THE NEVANLINNA ERROR TERM FOR COVERINGS GENERICALLY SURJECTIVE CASE

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Nevanlinna theory **[Ne]** started as the theory of the value distribution of meromorphic functions. The so-called Second Main Theorem is a theorem relating how often a function is equal to a given value compared with how often, on average, it is close to that value. This theorem takes the form of an inequality relating the counting function and the mean proximity function by means of an error term. Historically, only the order of the error term was considered important, but motivated by Vojta's **[Vo]** dictionary between Nevanlinna theory and Diophantine approximations, Lang and others, see **[La]** and **[L-C]** for instance, have started to look more closely at the form of this error term.

Vojta has a number theoretic conjecture, analogous to the Second Main Theorem, where the absolute height of an algebraic point is bounded by an error term, which is independent of the degree of the point. This caused Lang to raise the question, "how does the degree of an analytic covering of **C** come into the error term in Nevanlinna theory?" The second part of **[L-C]** looks at Nevanlinna theory on coverings in order to answer this question. Noguchi **[No1]**, **[No2]**, and **[No3]** and Stoll **[St]** are among those who have previously looked at the Nevanlinna theory of coverings.

As part of Vojta's dictionary, the Nevanlinna characteristic function corresponds to the height of a rational point in projective space. For a number field F, there are two notions of height. There is a relative height and an absolute height. Given a point $P = (x_0, \ldots, x_n)$ in $\mathbf{P}^n(F)$, the *relative* height, $h_F(P)$ is defined by

$$h_F(P) = \sum_{v \in S} [F_v : \mathbf{Q}_v] \log \max_j |x_j|_v,$$

where S is the set of absolute values on F, and $[F_v : \mathbf{Q}_v]$ is the local degree. The *absolute* height $h(P)$ is the relative height divided by the global degree $[F : \mathbf{Q}]$ and is independent of the field F. The Nevanlinna characteristic function T_f, as defined in Part II of **[L-C]**, corresponds to the relative height. As such, one wanted a second main theorem where the degree enters into the error term only as a factor multiplied by a universal expression independent of the degree. This is more or less what was achieved when $T_f(r)$ was larger than the degree, but when $T_f(r)$ was less than the degree, we could not get such a result, and it appeared that the error term depended on the degree in a more subtle way. However, this is to be expected because the classical second main theorem only holds when $T_f(r)$ is greater than one, and the condition that the *relative* T_f be greater than the degree is precisely the condition that the *absolute* T_f be greater than one. The main objective of this note is to show that when the Nevanlinna functions on coverings are normalized from the beginning by dividing by the degree, then the error term is independent of the degree, completely in line with Vojta's conjecture in the number theoretic case, and all the extraneous terms in **[L-C]** disappear.

Furthermore, by making two minor changes to the method in **[L-C]**, following Griffiths-King **[G-K]**, we are able to work with non-degenerate holomorphic maps from an analytic covering of $\mathbf{C}^m$ into an n-complex dimensional manifold, where $m \geq n$. This is more general than the equidimensional case treated in **[L-C]** and shows that the error term retains the same structure when the dimension of the domain space is larger than that of the range.

The main result of this note is the following Second Main Theorem:

Theorem. *Let $p: Y \to \mathbf{C}^m$ be a finite normal analytic covering of $\mathbf{C}^m$ which is unramified and non-singular above zero. Let X be an n-complex dimensional manifold, and let $f: Y \to X$ be a non-degenerate holomorphic map such that the "ramification" divisor R_f does not intersect Y<0>.*

Let:

$D = \sum_{j=1}^{q} D_j$ *be a divisor with simple normal crossings of complexity k;*

$L_j = L_{D_j}$ *be the line bundle associated to D_j;*

ρ_j *be a hermitian metric on L_j;*

Ω be a volume form on X;
κ be the metric on the canonical bundle associated to Ω;
η be a positive $(1,1)$ form on X such that $\eta^n/n! \geq \Omega$ and $\eta \geq c_1(\rho_j)$ for all j;
Assume that $f(y) \notin D$ for all $y \in Y\langle 0\rangle$.
Let:

$$B = \frac{b^{1/n}}{n}((q+1)q^{k/n} + \frac{1}{2}q^{2+k/n}\log 2);$$
$$b_1 = b_1(F_{\gamma_f^{1/n}}) \quad \text{and} \quad r_1 = r_1(F_{\gamma_f^{1/n}}),$$

where b is the constant of Lemma II.7.4 in **[L-C]**, *and depends only on Ω, D and η. Then, one has*

$$T_{f,\kappa}(r) + \sum_{j=1}^{q} T_{f,\rho_j}(r) - N_{f,D}(r) + N_{R_f}(r) - N_{p,\mathrm{Ram}}(r)$$
$$\leq \frac{n}{2}S(BT_{f,\eta}^{1+k/n}, \psi, b_1, r) - \frac{1}{2}\sum_{y\in\langle 0\rangle} \frac{\log \gamma_f(y)}{[Y:\mathbf{C}^m]} + 1,$$

for $r \geq r_1$ outside of a set of measure $\leq 2b_0(\psi)$.

Remarks. The symbols above, including the divisor R_f, which is the Griffiths-King ramification term, will be precisely defined in the sequel. Note that except for an additive term, which can be made to disappear by normalizing f, the Jacobian of f and the Jacobian of p at the points which are above zero, the error term is completely uniform in the functions p and f as well as in the degree of the covering. Also, the extraneous terms involving the degree which appear in **[L-C]** are not present here. Furthermore, when the error term function is expanded out, the constant which appears in front of the $\log T_{f,\eta}$ term is

$$\frac{n}{2}\left(1+\frac{k}{n}\right)$$

which is better than the constant $n(n+1)$ appearing in Stoll **[St]**. The larger constants in Stoll result from his method of summing up projections onto Grassmannians via the "associated maps." By combining the equidimensional method used by Wong **[Wo]** and improved by Lang, with the ramification terms in Griffiths-King which depend on the choice of Jacobian section, rather than the Wronskian determinant which appears in Stoll, the error term obtained in the generically surjective case is identical to that of the equidimensional case, and, in particular, does not contain the unnecessary factors which arise from projective linear algebra.

For the proof of the above theorem, we follow Chapter IV of **[L-C]**.

1. Preliminaries. Let $p : Y \to \mathbf{C}^m$ be a finite normal analytic covering of $\mathbf{C}^m$, and assume that Y is non-singular at the points above zero and that p is also unramified above zero.
Let:

$$[Y : \mathbf{C}^m] = \text{the degree of the covering;}$$
$$z = (z_1, \dots, z_m) \text{ be the complex coordinates of } C^m;$$
$$\|z\|^2 = \sum_{j=1}^{m} z_j \bar{z}_j;$$
$$Y(r) = \{y \in Y : \|p(y)\| < r\};$$
$$Y[r] = \{y \in Y : \|p(y)\| \leq r\};$$
$$Y\langle r\rangle = \{y \in Y : \|p(y)\| = r\}.$$

Consider the following differential forms on $\mathbf{C}^m$:

$$\omega(z) = dd^c \log \|z\|^2;$$
$$\varphi(z) = dd^c \|z\|^2;$$
$$\sigma(z) = d^c \log \|z\|^2 \wedge \omega^{m-1}(z);$$
$$\Phi(z) = \prod_{j=1}^{m} \left(\frac{\sqrt{-1}}{2\pi} dz_j \wedge d\bar{z}_j \right).$$

The pullback of these forms to Y via p will be denoted by a subscript Y:

$$\omega_Y = p^*\omega, \quad \varphi_Y = p^*\varphi, \quad \sigma_Y = p^*\sigma, \quad \Phi_Y = p^*\Phi.$$

Note that σ_Y is closed and C^∞ away from $Y\langle 0\rangle$ and that

$$\int\limits_{Y\langle r\rangle} \sigma_Y = [Y : \mathbf{C}^m].$$

The following form of the Green-Jensen integral formula will be needed. For a proof, see **[L-C]** Theorem IV.1.2.

Theorem 1 (Green-Jensen Formula). *Let α be a C^2 function from $Y \to \mathbf{C}$ except on a negligible set of singularities Z such that $Z \cap Y\langle 0\rangle = \emptyset$. Assume, in addition, that the following three conditions are satisfied:*

i) *$\alpha\sigma_Y$ is absolutely integrable on $Y\langle r\rangle$ for all $r > 0$.*
ii) *$d\alpha \wedge \sigma_Y$ is absolutely integrable on $Y[r]$ for all r.*
iii) $\displaystyle\lim_{\varepsilon\to 0} \int\limits_{S(Z,\varepsilon)(r)} \alpha\sigma_Y = 0$ *for all r,*

where for sufficiently small ε, $S(Z,\varepsilon)(r)$ denotes the boundary of the tubular neighborhood of radius ε around the singularities $Z \cap Y[r]$, which is regular for all but a discrete set of values ε. Then

$$(A)\quad \int\limits_0^r \frac{dt}{t} \int\limits_{Y\langle t\rangle} d^c\alpha \wedge \omega_Y^{m-1} = \frac{1}{2} \int\limits_{Y\langle r\rangle} \alpha\sigma_Y - \frac{1}{2} \sum_{y \in Y\langle 0\rangle} \alpha(y),$$

and

$$(B)\quad \int\limits_0^r \frac{dt}{t} \int\limits_{Y(t)} dd^c\alpha \wedge \omega_Y^{m-1} + \int\limits_0^r \frac{dt}{t} \lim_{\varepsilon\to 0} \int\limits_{S(Z,\varepsilon)(t)} d^c\alpha \wedge \omega_Y^{m-1}$$
$$= \frac{1}{2} \int\limits_{Y\langle r\rangle} \alpha\sigma_Y - \frac{1}{2} \sum_{y \in Y\langle 0\rangle} \alpha(y).$$

Let $f : Y \to X$ be a non-degenerate (i.e. not contained in any divisor on X) holomorphic map, where X is a compact n-complex dimensional manifold and n is assumed less than or equal to m.

Remark. It is not necessary for the function f to be defined on all of Y. Everything in the sequel remains true for a function $f : Y(R) \to X$ provided that $r < R$.

We define the **absolute** Nevanlinna functions as follows:

Height

If η is a $(1,1)$ form on X, then define

$$T_{f,\eta}(r) = \frac{1}{[Y : \mathbf{C}^m]} \int_0^r \frac{dt}{t} \int_{Y(t)} f^*\eta \wedge \omega_Y^{m-1},$$

and similarly, given a hermitian metric ρ on a holomorphic line bundle L on X, define

$$T_{f,\rho}(r) = \frac{1}{[Y : \mathbf{C}^m]} \int_0^r \frac{dt}{t} \int_{Y(t)} f^*c_1(\rho) \wedge \omega_Y^{m-1},$$

where $c_1(\rho) = dd^c \log \rho$ is the **Chern form** of ρ.

Counting functions

Given a divisor D on Y, let

$$\mathbf{n}_D(t) = \frac{1}{[Y : \mathbf{C}^m]} \int_{D(t)} \omega_Y^{m-1} \quad \text{and} \quad N_D(r) = \int_0^r \mathbf{n}_D(t) \frac{dt}{t},$$

and given a divisor D on X, let $N_{f,D} = N_{f^*D}$. The counting function for the ramification divisor of p, defined locally by the zeros of the Jacobian matrix, will be denoted $N_{p,\mathrm{Ram}}(r)$.

A volume form Ω on X defines a metric κ on the canonical line bundle K of X. Since,

$$f^*\mathrm{Ric}\,\Omega = f^*c_1(\kappa),$$

the **height associated to the volume form** Ω is defined as:

$$T_{f,\kappa}(r) = \frac{1}{[Y : \mathbf{C}^m]} \int_0^r \frac{dt}{t} \int_{Y(t)} f^* c_1(\kappa) \wedge \omega_Y^{m-1}.$$

2. Ramification. Let Φ be the Euclidean volume form on $\mathbf{C}^m$ and let $\Phi_Y = p^*(\Phi)$ be the pullback to a pseudo-volume form on Y. Let Ω be a volume form on X. Because f is non-degenerate, we can assume that the coordinates on $\mathbf{C}^m$ were chosen so that

$$f^*\Omega \wedge p^*\left(\prod_{j=n+1}^{m} \frac{\sqrt{-1}}{2\pi} dz_j \wedge d\bar{z}_j \right)$$

is not identically zero. Following Griffiths and King [**G-K**], let γ_f be the non-negative function such that

$$f^*\Omega \wedge p^*\left(\prod_{j=n+1}^{m} \frac{\sqrt{-1}}{2\pi} dz_j \wedge d\bar{z}_j \right) = \gamma_f \Phi_Y.$$

Note that γ_f is singular along the ramification divisor of p and vanishes along the divisor R_f given by the equation

$$f^*\Omega \wedge p^*\left(\prod_{j=n+1}^{m} \frac{\sqrt{-1}}{2\pi} dz_j \wedge d\bar{z}_j \right) = 0.$$

Remark. When $n = m$, the divisor R_f is the ramification divisor associated to the map f. In general, the divisor R_f depends not only on the ramification of f, but also on the choice of coordinates on $\mathbf{C}^m$. However, this dependence on the choice of coordinates is omitted from the notation.

Note that because $f^* c_1(\kappa) = dd^c \log \gamma_f$, one has

$$T_{f,\kappa}(r) = \frac{1}{[Y : \mathbf{C}^m]} \int_0^r \frac{dt}{t} \int_{Y(t)} dd^c \log \gamma_f \wedge \omega_Y^{m-1}.$$

Theorem 2. *Assume that $p : Y \to \mathbf{C}^m$ is unramified above zero, and let $f : Y \to X$ be a non-degenerate holomorphic map such that the divisor R_f does not intersect Y<0>. Then*

$$T_{f,\kappa}(r) + N_{R_f}(r) - N_{p,\mathrm{Ram}}(r) = \frac{1}{2} \int\limits_{Y<r>} (\log \gamma_f) \frac{\sigma_Y}{[Y : \mathbf{C}^m]} - \frac{1}{2} \sum_{y \in Y<0>} \frac{\log \gamma_f(y)}{[Y : \mathbf{C}^m]}.$$

Proof: With new notation, this is simply Theorem 1 (**B**) combined with the fact that

$$\lim_{\varepsilon \to 0} \int\limits_{S(Z,\varepsilon)(t)} d^c \log \gamma_f \wedge \omega_Y^{m-1} = \int\limits_{Z(t)} \omega_Y^{m-1},$$

where Z is the set of singularities for $\log \gamma_f$, and then divided by the degree.

3. Calculus Lemmas. Let ψ be a positive increasing function, such that

$$\int_e^\infty \frac{du}{u\psi(u)} = b_0(\psi)$$

is finite. Such a function is called a **type function.** Given a positive increasing function F, let $r_1(F)$ be the smallest number such that $F(r) \geq e$ for $r \geq r_1(F)$, and let $b_1(F)$ be the smallest number greater than or equal to one, such that

$$b_1 r^{2m-1} F'(r) \geq e \quad \text{for all} \quad r \geq r_1(F).$$

Define the **error term** function to be

$$S(F, b_1, \psi, r) = \log \{F(r)\psi(F(r))\psi(r^{2m-1} b_1 F(r)\psi(F(r)))\}.$$

Given a function α on Y, define the **height transform**:

$$F_\alpha(r) = \frac{1}{[Y : \mathbf{C}^m]} \int_0^r \frac{dt}{t^{2m-1}} \int_{Y(t)} \alpha \Phi_Y$$

for $r > 0$.

Let α be a function on Y such that the following conditions are satisfied:

(a) α is continuous and > 0 except on a divisor of Y.

(b) For each r, the integral $\int_{Y<r>} \alpha \sigma_Y$ is absolutely convergent and $r \mapsto \int_{Y<r>} \alpha \sigma_Y$ is a piecewise continuous function of r.

(c) There is an $r_1 \geq 1$ such that $F_\alpha(r_1) \geq e$.

Note: F_α has positive derivative, so is strictly increasing.

Lemma 3. *If α satisfies* (a), (b) *and* (c) *above, then F_α is C^2 and*

$$\frac{1}{r^{2m-1}} \frac{d}{dr}(r^{2m-1} F'_\alpha(r)) = \frac{2}{(m-1)!} \int_{Y<r>} \alpha \frac{\sigma_Y}{[Y : \mathbf{C}^m]}.$$

Proof: Use Fubini's Theorem and the fact that

$$\Phi_Y = \frac{||p||^{2(m-1)}}{(m-1)!} d||p||^2 \wedge \sigma_Y,$$

as in Chapter IV §3, and then divide by the degree.

The standard Nevanlinna calculus lemma then gives

Lemma 4. *If α satisfies* (a), (b) *and* (c) *above, then*

$$\log \int_{Y<r>} \alpha \frac{\sigma_Y}{[Y : \mathbf{C}^m]} \leq S(F_\alpha, b_1(F_\alpha), \psi, r) + \log \frac{(m-1)!}{2}$$

for all $r \geq r_1(F_\alpha)$ outside a set of measure $\leq 2b_0(\psi)$.

4. Trace and Determinant. Given a $(1,1)$ form η on Y, define the **trace** and **determinant** outside the ramification points of p as follows:

$$(\det(\eta))\Phi_Y = \frac{1}{m!}\eta^m$$
$$(m-1)!\,\mathrm{tr}(\eta)\Phi_Y = \eta \wedge \varphi_Y^{m-1}.$$

Furthermore, define the **n** $\times$ **n trace** and **determinant** outside the ramification points of p as follows:

$$(\det_n(\eta))\Phi_Y = \frac{1}{n!}\eta^n \wedge p^*\left(\prod_{j=n+1}^{m} \frac{\sqrt{-1}}{2\pi} dz_j \wedge d\bar{z}_j\right)$$
$$(n-1)!\,\mathrm{tr}_n(\eta)\Phi_Y = \eta \wedge \varphi_Y^{n-1} \wedge p^*\left(\prod_{j=n+1}^{m} \frac{\sqrt{-1}}{2\pi} dz_j \wedge d\bar{z}_j\right).$$

In the case when Y is $\mathbf{C}^m$ and p is the identity, the $n \times n$ trace and determinant are simply the trace and determinant of the $n \times n$ block in the upper-left of the matrix corresponding to η.

The following lemma is simply the pull-back to Y of some relations on $\mathbf{C}^m$, which follow immediately from the elementary linear algebra of hermitian positive semi-definite matrices.

Lemma 5. *If η is a semi-positive $(1,1)$ form on Y, then*

$$(\det_n(\eta))^{1/n} \le \frac{1}{n}\,\mathrm{tr}_n(\eta) \quad \textit{and} \quad \mathrm{tr}_n(\eta) \le \mathrm{tr}\,(\eta)$$

for the regular points in Y which are not ramification points of p.

Let η be a closed, positive $(1,1)$ form such that

$$\Omega = \frac{1}{n!}\eta^n.$$

Since

$$f^*\Omega \wedge p^*\left(\prod_{j=n+1}^{m} \frac{\sqrt{-1}}{2\pi} dz_j \wedge d\bar{z}_j\right) = \gamma_f \Phi_Y,$$

one finds that $\gamma_f = \det_n(f^*\eta)$.

Proposition 6. *Let* $\tau_f = \mathrm{tr}_n(f^*\eta)$. *Then*

$$(m-1)!F_{\tau_f} \leq T_{f,n}.$$

Proof: Let $\tau_f' = \mathrm{tr}\,(f^*\eta)$. All the symbols have been defined so that the proof of Proposition II.6.2 in [**L-C**], after dividing through by the degree, gives

$$T_{f,\eta} = (m-1)!F_{\tau_f'}.$$

But, since $\mathrm{tr}_n(f^*\eta) \leq \mathrm{tr}\,(f^*\eta)$, one has $F_{\tau_f} \leq F_{\tau_f'}$.

5. Second Main Theorem. Replacing the counterparts to the statements above in the proof of Theorem IV.4.3 in [**L-C**] gives the following Second Main Theorem.

Theorem 7. *Assume that* $p : Y \to \mathbf{C}^m$ *is unramified above zero, and let* $f : Y \to X$ *be a non-degenerate holomorphic map such that the divisor* R_f *does not intersect* $Y{<}0{>}$. *Let* $T_{f,\kappa}$ *be the height associated to the volume form* $\Omega = \eta^n/n!$ *on* X. *Then*

$$\begin{aligned} &T_{f,\kappa}(r) + N_{R_f}(r) - N_{p,\mathrm{Ram}}(r) \\ &\qquad \leq \frac{n}{2}S(T_{f,\eta}, b_1(F_{\tau_f}), \psi, r) - \frac{1}{2}\sum_{y\in Y<0>} \frac{\log\gamma_f(y)}{[Y:\mathbf{C}^m]} \end{aligned}$$

for all $r \geq r_1(F_{\tau_f})$ *outside a set of measure* $\leq 2b_0(\psi)$.

Proof:

$$\begin{aligned} &T_{f,\kappa}(r) + N_{f,\mathrm{Ram}}(r) - N_{p,\mathrm{Ram}}(r) + \frac{1}{2}\sum_{y\in Y<0>} \frac{\log\gamma_f(y)}{[Y:\mathbf{C}^m]} \\ &\quad = \frac{1}{2}\int\limits_{Y<r>} (\log\gamma_f)\frac{\sigma_Y}{[Y:\mathbf{C}^m]} \qquad \text{[Theorem 2]} \\ &\quad = \frac{n}{2}\int\limits_{Y<r>} \log\gamma_f^{1/n}\frac{\sigma_Y}{[Y:\mathbf{C}^m]} \end{aligned}$$

$$\leq \frac{n}{2}\log \int\limits_{Y\langle r\rangle} \gamma_f^{1/n} \frac{\sigma_Y}{[Y:\mathbf{C}^m]}$$

[concavity of the log]

$$\leq \frac{n}{2}\log \int\limits_{Y\langle r\rangle} \tau_f \frac{\sigma_Y}{[Y:\mathbf{C}^m]}$$

[Lemma 5]

$$\leq \frac{n}{2}S(F_{\tau_f}, b_1(F_{\tau_f}), \psi, r) + \frac{n}{2}\log(m-1)!$$

[Lemma 4]

$$\leq \frac{n}{2}S(T_{f,\eta}, b_1(F_{\tau_f}), \psi, r)$$

[Proposition 6]

for all $r \geq r_1(F_{\tau_f})$ outside a set of measure $\leq 2b_0(\psi)$.

Remarks. The term on the right involving $\log \gamma_f$ in the above inequality depends only on the values of f, the Jacobian of f, and the Jacobian of p above zero, so if these functions are normalized at the points above zero, then the right hand side is uniform in the functions f and p and in the degree. In the case when $m = n$, then $F_{\tau_f} = T_{f,\eta}/(n-1)!$, so $r_1(F_{\tau_f}) = r_1(T_{f,\eta}/(n-1)!)$.

Similar changes give the more general Second Main Theorem. Recall that a divisor is said to have simple normal crossings of complexity k if k is the minimal number such that there exist local coordinates $w_1, \ldots, w_n$ around each point of D, such that D is defined locally by $w_1 \ldots w_l = 0$, with $l \leq k$.

For the rest of this section, let:

$D = \sum_{j=1}^{q} D_j$ be a divisor on X with simple normal crossings of complexity k;

$L_j = L_{D_j}$ the holomorphic line bundle associated to D_j with hermitian metric ρ_j;

η be a closed, positive $(1,1)$ form on X such that $\eta \geq c_1(\rho_j)$ for all j, and $\eta^n/n! \geq \Omega$;

s_j be a holomorphic section of L_j, such that $(s_j) = D_j$;

Since X is compact, after possibly multiplying s_j by a constant, assume without loss of generality that

$$|s_j|_{\rho_j} \leq 1/e \leq 1.$$

For convenience, also assume that $f(y) \notin D$ for all $y \in Y\langle 0\rangle$, and that $Y\langle 0\rangle$ does not intersect the ramification divisor of f.

If λ is a constant with $0 < \lambda < 1$, then define the **Ahlfors-Wong** singular volume form

$$\Omega(D)_\lambda = \left(\prod |s_j|_j^{-2(1-\lambda)}\right)\Omega,$$

and define

$$\gamma_\lambda = \prod |s_j \circ f|_j^{-2(1-\lambda)}\gamma_f.$$

Given Λ a positive decreasing function of r with $0 < \Lambda < 1$, define

$$\gamma_\Lambda = \prod |s_j \circ f|_j^{-2(1-\Lambda)}\gamma_f.$$

Note that because of the assumption $|s_j|_j \leq 1/e \leq 1$, one has $\gamma_f \leq \gamma_\Lambda$.

Using the fact that $\mathrm{tr}_n \leq \mathrm{tr}$ and dividing through by the degree in the proof of Lemma IV.5.1 in **[L-C]** gives the following lemma.

Lemma 8. *Let b be the constant of Lemma II.7.4 in* **[L-C]**, *which depends only on Ω, D and η. Then for any decreasing function Λ with $0 < \Lambda < 1$, one has*

$$F_{\gamma_\Lambda^{1/n}}(r) \leq (q+1)\frac{b^{1/n}}{n(m-1)!}\frac{T_{f,\eta}(r)}{(\Lambda(r))^{k/n}} + \frac{qb^{1/n}}{2n(m-1)!}\frac{\log 2}{(\Lambda(r))^{1+k/n}}$$

for all r.

Remark. Notice that the degree no longer appears in this estimate, and this is why the error term is now uniform. Also note that the $n!$ in the denominator has been replaced with $n(m-1)!$.

Let $r_1 = r_1(F_{\gamma_f^{1/n}})$ and let

$$\Lambda(r) = \begin{cases} \dfrac{1}{qT_{f,\eta}(r)} & \text{for } r \geq r_1 \\ \text{constant} & \text{for } r \leq r_1. \end{cases}$$

Note that since $\eta^n/n! \geq \Omega$, one has $F_{\gamma_f^{1/n}} \leq T_{f,\eta}/n!$. Therefore $r_1(F_{\gamma_f^{1/n}}) \geq r_1(T_{f,\eta}/n!)$, and hence one has $\Lambda \leq 1$.

Applying Lemma 8 to the function Λ proves the next lemma.

Lemma 9. *Let b be the constant of Lemma II.7.4 of* [**L-C**] *and let*

$$B = \frac{b^{1/n}}{n}((q+1)q^{k/n} + \frac{1}{2}q^{2+k/n}\log 2).$$

Then

$$F_{\gamma_\Lambda^{1/n}}(r) \leq \frac{B}{(m-1)!}T_{f,\eta}^{1+k/\eta}$$

for $r \geq r_1$.

Lemma 10. *One has*

$$\log \int\limits_{Y<r>} \gamma_\Lambda^{1/n} \frac{\sigma_Y}{[Y : \mathbf{C}^m]} \leq S(BT_{f,\eta}^{1+k/n}, b_1, \psi, r)$$

for all $r \geq r_1$, outside a set of measure $\leq 2b_0(\psi)$, where

$$B = \frac{b^{1/n}}{n}((q+1)q^{k/n} + \frac{1}{2}q^{2+k/n}\log 2)$$
$$b_1 = b_1(F_{\gamma_f^{1/n}}) \quad \textit{and} \quad r_1 = r_1(F_{\gamma_f^{1/n}}).$$

Proof: Because $\gamma_\Lambda \geq \gamma_f$, one has

$$F_{\gamma_\Lambda^{1/n}} \geq F_{\gamma_f^{1/n}} \quad \text{and} \quad F'_{\gamma_\Lambda^{1/n}} \geq F'_{\gamma_f^{1/n}}.$$

Hence $b_1 = b_1(F_{\gamma_f^{1/n}})$ and $r_1 = r_1(F_{\gamma_f^{1/n}})$ are such that for $r \geq r_1$,

$$F_{\gamma_\Lambda^{1/n}}(r) \geq e \quad \text{and} \quad b_1 r^{2n-1} F'_{\gamma_\Lambda^{1/n}}(r) \geq e.$$

From Lemma 4, one has

$$\log \int_{Y<r>} \gamma_\Lambda^{1/n} \frac{\sigma_Y}{[Y : \mathbf{C}^m]} \leq S(F_{\gamma_\Lambda^{1/n}}, b_1, \psi, r) + \log \frac{(m-1)!}{2}$$

for $r \geq r_1$ outside an exceptional set of measure $\leq 2b_0(\psi)$. Now from Lemma 9, one has

$$S(F_{\gamma_\Lambda^{1/n}}, b_1, \psi, r) + \log \frac{(m-1)!}{2} \leq S(BT_{f,\eta}^{1+k/n}, b_1, \psi, r)$$

for $r \geq r_1$.

Finally, we can prove the general Second Main Theorem.

Theorem 11. *One has*

$$\begin{aligned} T_{f,\kappa}(r) &+ \sum_{j=1}^{q} T_{f,\rho_j}(r) - N_{f,D}(r) + N_{R_f}(r) - N_{p,\mathrm{Ram}}(r) \\ &\leq \frac{n}{2} S(BT_{f,\eta}^{1+k/n}, \psi, b_1, r) - \frac{1}{2} \sum_{y \in Y<0>} \frac{\log \gamma_f(y)}{[Y : \mathbf{C}^m]} + 1, \end{aligned}$$

for $r \geq r_1$ outside of a set of measure $\leq 2b_0(\psi)$.

Proof: Let λ be a constant with $0 < \lambda < 1$. Using Theorem 1 (B), and the fact that $dd^c \log$ transforms products into sums, one obtains:

$$\begin{aligned} T_{f,\kappa}(r) + (1-\lambda) \sum_{j=1}^{q} T_{f,\rho_j}(r) - (1-\lambda) \sum_{j=1}^{q} N_{f,D_j}(r) \\ + N_{R_f}(r) - N_{p,\mathrm{Ram}}(r) \end{aligned}$$

$$= \frac{1}{[Y:\mathbf{C}^m]} \int_0^r \frac{dt}{t} \int_{Y(t)} dd^c \log \gamma_\lambda + \frac{1}{[Y:\mathbf{C}^m]} \int_0^r \frac{dt}{t} \lim_{\epsilon \to 0} \int_{S(Z,\epsilon)(t)} d^c \log \gamma_\lambda$$

$$= \frac{n}{2} \int_{Y<r>} \log \gamma_\lambda^{1/n} \frac{\sigma_Y}{[Y:\mathbf{C}^m]} - \frac{1}{2} \sum_{y \in Y<0>} \frac{\log \gamma_\lambda(y)}{[Y:\mathbf{C}^m]}.$$

Because of the assumption that $|s_j|_j \leq 1$, one also has

$$-\frac{1}{2} \sum_{y \in Y<0>} \log \gamma_\lambda(y) \leq -\frac{1}{2} \sum_{y \in Y<0>} \log \gamma_f(y).$$

Also, since Λ is constant on $Y<r>$, the function Λ can replace λ in the above equality. Furthermore, $N_{f,D_j} \geq 0$ and $-1 \leq -(1-\lambda)$, so the factor $(1-\lambda)$ in front can be deleted. When $r \geq r_1$, one has

$$-1 \leq -\Lambda(r) \sum_{j=1}^{q} T_{f,\rho_j}(r),$$

from the definition of Λ, and from the fact that η was chosen so that

$$T_f, \eta \geq T_{f,\rho_j} \quad \text{for all } j.$$

Finally, by moving the log out of the integral, one has

$$\int_{Y<r>} \log \gamma_{\Lambda(r)}^{1/n} \frac{\sigma_Y}{[Y:\mathbf{C}^m]} \leq \log \left(\int_{Y<r>} \gamma_{\Lambda(r)}^{1/n} \frac{\sigma_Y}{[Y:\mathbf{C}^m]} \right).$$

Applying the estimate in Lemma 10 for Λ to the term with the integral on the right and collecting terms concludes the proof of the theorem.

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