with $l \in L$, $c \in \mathbf{R}$ are called *L-affine functions.*

DEFINITION 2.3. Let L be a set of real-valued functions defined on a set X. Let $f : X \to \mathbf{R}_{+\infty}$ or $f = -\infty$. The function

$$f_L^*(l) = \sup\{l(x) - f(x) : x \in X\}$$

is called the *(Fenchel-Moreau) L-conjugate with respect to the function f.* The function

$$f_L^{**}(x) = \sup\{l(x) - f_L^*(l) : l \in L\}$$

is called the *second L-conjugate with respect to L.*

Theorem 2.1. *(see for example[4, 5, 11]) Let $f : X \to \mathbf{R}_{+\infty}$. Then $f = f_L^{**}$ if and only if f is H-convex where H is the set of all L-affine functions.*

In the remainder of this paper we shall consider functions defined on the cone $\mathbf{R}^n_+$ all of vectors with nonnegative coordinates in n-dimensional Euclidean space $\mathbf{R}^n$. We shall use the following notation:

- x_i is the i-th coordinate of a vector $x \in \mathbf{R}^n$;
- if $x, y \in \mathbf{R}^n$ then $x \geq y \iff x_i \geq y_i \quad \text{for all} \quad i \in I = \{1, 2, \ldots, n\}$;
- if $x, y \in \mathbf{R}^n$ then $x \gg y \iff x_i > y_i \quad \text{for all} \quad i \in I$;
- $\mathbf{R}^n_+ = \{x = (x_i) \in \mathbf{R}^I : \; x \geq 0\}$;
- $\mathbf{R}^n_{++} = \{x = (x_i) \in \mathbf{R}^I : \; x \gg 0\}$.

We shall study abstract convex functions with respect to the set H of all L-affine functions where L is the set of the so-called *min-type functions*, that is functions l defined on the cone $\mathbf{R}^n_+$ by

$$l(x) = \langle l, x \rangle \qquad (x \in \mathbf{R}^n_+) \tag{2.1}$$

where

$$\langle l, x\rangle = \min_{i\in\mathcal{T}(l)} l_i x_i; \qquad \mathcal{T}(l) = \{i : l_i > 0\}. \tag{2.2}$$

We assume that the minimum over the empty set is equal to zero. We denote the vector $(l_1, \cdots, l_n)$ by the same symbol l as the function generated by this vector using (2.1). In order to describe abstract convex functions with respect to the mentioned above set H, we need the following definition.

DEFINITION 2.4. A function $f : \mathbf{R}^n_+ \to \mathbf{R}_{+\infty}$ is called *convex-along-rays (CAR)* if, for each $y \geq 0$, the function $f_y(\lambda) = f(\lambda y)$ is convex on the ray $\{\lambda \in \mathbf{R} : \lambda > 0\}$.

We shall show that a function $f : X \to \mathbf{R}_{+\infty}$ is H-convex if and only if this function is increasing and CAR (briefly ICAR). A function f is called *increasing* if $x \geq y \implies f(x) \geq f(y)$. For finite functions this result was established in [1, 2], see also [10].

3. ICAR functions. In this section we shall study the simplest properties of ICAR functions $\mathbf{R}^n_+ \to \mathbf{R}_{+\infty}$. We need the following definitions.

DEFINITION 3.1. A set $U \in \mathbf{R}^n_+$ is called *normal* if $(x \in U,\ 0 \leq x' \leq x) \implies x' \in U$. A set U is called $\mathbf{R}^n_+$*-stable* if $(x \in U,\ x' \geq x) \implies x' \in U$.

Let f be an increasing function defined on $\mathbf{R}^n_+$. Then level sets $\{x \in \mathbf{R}^n_+ : f(x) \leq c\}$ are normal and level sets $\{x \in \mathbf{R}^n_+ : f(x) \geq c\}$ are $\mathbf{R}^n_+$-stable. In particular, the set $\operatorname{dom} f = \{x \in \mathbf{R}^n_+ : f(x) < +\infty\}$ is normal and the set $\{x : f(x) = +\infty\}$ is $\mathbf{R}^n_+$-stable.

Proposition 3.1. *Let f be an ICAR function and $x \in \mathbf{R}^n_{++}$. If there exists $\lambda > 1$ such that $\lambda x \in \operatorname{dom} f$ then the function f is continuous at the point x.*

Proof. Let $x_k \to x$. Take a positive number ε such that $1 + \varepsilon \leq \lambda$. For large k the inequality $(1 - \varepsilon)x \leq x_k \leq (1 + \varepsilon)x$ holds. Since the function f is increasing we have

$$f((1-\varepsilon)x) \leq f(x_k) \leq f((1+\varepsilon)x); \qquad f((1-\varepsilon)x) \leq f(x) \leq f((1+\varepsilon)x).$$

Since the convex function $f_x : \alpha \mapsto f(\alpha x)$ is continuous on the segment $(0, \lambda)$ it follows that $f((1+\varepsilon)x) - f((1-\varepsilon)x) \to 0$ as $\varepsilon \to 0$. □

REMARK 3.1. A finite ICAR function can be discontinuous at a boundary point of the cone $\mathbf{R}^n_+$. For example the function

$$g_1(x) = \begin{cases} \sum_i x_i & x \gg 0 \\ 0 & \text{otherwise} \end{cases}$$

is ICAR and discontinuous at each boundary point of $\mathbf{R}^n_+$ excluding the origin.

It was shown in [1, 2] that a finite ICAR function is l.s.c on $\mathbf{R}^n_+$. At the same time there exist ICAR functions $\mathbf{R}^n_+ \to \mathbf{R}_{+\infty}$ which are not l.s.c . For example the function

$$g_2(x) = \begin{cases} \sum_i x_i & \sum_i x_i < 1 \\ +\infty & \text{otherwise} \end{cases}$$

is not l.s.c .

We now present some examples of ICAR functions.

EXAMPLE 3.1. A positively homogeneous of degree $m \geq 1$ increasing function defined on $\mathbf{R}^n_+$ is ICAR; in particular a function

$$f(x) = x_1^{m_1} x_2^{m_2} \dots x_n^{m_n} \qquad x = (x_1, x_2, \dots, x_n) \in \mathbf{R}^n_+ \tag{3.1}$$

with $m_1 + \dots m_n \geq 1$ is ICAR.

EXAMPLE 3.2. A polynomial with nonnegative coefficients is ICAR.

Let H be the set of all L-affine functions, where L is the set of all min-type functions defined by (2.1). It is easy to check that the following assertion holds.

Proposition 3.2. *An H-convex function $f : \mathbf{R}^n_+ \to \mathbf{R}_{+\infty}$ is l.s.c and ICAR.*

The following statement shows that the class of ICAR functions is very large.

Proposition 3.3. *Let f be a l.s.c function defined on the unit simplex $S = \{x \in \mathbf{R}^n_+ : \sum_i x_i = 1\}$. Then there exists an ICAR extension of f, that is an ICAR function $\tilde{f} : \mathbf{R}^n_+ \to \mathbf{R}_{+\infty}$ such that $\tilde{f}(x) = f(x)$ for all $x \in S$.*

The proof is based on the following assertion.

Lemma 3.1. *Let Q be a compact topological space and H be a set of continuous functions defined on Q such that*
1) H is a conic set: $h \in H, \lambda > 0 \implies \lambda h \in H$;
2) for each $h \in H$ and $c > 0$ the function $x \mapsto h(x) - c$ belongs to H;
3) for any $\varepsilon > 0$, $z \in Q$ and any neighbourhood V of z there exists $h \in H$ which is a "support to an Urysohn peak", that is

$$h(z) > 1 - \varepsilon, \quad h(x) \leq 1 \quad \text{for all} \quad x \in Q, \quad h(x) \leq 0 \quad \text{for all} \quad x \notin V. \tag{3.2}$$

Then for each l.s.c function $f : Q \to \mathbf{R}_{+\infty}$ there exists a set $V \subset H$ such that $f(x) = \sup_{h \in V} h(x)$ for all $x \in Q$.

This lemma was proved in [4] with the following assumption instead of 2): H is a convex set and negative constants belong to H; actually these assumptions were used only in order to prove 2).

Proof. (of Proposition 3.3): Let H_S be the set of all functions h_S defined on the simplex S by $h_S(x) = \langle l, x \rangle - c$ with $l \in \mathbf{R}^n_+$, $c \in \mathbf{R}$. Clearly conditions 1) and 2) from Lemma 3.1 hold for the set H_S. Let us check that condition 3) holds as well.

Let $z \in S$. Consider the vector $l = 1/z$ where

$$\frac{1}{z} = \begin{cases} \dfrac{1}{z_i} & \text{if} \quad z_i > 0 \\ 0 & \text{if} \quad z_i = 0 \end{cases} \tag{3.3}$$

It is clear that $\langle l, z \rangle = 1$. Since

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} z_i$$

for $x \in S$ it follows that for $x \neq z$ there exists an index j such that $x_j < z_j$. Clearly $j \in \mathcal{T}(z)$. Therefore

$$\langle l, x \rangle = \min_{i \in \mathcal{T}(z)} \frac{x_i}{z_i} < 1.$$

Consider the function h' defined on S by $h'(x) = \langle l, x \rangle - 1$. We have $h(z) = 0$ and $h(x) < 0$ for $x \neq z$. Let V be an open neighbourhood of a point z and $\eta = -\max\{h(x) : x \in S \setminus V\} > 0$. Consider the functions $h''(x) = h'(x) + \eta'$ with $0 < \eta' < \eta$ and $h(x) = h''(x)/\eta'$. We have

$$h''(z) = \eta', \quad h''(x) < \eta' \quad \text{for all} \quad x \neq z, \quad h''(x) < 0 \quad \text{for all} \quad x \notin V,$$

so

$$h(z) = 1, \quad h(x) < 1 \quad \text{for all} \quad x \neq z, \quad h(x) < 0 \quad \text{for all} \quad x \notin V.$$

Thus the condition 3) from Lemma 3.1 holds. Let $f : S \in \mathbf{R}_{+\infty}$ be a l.s.c function. It follows from Lemma 3.1 that there exists a set $U \subset H$ such that $f(x) = \sup_{h \in U} h(x)$ for all $x \in S$. Consider now the function $\tilde{f}$ defined on $\mathbf{R}^n_+$ by

$$\tilde{f}(x) = \sup\{h(x) : x \in U\}.$$

It follows from Proposition 3.2 that $\tilde{f}$ is an ICAR function. We have also $\tilde{f}(x) = f(x)$ for $x \in S$. $\square$

REMARK 3.2. Proposition 3.1 shows that the following assertion is valid: if a finite l.s.c function f is discontinuous at a point $x \in S$ then $\tilde{f}(\lambda x) = +\infty$ for any extension $\tilde{f}$ of this function and for any $\lambda > 1$. Thus $\tilde{f}(y) = +\infty$ for all $y \gg x$.

It can be shown (see [8]) that each positive Lipschitz function defined on S has a locally Lipschitz (hence finite) extension $\tilde{f}$.

4. H-convex functions. Proposition 3.2 shows that each H-convex function $f : \mathbf{R}^n_+ \to \mathbf{R}_{+\infty}$ is l.s.c and ICAR. The following result was established in [1, 2], see also [10].

Theorem 4.1. *A real-valued function f defined on $\mathbf{R}^n_+$ is H-convex if and only if f is an ICAR function.*

The proof of Theorem 4.1 (see [2]) is based on the following construction. For a function $f : \mathbf{R}^n_+ \to \mathbf{R}$ consider its positively homogeneous extension $\hat{f}$ defined by

$$\hat{f}(x, \lambda) = \lambda f\left(\frac{x}{\lambda}\right), \qquad (x, \lambda) \in Z \tag{4.1}$$

where $Z = \mathbf{R}^n_+ \times (0, +\infty)$. It can be shown that a real-valued function f is ICAR if and only if $\hat{f} \in \mathcal{F}$ where $\mathcal{F}$ is the set of all finite functions F defined on the set Z such that

a_1) F is positively homogeneous of the first degree;

a_2) the function $x \mapsto F(x, \lambda)$ is increasing on $\mathbf{R}^n_+$ for each $\lambda > 0$;

a_3) for each (x, λ) the function $g(\mu_1, \mu_2) = F(\mu_1 x, \mu_2 \lambda)$ is sublinear on the cone $\{\mu_1 \geq 0, \mu_2 > 0\}$.

It can be shown that a function $f : \mathbf{R}^n_+ \to \mathbf{R}$ is H-convex if and only if its positively homogeneous extension $\hat{f}$ is H_*-convex where H_* is the set of all functions h_* defined on the set Z by the formula

$$h_*(x, \lambda) = \langle l, x \rangle - c\lambda \tag{4.2}$$

with $l \in \mathbf{R}^n_+$, $c \in \mathbf{R}$. The following assertions hold:

Proposition 4.1. *If $F \in \mathcal{F}$ then for all $(y, \nu) \in Z$ the set $\partial F(y, \nu) = \{h_* \in H_* : h_* \leq F,\ h_*(y, \nu) = F(y, \nu)\}$ is not empty.*

It follows from this proposition that each $F \in \mathcal{F}$ is H_*- convex, hence each ICAR real-valued function is H-convex. H-convexity of a real-valued ICAR function implies its lower semicontinuity.

As it was mentioned above, an ICAR function $f : \mathbf{R}^n_+ \to \mathbf{R}_{+\infty}$ is not necessary l.s.c, so we can not extend Theorem 4.1 for all ICAR functions. We will extend it only for l.s.c ICAR functions mapping into $\mathbf{R}_{+\infty}$. We shall use the construction described above.

The positively homogeneous extension $\hat{f}$ can be defined by (4.1) for an arbitrary function f mapping into $\mathbf{R}_{+\infty}$. Let $f : \mathbf{R}^n_+ \to \mathbf{R}_{+\infty}$ be a l.s.c function. Consider the function

$$\hat{f}_1(x, \lambda) = \begin{cases} \hat{f}(x, \lambda) & x \in Z \\ +\infty & \text{otherwise} \end{cases}$$

and its lower regularization $\mathrm{cl}\, \hat{f}$:

$$(\mathrm{cl}\, \hat{f})(x, \lambda) = \min(\hat{f}(x, \lambda), \liminf_{(x', \lambda') \to (x, \lambda),\, (x', \lambda') \neq (x, \lambda)} \hat{f}_1(x', \lambda')) \qquad (x, \lambda) \in \mathbf{R}^{n+1}_+.$$

It is clear that $\mathrm{cl}\, \hat{f}$ is a positively homogeneous function which maps $\mathbf{R}^{n+1}$ into $\mathbf{R}_{+\infty}$. Since f is l.s.c it follows that the function $\hat{f}$ is also l.s.c on the cone Z, so

$$\mathrm{cl}\, \hat{f}(x, \lambda) = \hat{f}(x, \lambda) = \lambda f(\frac{x}{\lambda}) \quad \text{for} \quad x \in \mathbf{R}^n_+,\ \lambda > 0. \tag{4.3}$$

Let us denote by $\mathcal{F}_1$ the set of all functions $F : \mathbf{R}^{n+1}_+ \to \mathbf{R}_{+\infty}$ such that

b_1) F is l.s.c and positively homogeneous of the first degree;

b_2) $F(0, 1) < +\infty$;

b_3) the function $x \mapsto F(x, \lambda)$ $(x \in \mathbf{R}^n_+)$ is increasing for each $\lambda > 0$.
b_4) for each (x, λ) with $x \in \mathbf{R}^n_+$ and $\lambda > 0$ the function

$$g(\mu_1, \mu_2) = F(\mu_1 x, \mu_2 \lambda) \qquad (\mu_1, \mu_2) \in \mathbf{R}^2_+ \tag{4.4}$$

is sublinear.
The following statements hold.

Lemma 4.1. *If $f : \mathbf{R}^n_+ \to \mathbf{R}_{+\infty}$ is a l.s.c ICAR function and $\operatorname{dom} f = \{x \in \mathbf{R}^n_+ : f(x) < +\infty\}$ is not empty then $\operatorname{cl} \hat{f} \in \mathcal{F}_1$.*

Proof. It is clear that $\operatorname{cl} \hat{f}$ is a l.s.c and positively homogeneous of the first degree function. Since $\operatorname{dom} f \neq \emptyset$ and f is increasing it follows that $0 \in \operatorname{dom} f$ so $\operatorname{cl} \hat{f}(0,1) \leq \hat{f}(0,1) = f(0) < +\infty$. It is easy to check that monotonicity of f implies monotonicity of the function $x \mapsto \operatorname{cl} \hat{f}(x, \lambda)$ $(x \in \mathbf{R}^n_+)$ for each $\lambda > 0$. Sublinearity of the function g defined on the set $\operatorname{dom} g = \{\mu = (\mu_1, \mu_2) : g(\mu) < +\infty\}$ by (4.4) with $F = \operatorname{cl} \hat{f}$ easily follows from convexity of the function $\alpha \mapsto f((\alpha/\lambda)x)$. □

Lemma 4.2. *Let H_* be the set of all functions (4.2) with $l \in L$ and $c \in \mathbf{R}$. If the extension $\hat{f}$ of a function f is H_*-convex then f is H-convex.*

Proof. There exists a set $U \subset L \times \mathbf{R}$ such that $\hat{f}(x, \lambda) = \sup_{(l,c) \in U}(\langle l, x\rangle - c\lambda)$ for $(x, \lambda) \in Z$. By applying (4.3), we have

$$f(x) = \hat{f}(x, 1) = \sup_{(l,c) \in U} (\langle l, x \rangle - c) = \sup_{h=(l,c) \in U} h(x).$$

Thus the desired result follows. □

Proposition 4.2. *Each function $F \in \mathcal{F}_1$ is H_*-convex.*

The scheme of the proof of Proposition 4.2 is similar to the scheme of the proof of Proposition 4.1 presented in [2]. We need the following assertion in order to realize this scheme.

Lemma 4.3. *Let $g : \mathbf{R}^2_+ \to \mathbf{R}_{+\infty}$ be a l.s.c sublinear function such that $g(0,1) < +\infty$ and the function $\mu_1 \mapsto g(\mu_1, \mu_2)$ is increasing for each $\mu_2 \geq 0$. Then there exists a closed convex set $V_+ \in \mathbf{R}^2$ such that $g(\mu_1, \mu_2) = \sup_{v=(v_1,v_2) \in V_+} v_1\mu_1 + v_2\mu_2$ for each $(\mu_1, \mu_2) \in \mathbf{R}^2_+$ and $v_1 \geq 0$ for each $v \in V_+$.*

Proof. Let g_+ be a function defined on $\mathbf{R}^2$ by

$$g_+(\mu_1, \mu_2) = \begin{cases} g(\mu_1^+, \mu_2) & (\mu_1, \mu_2) \in \mathbf{R}^2, \ \mu_2 \geq 0 \\ +\infty & (\mu_1, \mu_2) \in \mathbf{R}^2, \ \mu_2 < 0 \end{cases}$$

where $\mu_1^+ = \max(\mu_1, 0)$. It is easy to check that g_+ is a sublinear l.s.c function defined on $\mathbf{R}^2$. Thus there exists a convex closed set $V_+ = \partial g_+(0)$ such that $g_+(\mu_1, \mu_2) = \sup_{v \in V} v_1\mu_1 + v_2\mu_2$. Let $v \in V_+$. Then for each $\mu_1 < 0$ we have (with $\mu_2 = 1$)

$$v_1\mu_1 + v_2 \leq g_+(\mu_1, 1) = g(0,1) < +\infty.$$

Hence $v_1 \geq 0$. For any vector $(\mu_1, \mu_2) \in \mathbf{R}^2_+$ we have:

$$g(\mu_1, \mu_2) = g_+(\mu_1, \mu_2) = \sup_{v \in V_+} v_1\mu_1 + v_2\mu_2.$$

Since $v_1 \geq 0$ for each $v \in V_+$, the desired result follows. □

Proof. (of Proposition 4.2): Consider the set

$$U = \{(l, c) : \langle l, x\rangle - c\lambda \leq F(x, \lambda) \quad \text{for all} \quad (x, \lambda) \in \mathbf{R}^n_+ \times \mathbf{R}_+\} \tag{4.5}$$

We need to show that $F(y, \nu) = \sup_{(l,c)\in U}(\langle l, y\rangle - c\nu)$ for all $(y, \nu) \in \mathbf{R}^n_+ \times \mathbf{R}_+$.

Let $(y, \nu) \in \mathbf{R}^n_+ \times \mathbf{R}_+$ be a fixed vector. Let g be a sublinear function defined on the cone $\mathbf{R}^2_+$ by

$$g(\mu_1, \mu_2) = F(\mu_1 y, \mu_2 \nu).$$

Since F is a l.s.c sublinear function it follows that the function g is l.s.c sublinear as well. Since the function $x \mapsto F(x, \lambda)$ is increasing for each $\lambda > 0$ it follows that the function $\mu_1 \mapsto g(\mu_1, \mu_2)$ is increasing for each μ_2. We have also $g(0, 1) = F(0, \nu) = \nu F(0, 1) < +\infty$. It follows from Lemma 4.3 that there exists a set $V_+ \in \mathbf{R}^2$ such that

$$g(\mu_1, \mu_2) = \sup\{v_1\mu_1 + v_2\mu_2 : v = (v_1, v_2) \in V_+\}$$

and $v_1 \geq 0$ for each $(v_1, v_2) \in V_+$. For $v = (v_1, v_2) \in V_+$ let

$$h_v(x, \lambda) = v_1\langle \frac{1}{y}, x\rangle + v_2\frac{\lambda}{\nu}.$$

(For the definition of the vector $\frac{1}{y}$ see (3.3).)

Let us check that for all $(x, \lambda) \in \mathbf{R}^n_+ \times \mathbf{R}_+$ and for each $v = (v_1, v_2) \in V_+$:

$$F(x, \lambda) \geq h_v(x, \lambda). \tag{4.6}$$

First assume that $v_1 = 0$. Let $x \in \mathbf{R}^n_+$. Since the function $x \mapsto F(x, \lambda)$ is increasing, we have for $\lambda > 0$:

$$h_v(x, \lambda) = v_2\frac{\lambda}{\nu} \leq g(0, \frac{\lambda}{\nu}) = F(0, \lambda) \leq F(x, \lambda).$$

Now assume that $v_1 > 0$. In such a case $y \neq 0$. In fact if $y = 0$ then for all $\mu_1 > 0$ we have, with $\mu_2 = \lambda/\nu$:

$$v_1\mu_1 + v_2\mu_2 \leq g(\mu_1, \mu_2) = F(0, \mu_2\nu) = F(0, \lambda) = \lambda F(0, 1) < +\infty$$

and we obtain a contradiction to the inequality $v_1 > 0$. If $F(x, \lambda) = +\infty$ then the inequality (4.6) holds. Assume now that $F(x, \lambda) < +\infty$ and (4.6) does not hold for the vector (x, λ). Then there exists a number β such that $h_v(x, \lambda) > \beta > F(x, \lambda)$. We have

$$v_2\frac{\lambda}{\nu} \leq g(0, \frac{\lambda}{\nu}) = F(0, \lambda) \leq F(x, \lambda) < \beta.$$

Since

$$h_v(x,\lambda) = v_1\langle \frac{1}{y}, x\rangle + v_2\frac{\lambda}{\nu} = v_1 \min_{i\in \mathcal{T}(y)} \frac{x_i}{y_i} + v_2\frac{\lambda}{\nu} \tag{4.7}$$

and $h_v(x,\lambda) > \beta$, it follows that

$$v_1\frac{x_i}{y_i} > \beta - v_2\frac{\lambda}{\nu} \quad \text{for all} \quad i \in \mathcal{T}(y).$$

Therefore

$$x \geq \frac{1}{v_1}(\beta - v_2\frac{\lambda}{\nu})y \geq 0.$$

Since the function $x \mapsto F(x,\lambda)$ is increasing we have

$$\beta > F(x,\lambda) \geq F(\frac{1}{v_1}(\beta - v_2\frac{\lambda}{\nu}y, \lambda) = g(\frac{1}{v_1}(\beta - v_2\frac{\lambda}{\nu}y, \frac{\lambda}{\nu})$$

$$\geq v_1(\frac{1}{v_1}(\beta - v_2\frac{\lambda}{\nu}) + v_2\frac{\lambda}{\nu} = \beta.$$

Thus we have a contradiction which shows that (4.6) holds. Let $l = v_1/y$ and $c = -v_2/\nu$. It follows from (4.6) that $(l,c) \in U$ where U is defined by (4.5). Hence we have

$$F(y,\nu) = g(1,1) = \sup_{v=(v_1,v_2)\in V_+} v_1 + v_2 = \sup_{v\in V_+} h_v(y,\nu) \leq \sup_{(l,c)\in U} \langle l,y\rangle - c\nu.$$

On the other hand the definition of the set U shows that

$$\sup_{(l,c)\in U} \langle l,y\rangle - c\nu \leq F(y,\nu).$$

Thus the desired result follows. □

Theorem 4.2. *A function $f : \mathbf{R}^n_+ \to \mathbf{R}_{+\infty}$ with $\operatorname{dom} f \neq \emptyset$ is H- convex if and only if f is a l.s.c ICAR function.*

Proof. The proof directly follows from Proposition 3.2, Lemma 4.1, Lemma 4.2 and Proposition 4.2. □

REMARK 4.1. Clearly the functions $f \equiv +\infty$ and $f \equiv -\infty$ are H-convex.

Theorem 4.3. *Let $f : \mathbf{R}^n_+ \to \mathbf{R}_{+\infty}$. The equality $f = f^{**}_L$ holds if and only if f is an ICAR function.*

Proof. It follows immediately from Theorem 2.1, Theorem 4.2 and Remark 4.1. □

Consider now L-conjugate functions f_L^*. By definition

$$f_L^*(l) = \sup_{x \in \mathbf{R}_+^n} \left(\min_{i \in \mathcal{T}(l)} l_i x_i - f(x) \right). \tag{4.8}$$

We indicate some simple properties of the conjugate functions. For each nonempty subset I of the set $N = \{1, 2, \ldots, n\}$ consider the cone

$$\mathbf{R}_{++}^I = \{x \in \mathbf{R}_+^n : x_i > 0, \quad (i \in I), \quad x_i = 0 \ (i \notin I)\}. \tag{4.9}$$

The restriction of a function $g : \mathbf{R}_+^n \to \mathbf{R}_{+\infty}$ on the cone $\mathbf{R}_{++}^I$ is denoted by g_I. Let $f : \mathbf{R}_+^n \to \mathbf{R}_{+\infty}$. The following assertions hold.

1) The function f_L^* is CAR;
2) The restriction of f_L^* on the cone R_{++}^I is ICAR for each I.
3) Let $l \in \mathbf{R}_+^n$, $I \subset \mathcal{T}(l)$ and the vector l_I be defined as follows:

$$(l_I)_i = \begin{cases} l_i & \text{if } \ i \in I \\ 0 & \text{if } \ i \notin I. \end{cases}$$

Then $f_L^*(l_I) \geq f_L^*(l)$. Indeed since $I = \mathcal{T}(l_I) \subset \mathcal{T}(l)$ we have

$$f_L^*(l_I) = \sup_x \min_{i \in I} l_i x_i \geq \sup_x \min_{i \in \mathcal{T}(l)} l_i x_i = f_L^*(l).$$

Thus if f_L^* is increasing then $f_L^*(l_I) = f_L^*(l)$ for all $I \subset N$. The following example shows that the function f_L^* is not necessarily increasing and therefore not necessarily ICAR.

EXAMPLE 4.1. Let f be a function defined on $\mathbf{R}^2$ by $f(x) = \min(x_1, x_2)$, $e_1 = (1, 0)$, $l = (1, 1)$. It is clear that $l > e_1$. We have

$$f_L^*(e_1) = \sup_x (x_1 - \min(x_1, x_2)) = \sup_x \max(0, x_1 - x_2) = +\infty.$$

$$f_L^*(l) = \sup_x (\min(x_1, x_2) - \min(x_1, x_2)) = 0.$$

Thus $f_L^*(e_1) > f_L^*(e_2)$.

In the next section we give a description of the L-conjugate function for increasing positively homogeneous (IPH) functions.

5. IPH functions and their support sets. A function $p : \mathbf{R}_+^n \to \mathbf{R}_{+\infty}$ is called *positively homogeneous* if $p(\lambda x) = \lambda p(x)$ for all $x \in \mathbf{R}_+^n$ and $\lambda > 0$. Clearly an increasing positively homogeneous (IPH) function is ICAR. It follows from Theorem 4.2 that for each l.s.c IPH function there exists a set $U \subset \mathbf{R}_+^n \times \mathbf{R}_{+\infty}$ such that

$$p(x) = \sup_{(l,c) \in U} (\langle l, x \rangle - c) \quad \text{for all} \quad x \in \mathbf{R}_+^n.$$

We have for each $\lambda > 0$:

$$\lambda p(x) = p(\lambda x) = \sup_{(l,c) \in U} \langle l, \lambda x \rangle - c = \lambda \sup_{(l,c) \in U} \left(\langle l, x \rangle - \frac{c}{\lambda} \right) \quad \text{for all} \quad x \in \mathbf{R}_+^n.$$

Thus

$$p(x) = \sup_{(l,c)\in U} (\langle l, x\rangle - \frac{c}{\lambda}) \quad \text{for all} \quad x \in \mathbf{R}^n_+. \tag{5.1}$$

It follows from (5.1) that there exists a set $V \subset \mathbf{R}^n_+$ such that $p(x) = \sup_{l\in V}\langle l, x\rangle$ for all $x \in \mathbf{R}^n_+$ so

$$p(x) = \sup\{\langle l, x\rangle : l \in \mathrm{s}(p, L)\} \quad \text{for all} \quad x \in \mathbf{R}^n_+ \tag{5.2}$$

where $\mathrm{s}(p, L)$ is the support set of the function p (see Definition 2.2):

$$\mathrm{s}(p, L) = \{l \in \mathbf{R}^n_+ : \langle l, x\rangle \leq p(x) \quad \text{for all} \quad x \in \mathbf{R}^n_+\}.$$

The equality (5.2) shows that each IPH function p is abstract convex with respect to the set L of all functions of the form (2.1). Clearly the converse is also true: an abstract convex with respect to L function is IPH.

Proposition 5.1. *Let p be an IPH function. Then $p^*_L = \delta_{s(p)}$ where $\delta(U)$ is the indicator function of a set $U \subset \mathbf{R}^n_+$:*

$$\delta_U(l) = \begin{cases} 0 & \textit{if} \quad l \in U \\ +\infty & \textit{if} \quad l \notin U. \end{cases}$$

Proof. It easily follows from the positive homogeneity of p. □

Thus a description of L-conjugate with respect to an IPH function p is reduced to a description of abstract convex with respect to L sets, that is (see Definition 2.2 and Remark 2.1) subsets U of the set L which enjoy the following property: there exists an IPH function p such that $U = \mathrm{s}(p)$.

First we discuss some properties of the set L. Of course we can identify this set with the cone $\mathbf{R}^n_+$. However we have to distinguish the algebraic, ordering and topological properties of the set L of vectors $l \in \mathbf{R}^n_+$ and the set of min-type functions belonging to L which are generated by vectors $l \in L$ using (2.1). Note that the conic structure of the set $\mathbf{R}^n_+$ is isomorphic to the conic structure of the set L. Thus for $\lambda > 0$ the function $x \mapsto \langle \lambda l, x\rangle$ which is generated by the vector λl is equal to the function λl where $l(x) = \langle l, x\rangle$. (Recall that we use the same notation for both a vector and the function generated by the vector.) So we can identify L and $\mathbf{R}^n_+$ only as conic sets.

Let us consider the usual 'functional' order relation $\succeq$ on the set L:

DEFINITION 5.1. For $l^1, l^2 \in L$

$$l^1 \succeq l^2 \iff l^1(x) \geq l^2(x) \quad \text{for all} \quad x \in \mathbf{R}^n_+.$$

Proposition 5.2. *For $l^1, l^2 \in L$ we have $l^1 \succeq l^2$ if and only if*

$$\mathcal{T}(l^1) \subset \mathcal{T}(l^2) \quad \textit{and} \quad l^1_i \geq l^2_i \quad \textit{for all} \quad i \in \mathcal{T}(l^1). \tag{5.3}$$

Proof. 1) Let $l^1 \succeq l^2$. Assume $\mathcal{T}(l^1) \not\subset \mathcal{T}(l^2)$. Then there exists $j \in \mathcal{T}(l^1)$ such that $j \notin \mathcal{T}(l^2)$. Take a vector $x \in \mathbf{R}^n_+$ such that $x_i = 1$ for $i \in \mathcal{T}(l^2)$ and $x_j = 0$. Then $l^1(x) = 0$ and $l^2(x) = \min_{i \in \mathcal{T}(l^2)} l^2_i > 0$. Since $l^1(x) < l^2(x)$ it follows that the inequality $l^1 \succeq l^2$ is not valid. We have a contradiction which shows that $\mathcal{T}(l^1) \subset \mathcal{T}(l^2)$. Now assume that there is $k \in \mathcal{T}(l^1)$ such that $l^1_k < l^2_k$. Take a vector y such that $y_k = 1$ and $y_i > \frac{l^1_k}{l^1_i}$ for all $i \in \mathcal{T}(l^1)$, $i \neq k$, and $y_k > \dfrac{l^2_k}{l^2_i}$ for all $i \in \mathcal{T}(l^2)$. Then

$$l^1(y) = l^1_k < l^2_k = l^2(y)$$

and we have a contradiction again. Thus (5.3) holds.

2) Now assume that (5.3) is valid for vectors l^1 and l^2. For $x \in \mathbf{R}^n_+$ we have

$$l^1(x) = \min_{i \in \mathcal{T}(l^1)} l^1_i x_i \geq \min_{i \in \mathcal{T}(l^2)} l^1_i x_i \geq \min_{i \in \mathcal{T}(l^2)} l^2_i x_i = l^2(x).$$

So $l^1 \succeq l^2$. □

In order to describe support sets we need the following definitions.

Definition 5.2. A subset U of the ordered set L is *normal* if

$$l^1 \in U,\ l^2 \in L,\ l^1 \succeq l^2 \implies l^2 \in U.$$

(Compare this definition with Definition 3.1.)

Definition 5.3. A subset U of the set L is *closed-along-rays* if

$$\lambda_n > 0,\ \ \lambda_n x \in U\ (n = 1, 2, \dots)\ \text{ and }\ \lambda_n \to \lambda \implies \lambda x \in U.$$

Definition 5.3 is consistent with the conic structure of the set L which is isomorphic to the conic structure of the set $\mathbf{R}^n_+$.

Proposition 5.3. *A subset U of the ordered set L is L-convex if and only if U is closed-along-rays and normal.*

Proof. It is easy to check that an L-convex set is closed-along-rays and normal. Now let U be a closed-along-rays and normal subset of the cone L. We have to show that the inequality

$$l(x) \leq \sup_{l' \in U} l'(x) \quad \text{for all} \quad x \in \mathbf{R}^n_+$$

implies the inclusion $l \in U$. Equivalently we need to show (see Proposition 2.1) that if $l \in L$ and $l \notin U$ then there is a $x \in \mathbf{R}^n_+$ such that $l(x) > \sup_{l' \in U} l'(x)$. Let us consider such a vector $l \notin U$. Since U is closed-along-rays there is an $\varepsilon > 0$ such that $(1-\varepsilon)l \notin U$. Let $\bar{x} = (\bar{x}_1, \dots, \bar{x}_n) = \dfrac{1}{(1-\varepsilon)l}$ that is

$$\bar{x}_i = \frac{1}{(1-\varepsilon)l_i} \quad \text{for all} \quad i \in \mathcal{T}(l)), \qquad \bar{x}_i = 0 \quad \text{for all} \quad i \notin \mathcal{T}(l).$$

We have $l(\bar{x}) = \min_{i \in \mathcal{T}(l)} l_i \bar{x}_i = 1/(1-\varepsilon) > 1$. Now let $l' \in U$. Since U is normal the inequality $l' \succeq (1-\epsilon)l$ is not true. Applying Proposition 5.2 we can conclude that either

$$\mathcal{T}(l') \not\subset \mathcal{T}((1-\varepsilon)l) = \mathcal{T}(l) \tag{5.4}$$

or

$$\mathcal{T}(l') \subset \mathcal{T}(l) \text{ but } \exists\, i_o \in \mathcal{T}(l) \text{ such that } l'_{i_o} < (1-\varepsilon)l_{i_o}. \tag{5.5}$$

Assume (5.4) holds. Then we can find an index $i' \in \mathcal{T}(l')$ such that $i' \notin \mathcal{T}(l)$. Since $\bar{x}_{i'} = 0$ we have $\langle l', \bar{x} \rangle = 0 < \langle l, \bar{x} \rangle$. Now assume that (5.5) is valid. Then $\bar{x}_{i_o} > 0$. Hence

$$l'(\bar{x}) = \min_{i \in \mathcal{T}(l')} l'_i \bar{x}_i \le l'_{i_o} \bar{x}_{i_o} < (1-\epsilon) l_{i_o} \bar{x}_{i_o} = 1.$$

Thus we have constructed a vector $\bar{x}$ with the property

$$l(\bar{x}) > 1 \ge \sup_{l' \in U} l'(\bar{x}).$$

$\square$

REMARK 5.1. We say that a subset U of the set L is pointwise closed if $l^k \in U\,(k = 1, 2, \dots)$ and $l^k \to_{k \to +\infty} l$ implies $l \in U$. It follows directly from the definition of abstract convex sets that an L-convex set is pointwise closed. So Proposition 5.3 shows that a normal closed-along-rays subset of L is pointwise closed.

Theorem 5.1. *A function $g : L \to \mathbf{R}_{+\infty}$ is L-conjugate with respect to an IPH function p if and only if g coincides with the indicator function of a normal closed-along-rays subset of L.*

Proof. It follows directly from Proposition 5.1 and Proposition 5.3. $\square$

Consider now IPH functions defined on the cone $\mathbf{R}^n_{++}$. Let $\tilde{L}$ be the set of all functions of the form $x \to \langle l, x \rangle$ with $l \ge 0$.

Theorem 5.2. *[9] Let p be an IPH function defined on $\mathbf{R}^n_{++}$. Then*

$$s(p, \tilde{L}) = \{x \in \mathbf{R}^n_{++} : p(\frac{1}{x}) \ge 1\}.$$

Let p be an IPH function defined on $\mathbf{R}^n_+$. For each nonempty $I \subset N = \{1, \dots, n\}$ consider the restriction p_I of the function p on the cone R^I_{++} defined by (4.9). Let $L_I = \{l \in L : \mathcal{T}(l) = I\}$.

Proposition 5.4. *Let p be an IPH function defined on $\mathbf{R}^n_+$. Then*

$$s(p, L) = \bigcup_{I \subset N, I \neq \emptyset} \{x \in \mathbf{R}^I_{++} : p_I(\frac{1}{x}) \geq 1\} \cup \{0\}.$$

Proof. By applying Theorem 5.2 we have:

$$\begin{aligned} s(p, L) &= \{l \in L : \langle l, x\rangle \leq p(x) \quad \text{for all} \quad x \in \mathbf{R}^n_+\} \\ &= \bigcup_{I \subset N, I \neq \emptyset} \{l \subset L, \mathcal{T}(l) = I : \langle l, x\rangle \leq p(x) \quad \text{for all} \quad x \in \mathbf{R}^I_{++}\} \cup \{0\} \\ &= \bigcup_{I \subset N, I \neq \emptyset} \{l \in L_I : l \in s(p_I, L_I)\} \cup \{0\} \\ &= \bigcup_{I \subset N, I \neq \emptyset} \{x \in \mathbf{R}^I_{++} : p_I(\frac{1}{x}) \geq 1\} \cup \{0\}. \end{aligned}$$

The proof is complete. □

Let us give an example.

EXAMPLE 5.1. Let $p(x) = \sum_{i \in N} a_i x_i$ with $a_i > 0$ for all $i \in N$. We have for nonempty $I \subset N$: $p_I(x) = \sum_{i \in I} a_i x_i$. Thus

$$s(p_I, L_I) = \{x \in \mathbf{R}^I_{++} : \sum_{i \in I} \frac{a_i}{x_i} \geq 1\}.$$

and

$$s(p, L) = \bigcup_{I \subset N, I \neq \emptyset} \{x \in \mathbf{R}^I_{++} : \sum_{i \in I} \frac{a_i}{x_i} \geq 1\} \cup \{0\}.$$

In particular if $n = 2$ then $s(p, L)$ is the union of zero and three sets: two of them are segments on the coordinate axes:

$$s(p_{\{1\}}, L_{\{1\}}) = \{x = (x_1, x_2) : 0 < x \leq a_1, \, x_2 = 0\};$$

$$s(p_{\{2\}}, L_{\{2\}}) = \{x = (x_1, x_2) : x_1 = 0, \, 0 < x_2 \leq a_2\}.$$

The third set is

$$s(p_{\{1,2\}}, L_{\{1,2\}}) = \{x = (x_1, x_2) \in \mathbf{R}^2_{++} : \frac{a_1}{x_1} + \frac{a_2}{x_2} \geq 1\}.$$

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