AN ALGEBRAIC APPROACH TO OPTIMAL OUTPUT DECOUPLING BY OUTPUT FEEDBACK CONTROL

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Abstract. A controller which minimises the $\mathcal{H}_2$ norm of the closed loop transfer function for a linear time invariant plant, subject to an output disturbance decoupling constraint is found. The problem formulation results in an optimization over a parameter $Q(s)$ which is constrained to be a diagonal transfer matrix. The solution method relies on an $\mathcal{H}_2$ isometric isomorphism between transfer matrices and transfer vectors.

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Key words. $\mathcal{H}_2$ optimization, decoupling, diagonalizing, algebraic framework.

1. Introduction. Decoupling controllers which, in closed loop, eliminate interactions between the various reference and output signals are interesting both from a theoretical and a practical viewpoint. Earlier work includes [5], [6], and [2]. Recent theoretical results in [11] give algebraic conditions for the solvability of the output decoupling problem, as well as a characterisation of all possible stabilising controllers which simultaneously satisfy a decoupling constraint. The practical problem of choosing a *particular* decoupling controller, however, still remains.

Decoupling may be a desirable design goal for a number of reasons. It eliminates the effect of interactions between outputs so that each output may be controlled independently so that once a decoupling precompensator has been found its possible to undertake independent controller design for each loop.

This paper presents a method for determining an $\mathcal{H}_2$ optimal [14], [1] controller, subject to a decoupling constraint on the output. The method may potentially be extended to the case of non-zero closed loop off-diagonal element constraints. Solving for an optimal decoupling controller lends insight into the *performance costs* associated with a decoupling restriction. The unconstrained optimal cost can be compared to the constrained one and hence that associated with the decoupling constraint evaluated. Quantifying the cost associated with decoupling enables an understanding of the trade-offs involved, and hence whether decoupling is desirable.

Work on a similar problem [15], [16], is motivated by the observation that an $\mathcal{H}_2$ optimisation criterion may be useful for selecting a decoupling controller. The solution presented here exploits an alternative method of solution and overcomes the hurdle [15] of performing quadratic optimisation over a diagonal transfer matrix parameter. Specifically, we rely on and extend the work in [11].

The structure of this paper follows. Firstly we define decoupling (diagonalizing) control, and restate results related to all decoupling controllers. We discuss the motivation for solving the $\mathcal{H}_2$ optimal decoupling problem before defining the problem rigourously and presenting the solution. We finish with an example and a conclusion.

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2. Preliminaries.

2.1. Notation and Definitions. A column vector is indicated by an unscripted lower case variable, for example, v. The hermitian (conjugate transpose) and transpose operators allow row vectors to be denoted as v^* or v^T. A subscripted lower case variable v_j indicates the scalar j_{th} element of v. Superscripts distinguish different variables with the same base name e.g. v^1, v^2 etc.

The $(i,j)_{th}$ element of a matrix A is denoted by $[A]_{i,j}$. A diagonal transfer matrix $A_d(s)$ is defined as a transfer matrix where the off-diagonal elements are zero. We denote as $\mathcal{D}$, the set of all proper and stable diagonal transfer functions, that is $\mathcal{D} = \left\{ A_d(s) : [A_d(s)]_{ij} = 0 \ \forall i \neq j, A_d(s) \in \mathcal{H}_\infty, \right\}$.

The symbols $\mathcal{H}_2$ and $\mathcal{H}_2^\perp$ represent [24] the Hilbert spaces [23] corresponding to (finitely dimensioned) transfer matrices (functions) which are strictly proper and analytic in respectively the closed right half and closed left half complex plane $\mathbb{C}$. The symbol $\mathcal{L}_2$ denotes the Hilbert space corresponding to the direct sum $\mathcal{H}_2 \bigoplus \mathcal{H}_2^\perp$, equipped with the inner product $\langle F, G \rangle :\overset{\text{def}}{=} \frac{1}{2\pi j} \int_{-\infty}^{\infty} F(j\omega)^* G(j\omega) \quad d\omega$, and corresponding norm. For $F \in \mathcal{H}, G \in \mathcal{H}^\perp$ it is well known [24] that $\langle F, G \rangle = 0$, and thus $\|F+G\|_2^2 = \|F\|_2^2 + \|G\|_2^2$. This result will be exploited in some of the proofs below.

In addition [24], the symbol $\mathcal{L}_\infty$ denotes the Banach space of matrix-valued functions on $\mathbb{C}$, that are (essentially) bounded on the imaginary axis. The symbol $\mathcal{H}_\infty$, denotes the (closed) subspace of $\mathcal{L}_\infty$ of functions which are analytic and bounded in the open right hand plane (ORHP). The symbol $\mathcal{H}_\infty^-$, denotes the (closed) subspace of $\mathcal{L}_\infty$ of functions that are analytic and bounded in the open *left* hand plane (OLHP). When the symbols for various transfer matrix vectorspaces are superscripted as in $\mathcal{H}_2^{a\times b}$, $\mathcal{H}_\infty^{-\,a\times b}$, $\mathcal{D}^{a\times b}$ or $\mathcal{L}_\infty^{a\times b}$ they refer specifically to the corresponding space of transfer matrices with dimensions $a \times b$. Coprime factorisations [19], [18] are over the ring of proper and stable transfer matrices.

2.2. MIMO Decoupling Control. Consider a linear time invariant (LTI) system which is multiple-input multiple-ouput (MIMO) $(p \times m)$ open loop system, and the problem of finding a set of control signal vectors $u^i(t)$, each of which has dimension $m{\times}1$, which cause the p output signal vectors $y^i(t)$, each of which has dimension $p{\times}1$, to asymptotically track n_r given reference signal vectors $r^i(t)$, with dimension $p \times 1$. We consider an output feedback (single degree of freedom) controller structure as in Figure 2.1 where the controller has access only to the reference error. The control must stabilise the closed loop and is allowed to depend only on past observations, that is, the output error. We assume also that the controller is linear and time-invariant.

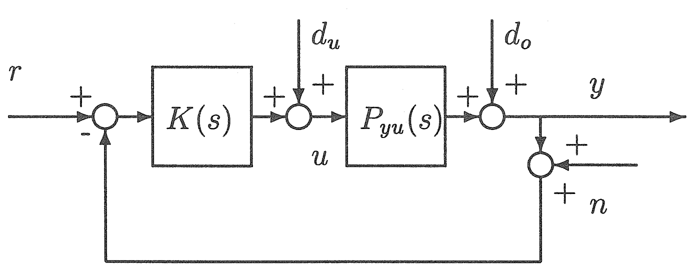

FIG. 2.1. *Single Degree of Freedom Controller Structure*

Associated with this controller structure are several important transfer functions

[8]. The output open loop transfer function is $L_O(s) \stackrel{\text{def}}{:=} \mathcal{P}_{yu}(s)\mathcal{K}_{uy}(s)$ and the (closed loop) output sensitivity and *complementary* sensitivity functions are respectively, $S_O(s) \stackrel{\text{def}}{:=} (I + L_O(s))^{-1}$ and $T_O(s) \stackrel{\text{def}}{:=} L_O(s)(I + L_O(s))^{-1}$.

A decoupled closed loop system is one in which there is no interaction between various references, so that a reference change for one output results in a change in only that output. Provided that we restrict our attention to reference signals $r^i(t)$ such that they have a non-zero component in the i_{th} entry $r^i_i(t)$ and is zero for $r^i_j, i \neq j$ elsewhere, then we may be led to a meaningful definition of the decoupling requirement.

A control is "output decoupling" if the control vector $u^i(t)$ generated by the controller in response to a particular reference signal $r^i(t)$ for the i_{th} output is such that $y^i_j(t) = 0$ for all $i \neq j$. This is equivalent to requiring that the transfer function from the reference to the output (the output complementary sensitivity function $T_O(s)$) is diagonal. Since the output sensitivity and complementary sensitivity satisfy $S_O(s) + T_O(s) = I$, then this is also equivalent to requiring that the output sensitivity function, $S_O(s)$, the transfer function from the reference to the error, is diagonal. For a single degree of freedom controller structure as in Figure 2.1, we also have that a decoupling constraint implies that an output disturbance $d_o(t)$ on one channel affects (via the control loop) only that output. It is also equivalent to requiring that the open loop transfer function $L_O(s)$ is diagonal.

2.3. Gómez-Goodwin Parametrisation of All Decoupling Controllers. A necessary requirement for the existence of a decoupling controller is that the open loop plant be (right) invertible, that is, it must have no more outputs than inputs ($p \leq m$) and have full normal-rank. Results in [11] give sufficient and (other) necessary algebraic conditions for the existence of a decoupling controller, in terms of unstable zeros and poles and their directions. This leads to a Youla-type parametrisation [24] of all decoupling controllers and all achievable corresponding decoupled transfer functions. The main results are reiterated here briefly for completeness.

We assume that the open loop plant $\mathcal{P}_{yu}$ is single degree of freedom output decouplable and has coprime factorisation $N(s)D(s)^{-1} = \tilde{D}(s)^{-1}\tilde{N}(s)$. Note that right-invertible plants are generically decouplable with a single degree of freedom. Then, as before, there exists [11] a double coprime factorisation of a *decoupling* controller $\mathcal{K}_{uy} = P(s)L(s)^{-1} = \tilde{L}(s)^{-1}\tilde{P}(s)$ which both satisfies the double Bezout identity $\tilde{D}(s)L(s) + \tilde{N}(s)P(s) = \tilde{L}(s)D(s) + \tilde{P}(s)N(s) = I$ *and* for which the output sensitivity and complementary sensitivity functions, respectively, $S_O(s) = N(s)\tilde{P}(s)$ and $T_O(s) = L(s)\tilde{D}(s)$, are both diagonal.

All stabilising (but not necessarily decoupling) output feedback controllers are

$$\begin{aligned} \mathcal{K}_{uy}(s) &= \left[\tilde{D}(s) - Q(s)\tilde{N}(s)\right]^{-1}\left[\tilde{P}(s) + Q(s)\tilde{L}(s)\right], \\ &= \left[P(s) + D(s)Q(s)\right]\left[L(s) - N(s)Q(s)\right]^{-1}, \qquad (2.1) \\ \text{with} \quad & Q(s) \in \mathcal{H}_\infty. \end{aligned}$$

Furthermore, all achievable sensitivity functions $S_O(s)$ from output disturbance to error are given by

$$S_O(s) = L(s)\tilde{D}(s) - N(s)Q(s)\tilde{D}(s). \qquad (2.2)$$

Since $L(s)\tilde{D}(s)$ is already diagonal, a diagonal restriction on the closed loop $S_O(s)$ is equivalent to requiring $N(s)Q(s)\tilde{D}(s)$ to be diagonal.

The results in [11] show that *all* stabilising *and decoupling* controllers can be written as (2.1) above but with the additional restrictions that

$$\begin{aligned} Q(s) &= R_N(s)Q_d(s)R_{\tilde{D}}(s) + Z(s), \\ \text{where } Q_d(s) \in \mathcal{D}, &\text{ and } R_N, R_{\tilde{D}} \text{ are such that} \\ N(s)R_N(s) &= S_N(s) \\ \text{and } R_{\tilde{D}}(s)\tilde{D}(s) &= \tilde{S}_{\tilde{D}}(s). \end{aligned} \tag{2.3}$$

In the above $S_N(s)$, $\tilde{S}_{\tilde{D}}(s)$ are respectively, the right and left diagonal structures [6] of respectively, $N(s)$ and $\tilde{D}(s)$. Also, $Z(s) \in \mathcal{H}_\infty$ is such that $N(s)Z(s) = 0$. It is straightforward to find a $P(s), L(s)$ satisfying the Bezout identity *and* the decoupling constraint once any stabilising decoupling controller $\mathcal{K}_{yu}(s)$ has been found.

Proof. See [11], theorem 5.3 and remark 5.5. □

2.3.1. Decoupling Control: Spreading of Poles and Zeros. It is well known [5], [12], [10] that a decoupling requirement may result in the spreading of poles and zeroes in the closed loop system. Specifically, **for each** non-minimum phase zero z_k in the plant, with output direction $\bar{v}_k^*$, there is a non-minimum phase zero in the decoupled *complementary* sensitivity function at each (output) channel corresponding to each *non-zero* element in $\bar{v}_k^*$. Also **for each** unstable pole p_j in the plant with (output) direction $\hat{w}_j$, there is a non-minimum phase zero in the decoupled sensitivity function at each (output) channel corresponding to each *non-zero* element in $\hat{w}_j$.

Hence, for decoupled stabilising control there is spreading of unstable poles and nonminimum phase zeros into each channel where there is non-zero component in the (output) zero or pole direction. It is also well known [4], [7], [9] that the existence of unstable poles and non-minimum phase zeros makes control difficult in terms of the minimum acheivable disturbance sensitivity peak. It also places limitations on the minimum $\mathcal{H}_2$ norm of the sensitivity function [4], [21], [2].

2.3.2. Existence of Output Decoupling Controllers: Necessary Conditions. If there are coincident unstable poles and zeroes for which the corresonding direction has a non-zero component in the same output channel, then decoupling is impossible. This can be seen as an extreme cost of decoupling. Restated, a necessary condition for decoupling is that no coincident unstable pole and zeroe for which the corresonding direction have a non-zero component in the same output.

This necessary condition is also sufficient if a certain technical condition, namely that the geometric and algebraic multiplicity of the poles and zeros is the same, is true [11]. If the multiplicity condition holds, then if (and only if) there are no coincident unstable poles and zeroes for which the corresponding directions have a non-zero component in the same output channel, decoupling *is* possible.

2.4. Costs of Decoupling. It is clear that the restriction of the set of controllers from a free parameter $Q(s) \in \mathcal{H}_\infty$ to one which is diagonal $Q_d(s) \in \mathcal{D}$ will affect achievable performance. The previous section revealed that decoupling may be costly in terms of stability when there are coincident right hand plane (RHP) poles and zeroes. In addition it is shown in [2], [9] that even non-coincident open loop RHP poles and zeroes result in performance costs in terms of achievable $\mathcal{H}_2$ performance or peak sensitivity minimisation. Are there still other $\mathcal{H}_2$ performance costs of decoupling-even in the absence of RHP poles or zeroes? How would we go about quantifying them to and thus find out whether these costs are severe? Equation (2.1) gives a parametrisation of *all* decoupling controllers in terms of an unknown parameters $Q_d(s)$ and $Z(s)$. This suggests the question "What would be a sensible way to choose particular values for those parameters?"

2.4.1. Quadratic Tracking Error Cost. The average quadratic tracking error [4] is a particular example of a (quadratic) cost function defined in terms of a set of n_r reference signals $r^i(t)$ and the corresponding controls $u^i(t)$ as follows.

$$J(r(t), u^i(t) : i = 1 \dots n) :\stackrel{\text{def}}{=} \sum_{i=1}^{n_r} \int_0^{\infty} e^{i^T} e^i dt = \sum_{i=1}^{n_r} \left\| e^i(t) \right\|_2^2 . \tag{2.4}$$

Here $e^i(t) = r^i(t) - y^i(t)$ is the error of the system output when the output vector, $y^i(t)$, is required to track the i_{th} reference signal vector $r^i(t)$. One design approach might be to seek the minimum quadratic tracking error cost over all allowable controls. This is the minimum achievable error for a given set of reference signals, with no penalty for, or restriction on, the size of the control effort $u(t)$ required [2].

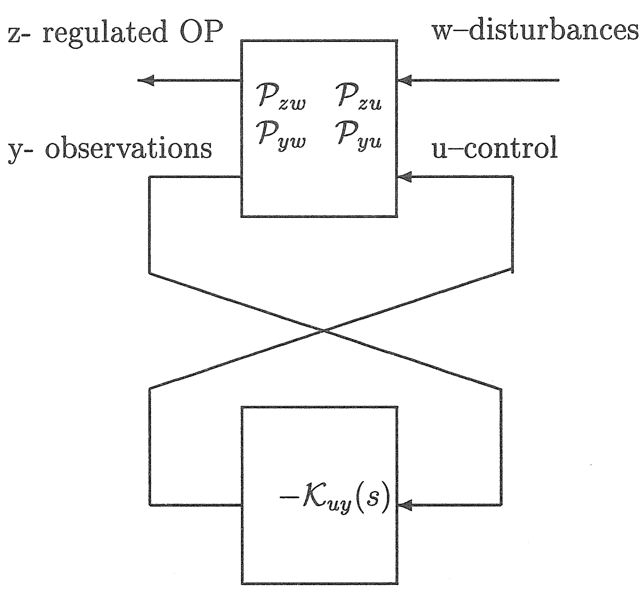

FIG. 2.2. *Generic Controller Structure*

This particular cost function may be expressed via the generic linear controller structure [24] of Figure 2.2. The generic plant $\mathcal{G}(s)$ is given as

$$\begin{aligned} \begin{bmatrix} z \\ y \end{bmatrix} &= \begin{bmatrix} \mathcal{P}_{zw}(s) & \mathcal{P}_{zu}(s) \\ \mathcal{P}_{yw}(s) & \mathcal{P}_{yu}(s) \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix} \\ &= \mathcal{G}(s) \begin{bmatrix} w \\ u \end{bmatrix} \end{aligned} \tag{2.5}$$

The exogenous signals are w $(q \times 1)$ and the control input is u $(m \times 1)$. The regulated output is z $(r \times 1)$, and the measured and diagonalised output by y $(p \times 1)$. The parametrization of all output decoupling controllers depends only on $\mathcal{P}_{yu}(s)$.

We define the regulated output to be the tracking error $z(t) :\stackrel{\text{def}}{=} e(t) = y(t) - r(t)$ and allow the exogenous disturbance $w(t)$ to be the reference signal. The quadratic tracking error is then $J :\stackrel{\text{def}}{=} \|z(t)\|_2^2 = \|\hat{z}(s)\|_2^2 = \|T_{zw}(s)\hat{w}(s)\|_2^2$ where the first norm is in the time domain and the latter two are in the frequency domain. The second equality follows from Parseval's theorem, and $T_{zw}(s)$ is the closed loop transfer function from the reference $\hat{w}(s)$ to the error $\hat{z}(s)$.

We are interested in $J^\star :\stackrel{\text{def}}{=} \inf_{\mathcal{K}_{uy}(s)} \|T_{zw}(s)\hat{w}(s)\|_2^2$ where $\mathcal{K}_{uy}(s)$ is a (LTI) stabilising controller. The infimal cost may be calculated for both where $\mathcal{K}_{uy}(s)$ is restricted to be diagonalising, as well as the unrestricted case. The latter may act as a benchmark against which the decoupled performance may be compared.

3. The Optimization Problem Definition: More General Cost Function. In light of the previous motivating discussion we propose the following more general, problem definition. Note that the quadratic tracking error results in only one particular choice of input disturbance to regulated output transfer function $T_{zw}(s)$. In general, any set of disturbances to any regulated output as in Figure 2.2 is possible. In the generic controller structure, exogenous disturbance signals $\hat{w}(s)$ can be expressed as a filtered impulse with this filter absorbed into the transfer matrix $T_{zw}(s)$. The choice of cost transfer function can be based upon design specifications or upon which disturbances it is important to reject from which outputs.

A physical interpretation of the $\mathcal{H}_2$ norm $\|\cdot\|_2$ is of relative energy in the output due to an impulse or white noise input. Minimising this functional has the effect of making the impact of disturbances on the regulated output small.

The general problem enables us to answer the two motivating questions. Because we minimise the effect of an arbitrary disturbance on an arbitrary output in an $\mathcal{H}_2$ norm sense. Thus, in terms of disturbance rejection, the choice of decoupling controller is a "good" one. Furthermore we can solve the problem *without* a decoupling constraint, and hence quantify the additional costs of the decoupling requirement.

The problem may now be stated formally as follows. We would like to design an observer-based controller $u = -\mathcal{K}_{uy}(s)y$. which observes the state via the measurement variable y and minimises the overall $\mathcal{H}_2$ norm of $T_{zw}(s)$ from w to z subject to the closed loop being stable and the off-diagonal elements of the sensitivity and complementary sensitivity transfer functions are each zero.

Note that the feedthrough term D_{zw} of $\mathcal{P}_{zw}$ must be such that $D_{zw} = 0$ for the $\mathcal{H}_2$ problem to be well-posed. Furthermore the feedthrough term of $\mathcal{P}_{yu}$ given by D_{yu} may as well be 0, since the control variable u is known.

3.1. All Achievable Transfer Functions. We next determine a parametrisation of all achievable $T_{zw}(s)$ from disturbance w to regulated output z.

Lemma 3.1. *Consider a generalised p-output, m-input ($p<m$) plant $\mathcal{G}(s)$ as in equation (2.5), with $\mathcal{P}_{yu}(s)$ of rank p, having coprime factorisation $N(s)D(s)^{-1}$. The set of all acheivable transfer functions $T_{zw}(s)$ from a q-input disturbance vector to an r-output cost vector, such that the controller is stabilising and results in output disturbance decoupling (that is, $T_O(s), S_O(s) \in \mathcal{D}^{p\times p}$) is given by an affine combination of transfer function matrices of the form*

$$T_{zw}(s) = F(s) - G_d(s)Q_d(s)H_d(s) - \bar{G}(s)\bar{Q}(s)\bar{H}(s) \tag{3.1}$$

where $F(s) \in \mathcal{H}_\infty^{r\times q}$, $G_d(s) \in \mathcal{H}_\infty^{r\times p}$, $H_d(s) \in \mathcal{H}_\infty^{p\times q}$, $\bar{G}(s) \in \mathcal{H}_\infty^{r\times n}$ and $\bar{H}(s) \in \mathcal{H}_\infty^{p\times q}$, are fixed and both $Q_d(s) = \mathcal{D}^{p\times p}$ and $\bar{Q}(s) \in \mathcal{H}_\infty^{n\times p}$ are free. In the above, $n :\stackrel{\text{def}}{=} m{-}p$.

Proof. Using a controller $\mathcal{K}_{uy}(s)$ gives $u = -\mathcal{K}_{uy}(s)\left[I + \mathcal{P}_{yu}(s)\mathcal{K}_{uy}(s)\right]^{-1}\mathcal{P}_{yw}(s)w$ and hence $T_{zw}(s) = \mathcal{P}_{zw}(s) - \mathcal{P}_{zu}(s)\mathcal{K}_{uy}(s)\left[I + \mathcal{P}_{yu}(s)\mathcal{K}_{uy}(s)\right]^{-1}\mathcal{P}_{yw}(s)$

For $\mathcal{K}_{uy}(s)$ as in (2.1) we have $\mathcal{K}_{uy}(s)\left[I + \mathcal{P}_{yu}(s)\mathcal{K}_{uy}(s)\right]^{-1} = \left[P(s) - D(s)Q(s)\right]\tilde{D}(s)$, where $Q(s)$ satisfies the constraints of equations (2.3) resulting in

$$\begin{aligned} T_{zw}(s) &= \mathcal{P}_{zw}(s) - \mathcal{P}_{zu}(s)\left[P(s) - D(s)Q(s)\right]\tilde{D}(s)\mathcal{P}_{yw}(s), \qquad (3.2)\\ \text{subject to } Q(s) &= R_N(s)Q_d(s)R_{\tilde{D}}(s) + Z(s),\\ Z(s) &\in \mathcal{H}_\infty^{m\times p},\\ 0 &= N(s)Z(s). \end{aligned}$$

Recall [11], that in general, for a non-square plant $\mathcal{P}_{yu}(s)$ with right coprime factorisation $N(s)D(s)^{-1}$ there exists a unimodular tranformation $U_m(s)$ such that $\mathcal{P}_{yu}(s)U_m(s) = \begin{bmatrix} \bar{\mathcal{P}}_{yu}(s) & 0 \end{bmatrix}$ and $N(s)U_m(s) = \begin{bmatrix} \bar{N}(s) & O^{p\times n} \end{bmatrix}$ with $\bar{\mathcal{P}}_{yu}(s), \bar{N}(s)$ non-singular for almost all s.

Furthermore, $U_m(s)$ can be decomposed into $U_m(s) = \begin{bmatrix} U_p(s) & U_n(s) \end{bmatrix}$ with $U_p(s) \in \mathcal{H}_\infty^{m\times p}$ being composed of the p leftmost columns of $U_m(s)$ and $U_n(s) \in \mathcal{H}_\infty^{m\times n}$ being composed of the n rightmost columns of $U_m(s)$.

A necessary and sufficient condition for $N(s)Z(s) = 0$ is that

$$U_m(s)^{-1}Z(s) = \begin{bmatrix} O_p \\ \bar{Q}(s) \end{bmatrix}$$

for some $\bar{Q}(s)$ where O_p is a $p{\times}p$ square zero matrix. Now the requirement that $Z(s)$ be stable is equivalent to $\bar{Z}(s)$ being stable, so all stable $Z(s)$ satisfying $N(s)Z(s) = 0$ can be parametrised as $Z(s) = U_n(s)\bar{Z}(s)$, with $\bar{Q}(s) \in \mathcal{H}_\infty^{n\times p}$.

Substitituting the above into equation (3.2) gives

$$T_{zw}(s) = \mathcal{P}_{zw} - \mathcal{P}_{zu}\left\{P\tilde{D} + D\left[R_N Q_d(s)S_{\tilde{D}} + U_n\bar{Q}(s)\tilde{D}\right]\right\}\mathcal{P}_{yw}.$$

This can be seen to be equal to the first expression (3.1) provided

$$\begin{aligned} F(s) &= \mathcal{P}_{zw}(s) - \mathcal{P}_{zu}(s)P(s)\tilde{D}(s)\mathcal{P}_{yw}(s),\\ G_d(s) &= \mathcal{P}_{zu}(s)D(s)R_N(s),\\ H_d(s) &= S_{\tilde{D}}(s)\mathcal{P}_{yw}(s),\\ \bar{G}(s) &= \mathcal{P}_{zu}(s)D(s)U_n(s),\\ \text{and } \bar{H}(s) &= \tilde{L}(s)\mathcal{P}_{yw}(s). \end{aligned}$$

This completes the proof. □

We then seek the $Q_d(s)$ and $\bar{Q}(s)$ to minimise the $\mathcal{H}_2$ norm of the MIMO transfer function $T_{zw}(s)$, that is

$$J^\star = \inf_{Q_d\in\mathcal{D},\bar{Q}\in\mathcal{H}_\infty} \|F(s) + G_d(s)Q_d(s)H_d(s) + \bar{G}(s)\bar{Q}(s)\bar{H}(s)\|_2 \qquad (3.3)$$

This has a known solution [12], [1], [24], [15] when there is no diagonal restriction. A standard method is via (Wiener-Hopf) spectral factorization [13], [22]. If the regulated output is the also the output error, then the problem reduces to a set of uncoupled single-input single-output $\mathcal{H}_2$ optimisation problems- the solution of which is trivial. However, the interpretation of the *results* is more interesting [2].

3.2. Solution With Diagonal Restriction. The solution to the optimisation problem (3.3) when there is a diagonal restriction relies on three key facts. Firstly, we note that the $\mathcal{H}_2$ norm of an arbitrary $p{\times}m$ matrix is the same as that of another $pm{\times}1$ *vector* formed from the matrix by stacking columns. Secondly, we can expand $G(s)Q_d(s)H(s)$ as the sum of products of fixed transfer matrices and free scalar transfer functions. We can then convert the problem into an $\mathcal{H}_2$ transfer matrix matching problem where the parameter vector may have a lower rank than the target function. Thirdly we also use the fact that for every tall matrix there exists a (stable) unitary matrix, some of the columns of which are orthogonal to the tall matrix, so that premultiplication yields a square matrix of entries above a matrix of null entries.

3.2.1. Main Results: Isometric Isomorphism. We formalise these key facts with three following lemmas. The first lemma is about the relationship between a matrix and a vector formed from that matrix by stacking columns.

Lemma 3.2. *Consider the map given by* $\mathcal{M} : \mathcal{H}_2^{a\times b} \to \mathcal{H}_2^{ab\times 1}$ *as* $[\mathcal{M}(G(s))]_{ai+j,1} = [G(s)]_{i,j}$. *The map* $\mathcal{M}$ *is an isometric isomorphism, so that* $||\mathcal{M}(G(s))||_2 = ||G(s)||_2$.

Proof. The proof follows easily from the definition of $\mathcal{M}$ and the $\mathcal{H}_2$ norm. See [3]. □

3.2.2. Diadic Expansion. The next lemma exploits the previous result and allows us to re-express the problem as an (extended) $\mathcal{H}_2$ model matching problem with a target function with more elements than the free parameter.

Lemma 3.3. *The* $\mathcal{H}_2$ *norm of* $T_{zw}(s)$ *of equation (3.1) in theorem 3.1 may be expressed in the following form:*

$$||T_{zw}(s)||_2 = ||f(s) + R(s)q(s)||_2$$

where $f(s)$ *is a* $qr\times 1$ *vector of transfer function matrices* $R(s)$ *is in* $\mathcal{H}_\infty^{qr\times(n+1)p}$ *and* $q(s)$ *is an arbitrary matrix in* $\mathcal{H}_\infty^{(n+1)p\times 1}$.

Proof. Follows readily from the application of lemmas 3.1 and 3.2. See [3]. □

The above result implies that we can translate the problem to an equivalent one of

$$J^\star = \min_{q\in\mathcal{H}_2} ||f(s) - R(s)q(s)||_2 \tag{3.4}$$

where $f(s)$ is a fixed transfer vector, $R(s)$ is a tall matrix and $q(s)$ is a free vector of stable transfer functions.

3.2.3. Unitary Premultiplier. This next lemma is required in order to solve the extended model matching problem (3.4).

Lemma 3.4. *For every tall matrix $E(s) \in \mathcal{L}_\infty^{a \times b}$, with $a > b$ there exists a unitary $U(s) \in \mathcal{H}_\infty^{a \times a}$ such that $U(s)^* U(s) = I$ and*

$$U(s)E(s) = \begin{bmatrix} \bar{E}(s) \\ O^{(a-b)\times b} \end{bmatrix},$$

where $\bar{E}(s) \in \mathcal{L}_\infty^{b \times b}$ and $O \in \mathcal{H}_\infty^{(a-b)\times b}$ is an identically zero matrix.

Proof. See [3] for a constructive proof. □

REMARK 3.1. The statement of this lemma is reminiscent of theorem 2 of [6], but with bicausal (unimodular) strengthened to unitary.

3.2.4. Conversion to Standard Problem. The above results allow us to describe a procedure for determining the optimal $\mathcal{H}_2$ cost subject to the decoupling constraint as well as finding a controller which acheives that optimal cost. This problem is solvable by an extension of, for example, the "tall matrix filtering problem" in [19]. The necessary algorithm extension follows.

Note that $R(s)$ has rank at most $(n+1)p$. Since $R(s)$ is a tall matrix then by lemma 3.4, there exists a unitary $qr \times qr$ matrix, $V(s) \in \mathcal{H}_\infty$, such that $V(s)R(s)$ has its last $qr-(n+1)p$ rows identically zero and first $(n+1)p$ rows equal to some $\bar{R}(s)$. Multiplication by a unitary matrix $V(s)$ preserves the $\mathcal{H}_2$ norm, so that

$$J^\star = \inf_{q(s)\in\mathcal{H}_2} \left\| V(s)f(s) - \begin{bmatrix} \bar{R}(s) \\ O^{qr-(n-1)p} \end{bmatrix} q(s) \right\|_2$$

Now let $f^v(s) :\overset{\text{def}}{=} V(s)f(s)$ be decomposed compatibly as $f^v(s) = \begin{bmatrix} f_u^v(s)^* & f_l^v(s)^* \end{bmatrix}^*$. Here $f_u^v(s)$ is composed of the upper $(n+1)p$ elements of $f^v(s)$ and $f_l^v(s)$ is composed of the lower elements. It is then obvious that

$$\begin{aligned} J^\star &= \inf_{q(s)\in\mathcal{H}_2} \left\| \begin{bmatrix} f_u^v(s) \\ f_l^v(s) \end{bmatrix} - \begin{bmatrix} \bar{R}(s) \\ O^{qr-(n-1)p} \end{bmatrix} q(s) \right\|_2, \\ &= \|f_l^v(s)\|_2 + \inf_{q(s)\in\mathcal{H}_2} \left\| f_u^v(s) - \bar{R}(s)q(s) \right\|_2 . \end{aligned}$$

The first term in the above is independent of $q(s)$ so we now seek the solution to a new problem, namely $\bar{J}^\star :\overset{\text{def}}{=} \inf_{q(s)\in\mathcal{H}_2} \|f_u^v(s) - \bar{R}(s)q(s)\|_2$. We have thus converted a problem where the number of parameter elements was less than the number of target function elements, into a more standard problem [19] where this is no longer true.

3.2.5. Solution of the Standard Problem. This problem is now soluble by the techniques in [19] as follows. We perform an inner-outer factorisation on $\bar{R}(s)$ leading to $\bar{R}(s) = R_i(s)R_o(s)$, where $R_i(s), R_o(s) \in \mathcal{H}_\infty$, $R_i^*(s)R_i(s) = I$ and $R_o(s)$ is outer. Premultiply the normed quantity in the previous expression by the norm-preserving $R_i^*(s) \in \mathcal{H}_2^\perp$ to give $\bar{J}^\star :\overset{\text{def}}{=} \inf_{q(s)\in\mathcal{H}_2} \|R_i^*(s)f_u^v(s) - \bar{R}(s)q(s)\|_2$. We decompose $f^u(s) :\overset{\text{def}}{=} R_i^*(s) f_u^v(s)$ into $f^u(s) = f_u^u(s) + f_s^u(s)$ where $f_u^u(s) \in \mathcal{H}_2^\perp$.

$$\begin{aligned} \text{It follows that } \bar{J}^\star :\overset{\text{def}}{=} & \inf_{q(s)\in\mathcal{H}_\infty} \|f_u^u(s) + f_s^u(s) - R_o(s)q(s)\|_2 \\ = & \|f_u^u(s)\|_2 + \inf_{q(s)\in\mathcal{H}_\infty} \|f_s^u(s) - R_o(s)q(s)\|_2. \end{aligned}$$

Since $R_o(s)$ is outer, it has a right inverse in $\mathcal{H}_\infty$ and $\inf_{q(s)\in\mathcal{H}_\infty} \|f_s^u(s) - R_o(s)q(s)\|_2 = 0$; the latter is established in [4]. One $q(s)$ which acheives the infimum is given by $q^\star(s) = R_o^{inv}(s) f_s^u(s)$ where $R_o^{inv}(s)$ is any right inverse of $R_o(s)$. This $q^\star(s)$ may be improper, however similarly to [4], it possible to approach $q^\star(s)$ arbitrarily closely in the $\mathcal{H}_2$ norm by a sequence of proper $q_k(s) = q^\star(s)(\epsilon_k s+1)^{-d_\infty}$ for some integer d_∞ and a sequence of positive $\epsilon_k \to 0$.

The optimal $Q_d(s)^\star, \bar{Q}^\star(s)$ for problem (3.3) corresponds to $q^\star(s)$ according to the inverse of the (isometric) map $\mathcal{M}_Q : \mathcal{H}_\infty^{n\times p} \bigoplus \mathcal{D}^{p\times p} \to \mathcal{H}_\infty^{(n+1)p\times 1}$, given by

$$\mathcal{M}_Q\left(\bar{Q}(s), Q_d(s)\right) = q(s),$$

$$\text{where } [q(s)]_{p(i-1)+j,1} = \begin{cases} \left[\bar{Q}(s)\right]_{ij} & \text{for} \quad 1\le i\le n, \quad 1\le j\le p \\ \left[Q_d(s)\right]_{jj} & \text{for} \quad i=n+1, \quad 1\le j\le p \end{cases}.$$

This means that $\left\{\bar{Q}(s), Q_d(s)\right\} = \mathcal{M}_Q^{-1}\left(q(s)\right)$

$$\text{is given by } \bar{Q}(s) = \begin{bmatrix} q_1(s) & q_2(s) & \cdots & q_p(s) \\ q_{p+1}(s) & q_{p+2}(s) & \cdots & q_{2p}(s) \\ \vdots & & \ddots & \vdots \\ q_{(n-1)p+1}(s) & q_{(n-1)p+2}(s) & \cdots & q_{np}(s) \end{bmatrix},$$

$$Q_d(s) = \operatorname{diag}\left\{q_{np+1}(s), q_{np+2}(s), \ldots, q_{(n+1)p}(s)\right\}.$$

We can now substitute an optimal $Q_d(s), \bar{Q}(s)$ into the expression for $\mathcal{K}_{uy}(s)$ to give us an optimal controller. This enables us to calculate the cost of requiring a controller to be decoupling in terms of achievable $\mathcal{H}_2$ performance. We have thus developed a solution to the "Optimal $\mathcal{H}_2$ Decoupling Problem".

4. Example. Consider the following open loop plant $\mathcal{P}_{yu}(s)$.

$$\begin{aligned} \mathcal{P}_{yu}(s) &= \begin{bmatrix} \frac{1-s}{(s+1)(s-2)} & \frac{s+3}{(s+1)(s-2)} \\ \frac{1-s}{(s+1)(s+2)} & \frac{2(s+4)}{(s+2)(s-2)} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1-s}{(s+3)(s+1)} & \frac{1}{s+1} \\ \frac{-(s-2)(s-1)}{(s+1)(s+2)(s+3)} & \frac{2(s+4)}{(s+2)(s+3)} \end{bmatrix} \begin{bmatrix} \frac{s-2}{s+3} & 0 \\ 0 & \frac{s-2}{s+3} \end{bmatrix}^{-1} \\ &= \begin{bmatrix} \frac{s-2}{s+3} & 0 \\ 0 & \frac{s-2}{s+3} \end{bmatrix}^{-1} \begin{bmatrix} \frac{(-s+1)}{(s+3)(s+1)} & \frac{1}{s+1} \\ \frac{-(s-2)(s-1)}{(s+1)(s+2)(s+3)} & \frac{2(s+4)}{(s+2)(s+3)} \end{bmatrix} \end{aligned}$$

The system has a non-minimum phase (NMP) zero at $s = 1$ with direction $\begin{bmatrix} 5 & -3 \end{bmatrix}$. It also has an unstable pole with geometric multiplicity two at $s = 2$ with directions $\begin{bmatrix} 1 & 0 \end{bmatrix}^*$ and $\begin{bmatrix} 0 & 1 \end{bmatrix}^*$ Decoupling does not result in spreading of poles because the associated directions are *canonical*, but it does result in zero spreading.

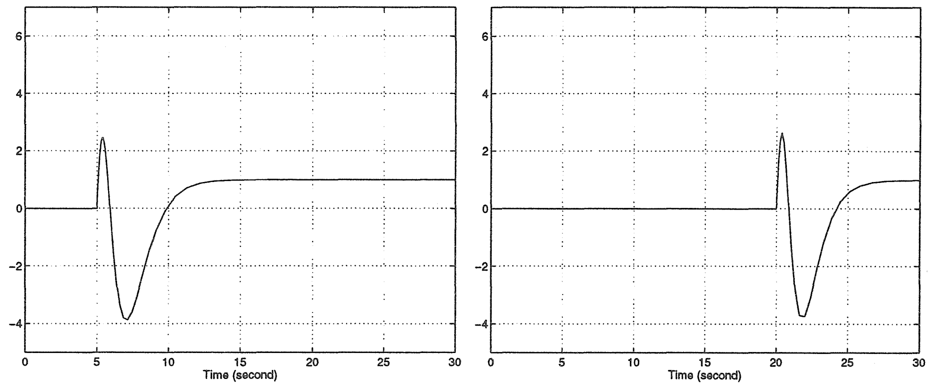

FIG. 4.1. *Optimum Decoupled Step Response*

4.1. A decoupling controller. We next find a particular (decoupling) controller $\mathcal{K}_{uy}(s)$. We let

$$\mathcal{K}_{uy}(s) = \frac{3(s-14)(s+1)}{(s+11)(s+7)} \begin{bmatrix} \frac{2(s+1)(s+4)}{(s+2)(s-2)} & \frac{(-s+3)}{(s-2)} \\ \frac{s-1}{(s-2)} & \frac{-(s-1)}{(s-2)} \end{bmatrix},$$

which gives $\mathcal{P}_{yu}(s)\mathcal{K}_{uy}(s)$ diagonal and $(I + \mathcal{P}_{yu}\mathcal{K}_{uy})^{-1}$ stable. This was found using the model reference dynamic decoupling MIMO design methods presented in [12] and enables us to find a parametrisation of all stabilizing and decoupling controllers in terms of a coprime factorisation of $\mathcal{K}_{uy}(s)$ and a free parameter $Q(s)$.

The cost function is given by $J = \sum_{i=1}^{2} \int_0^\infty y^i(t)^T y^i(t) + \alpha \cdot \bar{u}^i(t)^T \bar{u}^i(t)\, dt$, where $y^i(t)$ is the response to to a step disturbance on output channel i . Here $\bar{u}^i(t)$ is the difference between the control at each point in time and the steady state u^i_{ss} required to reject each disturbance and α is a weight on the control transient relative to the output transient. This can be expressed in the generic state space format as

$$\begin{bmatrix} z \\ \hline y \end{bmatrix} = \left[\begin{array}{c|c} s^{-1}I & -\mathcal{P}_{yu}(s) \\ \alpha \cdot s^{-1}[\mathcal{P}_{yu}(0)]^{-1} & -\alpha I \\ \hline s^{-1}I & -\mathcal{P}_{yu}(s) \end{array}\right] \begin{bmatrix} w \\ \hline u \end{bmatrix}.$$

In the above, the s^{-1} terms arise due to the step disturbances.

4.2. Optimal Responses. For $\alpha = 1$ and a decoupling constraint the optimal step response with reference steps at $t = 5$ and $t = 20$ appears as Figure 4.1. Notice that the responses are, in fact, decoupled from each other- so that a reference change for one output does not affect the other output. The minimum decoupled cost is calculated as $J^\star = 19.1$. For comparison, without a decoupling constraint, the optimal step response is as Figure 4.2 and the minimum cost is calculated a $J^\star = 17.0$.

Note that in both the diagonally constrained and unconstrained cases, the large undershoot and overshoot are due to the interactions of the unstable pole at $s = 2$ and the non-minimum-phase zero at $s = 1$. For this example, the cost is increased by 2.1, a percentage increase of a little more than 10%, by the decoupling requirement. In this case, we might say non-rigourously, that the predominant $\mathcal{H}_2$ cost is due to the pole-zero interaction, with little due to the decoupling requirement.

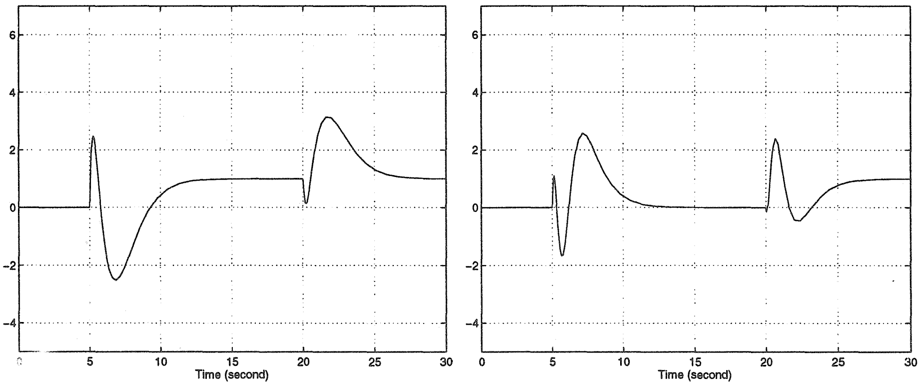

FIG. 4.2. *Unconstrained Optimum Step Response*

4.3. Further Research.

4.3.1. General Decoupling Problem. In the problem presented in this paper, the decoupling requirement was for the observed outputs. A more general problem would be to require the transfer function from a fixed but arbitrary set of input disturbances or references to a fixed but arbitrary set of decoupled outputs to be decoupled or partially decoupled and allow the set of observed available to the controller to be different from the decoupled outputs.

It is true that if one such decoupling controller exists then it is possible, in an algebraic framework, to parametrise all general decoupling controllers in terms of that controller and a diagonal parameter $Q_d(s)$ (again with a free $\bar{Q}(s)$ for a non-square plant). Again, all acheivable transfer functions from the cost disturbance set to the cost regulated output set, can be expressed as $F(s) + G_d(s)Q_d(s)H_d(s) + \bar{G}(s)\bar{Q}(s)\bar{H}(s)$, so the same techniques for solving the optimal output decoupling problem can be applied to the general optimal decoupling problem. Research into the algebraic conditions for the existence and solution of a decoupling controller, may be beneficial.

4.3.2. Optimal Decoupling Problem: A Geometric framework. In addition to an algebraic approach to the optimal decoupling problem, another may be to use a state space (geometric) framework in which to solve the problem.

The open loop plant $\mathcal{G}(s)$, for the general decoupling problem, may be expressed as

$$\begin{aligned}
\dot{x} &= Ax + B_d w_d + B_c w_c + B_u u, \\
z_d &= C_d x + D_{dd} w_d + D_{du} u, \\
z_c &= C_c x + D_{cc} w_c + D_{cu} u, \\
y &= C_y x + D_{yd} w_d + D_{yc} w_c,
\end{aligned}$$

where y is the observed output and z_c is the (cost) regulated output and w_c is the (cost) disturbance input and the transfer function from w_d to z_d is diagonally constrained.

The concepts of controllability and observability subspaces [20], [17] have the potential to be used to solve the (general) decoupling problem. It is certainly possible to find the optimal decoupling controller with full state-feedback in a geometric framework: results which will appear in a later publication. The optimal decoupling problem with output feedback problem, however, remains to be solved.

4.3.3. Other related problems. The "almost decoupling" optimal control problem is that of finding the $\mathcal{H}_2$ optimal controller subject to a finite (non-zero) constraint on the off diagonal elements. One way of tackling this problem might be to use Lagrange multipliers in conjuction with the transformation from the Hilbert space of matrix transfer functions to the Hilbert space of transfer vectors. It is also possible to investigate minimising the $\mathcal{H}_\infty$ norm of $T_{zw}(s)$ subject to a decoupling constraint on a particular set of inputs to a set of outputs, in an "Optimal $\mathcal{H}_\infty$ Decoupling Problem".

5. Conclusion. We have defined an $\mathcal{H}_2$ optimal control problem with an output decoupling constraint and demonstrated how to reduce the problem into a standard $\mathcal{H}_2$ frequency domain optimisation for which there are known solutions. It is now possible to quantify the costs of the output decoupling constraint in terms of the $\mathcal{H}_2$ norm criterion. The results presented in this paper may provide a useful starting point for research into other related issues.

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