We now discharge our obligation to show that various of the structures we encountered in §11 are iterable. We shall concentrate on proving Lemma 11.1, which states, in the language of §11, that $\mathcal{N}_\eta$ is reliable for all η. The other iterability lemmas from §11 are proved in almost the same way. A complete proof of these lemmas will be given in the paper [S?a].

As we observed in §11, it is enough to show

Theorem 12.1. *Let $\mathcal{N}_\eta$ be the ηth "$\mathcal{N}$-model" of the construction of §11. Let $0 \le k \le \omega$ and suppose $\mathfrak{C}_k(\mathcal{N}_\eta)$ exists. Then $\mathfrak{C}_k(\mathcal{N}_\eta)$ is k-iterable.*

PROOF. The proof of theorem 12.1 will take up all of this final section of the paper. Let

$$\mathcal{T} = \langle T, \deg, D, \langle E^{\mathcal{T}}_\alpha, \mathcal{P}^*_{\alpha+1} \mid \alpha + 1 < \theta \rangle \rangle$$

be a k-bounded, k-maximal iteration tree on

$$\mathcal{P}_0 = \mathfrak{C}_k(\mathcal{N}_\eta)\,.$$

The assumption of k-maximality is not necessary, but it simplifies the notation a bit, and we have never used non-maximal trees anyway. We let $\mathcal{P}_\alpha$ be the αth model of $\mathcal{T}$. Suppose that $\mathcal{T} \restriction \lambda$ is simple for all $\lambda < \theta$, and that θ is a limit ordinal. We shall show that $\mathcal{T}$ has a cofinal wellfounded branch.

For $\gamma < \eta$ such that Case 1 applied in the definition of $\mathcal{M}_{\gamma+1}$ from $\mathcal{M}_\gamma$, that is, such that $\mathcal{N}_{\gamma+1}$ is equal to $\mathcal{M}_\gamma$ expanded by a new predicate for a last extender, we let F^*_γ be the background extender for the last extender of $\mathcal{N}_{\gamma+1}$. Thus F^*_γ is $\nu + \omega$ strong, where $\nu = \nu^{\mathcal{N}_{\gamma+1}}$. Set

$$\mathbb{C} = (\langle \mathcal{N}_\gamma \mid \gamma \le \eta \rangle, \langle F^*_\gamma \mid \gamma < \eta \text{ and } F^*_\gamma \text{ defined} \rangle)\,.$$

Our strategy for the proof of theorem 12.1 is straightforward. We shall associate to $\mathcal{T}$ a tree $\mathcal{U}$ which will be an iteration tree on V in the sense of [MS]. As such the models of $\mathcal{U}$ will be well founded by results methods of [MS]. The tree ordering of $\mathcal{U}$ will be the same tree ordering, T, as $\mathcal{T}$, and we will define embeddings π_α from the models of $\mathcal{U}$ to those of $\mathcal{U}$. Thus the models of the tree $\mathcal{T}$ will also be well founded, which is what we need to show.

Since $\mathcal{U}$ is not a fine structure iteration tree it doesn't make sense to ask that $\vec{\pi}$ be a tree-embedding in the sense of section 5. However, if we let R_α be the αth model of $\mathcal{U}$ then the embeddings π_α will be embeddings from the αth model $\mathcal{P}_\alpha$ of $\mathcal{T}$ into $Q_\alpha = \mathfrak{C}_j(\mathcal{S}) \in R_\alpha$ where $\mathcal{S}$ is on the sequence of models of $i^{\mathcal{U}}_0(\mathbb{C})$ and $j = \deg(\alpha)$. If we modify the definition of a tree-embedding for this case by asking that π_α be a $(\deg^{\mathcal{T}}, Y_\alpha)$-embedding into Q_α instead of into R_α then $\vec{\pi}$ will satisfy this definition.

We must also maintain a certain amount of agreement between π_α and the π_β's for $\beta < \alpha$. We now state some definitions which allow us to describe this agreement.

DEFINITION 12.1.1. Let $\mathcal{M}$ be a premouse, and $\omega\lambda \leq \mathrm{OR}^{\mathcal{M}}$. Then the *$\lambda$-dropdown sequence of* $\mathcal{M}$ is the sequence $\langle\langle\beta_0, k_0\rangle, \ldots, \langle\beta_i, k_i\rangle\rangle$ defined as follows:

(1) $\langle\beta_0, k_0\rangle = \langle\lambda, 0\rangle$.
(2) $\langle\beta_{i+1}, k_{i+1}\rangle$ is the lexicographically least pair $\langle\beta, k\rangle$ such that $\lambda \leq \beta$, $\omega\beta \leq \mathrm{OR}^{\mathcal{M}}$, and $\rho_k(\mathcal{J}_\beta^{\mathcal{M}}) < \rho_{k_i}(\mathcal{J}_{\beta_i}^{\mathcal{M}})$.

If there is no such pair $\langle\beta, k\rangle$ then $\langle\beta_{i+1}, k_{i+1}\rangle$ is undefined. Let i be the largest integer such that $\langle\beta_i, k_i\rangle$ is defined.

Notice that if $\langle\langle\beta_e, k_e\rangle \mid e \leq i\rangle$ is the λ-dropdown sequence of $\mathcal{M}$, then $k_e < \omega$ for all $e \leq i$ and

$$\langle\beta_e, k_e\rangle <_{\mathrm{lex}} \langle\beta_{e+1}, k_{e+1}\rangle$$

for all $e < i$. Moreover every ordinal of the form $\rho = \rho_k(\mathcal{J}_\beta^{\mathcal{M}})$ for $k \in \omega$, $\beta\omega \leq \mathrm{OR}^{\mathcal{M}}$, and $\rho \leq \lambda \leq \beta$ is in the set $\{\, \rho_{k_e}(\mathcal{J}_{\beta_e}^{\mathcal{M}}) \mid e \leq i\,\}$.

Now we prepare to define the *(j, ξ)-resurrection sequence* for an extender E, where E is on the extender sequence of $\mathcal{M} = \mathfrak{C}_j(\mathcal{N}_\xi)$, the jth core of one of the models of our construction $\mathbb{C}$. We are allowing the possibility that $E = \dot{F}^{\mathcal{M}}$. The idea is just to trace E back to its origin as the last extender of some $\mathcal{N}_\gamma$ with $\gamma \leq \xi$.

Let $\lambda = \mathrm{lh}\, E$, and suppose that $\langle\langle\beta_0, k_0\rangle \cdots \langle\beta_i, k_i\rangle\rangle$ is the initial segment of the λ-dropdown sequence of $\mathcal{M}$ consisting of those pairs $\langle\beta, k\rangle$ on the sequence such that $\langle\beta, k\rangle \leq_{\mathrm{lex}} \langle\alpha, j\rangle$, where $\omega\alpha = \mathrm{OR}^{\mathcal{M}}$. Our first goal is to show that there is a unique $\gamma \leq \xi$ such that $\mathcal{J}_{\beta_i}^{\mathcal{M}} = \mathfrak{C}_{k_i}(\mathcal{N}_\gamma)$. Fix α such that $\omega\alpha = \mathrm{OR}^{\mathcal{M}}$ and let $\kappa = \rho_{k_i}(\mathcal{J}_{\beta_i}^{\mathcal{M}})$.

CLAIM 1. Let $\langle\gamma, e\rangle \leq_{\mathrm{lex}} \langle\xi, j\rangle$ and suppose $\mathcal{J}_{\beta_i}^{\mathcal{M}}$ is an initial segment of $\mathfrak{C}_e(\mathcal{N}_\gamma)$. Then for all $\langle\tau, n\rangle$ such that $\langle\gamma, e\rangle \leq_{\mathrm{lex}} \langle\tau, n\rangle \leq_{\mathrm{lex}} \langle\xi, j\rangle$, $\mathcal{J}_{\beta_i}^{\mathcal{M}}$ is an initial segment of $\mathfrak{C}_n(\mathcal{M}_\tau)$.

PROOF. Let $\kappa = \rho_{k_i}(\mathcal{J}_{\beta_i}^{\mathcal{M}})$, which is the minimum value of $\rho_k(\mathcal{J}_\beta^{\mathcal{M}})$ for pairs $\langle\beta, k\rangle$ satisfying $\langle\lambda, 0\rangle \leq_{\mathrm{lex}} \langle\beta, k\rangle \leq_{\mathrm{lex}} \langle\alpha, j\rangle$. Notice that $\mathcal{J}_{\beta_i}^{\mathcal{M}}$ is k_i-sound, since $k_i \leq j$ if $\beta_i = \alpha$. It will suffice then to show that $\rho_n(\mathcal{N}_\tau) \geq \kappa$ whenever $\langle\gamma, e\rangle \leq_{\mathrm{lex}} \langle\tau, n\rangle \leq_{\mathrm{lex}} \langle\xi, j\rangle$. (We leave the details here to the reader.) So suppose $\mu < \kappa$ and $\mu = \rho_n(\mathcal{N}_\tau)$ for some such $\langle\tau, n\rangle$. Let μ be the minimal value of such a $\rho_n(\mathcal{N}_\tau)$. Then $\mathfrak{C}_n(\mathcal{N}_\tau)$ is an initial segment of $\mathfrak{C}_j(\mathcal{N}_\xi) = \mathcal{M}$ by 8.1. The minimality of κ implies $\mathfrak{C}_n(\mathcal{N}_\tau)$ is a proper initial segment of $\mathcal{J}_\lambda^{\mathcal{M}}$. This contradicts that there is a subset of μ which is definable over $\mathfrak{C}_n(\mathcal{N}_\tau)$ but not a member of $J_\kappa^{\mathcal{M}}$. □

CLAIM 2. If $\langle\gamma, e+1\rangle \leq_{lex} \langle\xi, j\rangle$ and $\mathcal{J}_{\beta i}^{\mathcal{M}}$ is a proper initial segment of $\mathfrak{C}_{e+1}(\mathcal{N}_\gamma)$, then $\mathcal{J}_{\beta_i}^{\mathcal{M}}$ is a proper initial segment of $\mathfrak{C}_e(\mathcal{N}_\gamma)$.

PROOF. By the 1st claim $\rho_{e+1}(\mathcal{N}_\gamma) \geq \kappa$. But $\beta_i < (\kappa^+)^{\mathfrak{C}_{e+1}(\mathcal{N}_\gamma)}$ since $\mathcal{J}_{\beta_i}^{\mathcal{M}}$ has projectum κ. By 8.1, $\mathcal{J}_{\beta_i}^{\mathcal{M}}$ is a proper initial segment of $\mathfrak{C}_e(\mathcal{N}_\gamma)$. □

CLAIM 3. There is a unique $\gamma \leq \xi$ s.t. $\mathcal{J}_{\beta_i}^{\mathcal{M}} = \mathfrak{C}_{k_i}(\mathcal{N}_\gamma)$.

PROOF. Let $\langle\gamma, e\rangle$ be $\leq_{\text{lex}}$ least such that $\mathcal{J}_{\beta_i}^{\mathcal{M}}$ is an initial segment of $\mathfrak{C}_e(\mathcal{N}_\gamma)$.

Suppose first that $\mathcal{J}_{\beta_i}^{\mathcal{M}} = \mathfrak{C}_e(\mathcal{N}_\gamma)$. If $e \leq k_i$, then since $\mathcal{J}_{\beta_i}^{\mathcal{M}}$ is k_i sound, $\mathcal{J}_{\beta_i}^{\mathcal{M}} = \mathfrak{C}_{k_i}(\mathcal{N}_\gamma)$ as desired. To see that $e \leq k_i$, suppose toward a contradiction that $k_i < e$. For t, $u \leq e$ set

$$\rho_t^u = \rho_t(\mathfrak{C}_u(\mathcal{N}_\gamma)).$$

It will be enough to see that $\rho_{k_i}^{k_i} = \rho_e^e$, since this implies that $\mathfrak{C}_{k_i}(\mathcal{N}_\gamma) = \mathfrak{C}_e(\mathcal{N}_\gamma) = \mathcal{J}_{\beta_i}^{\mathcal{M}}$, contrary to the minimality of $\langle\gamma, e\rangle$. So suppose we have t s.t. $k_i \leq t < e$ and $\rho_{t+1}^{t+1} < \rho_t^t$. We may assume t is the largest such. Now the reader can easily check[1] that for any u, $\rho_{u+1}^u = \rho_{u+1}^{u+1}$, and $\rho_{u+1}^u < \rho_u^u \Rightarrow \rho_{u+1}^{u+1} < \rho_u^{u+1}$. Thus we have $\rho_{t+1}^{t+1} < \rho_t^{t+1} \leq \rho_{k_i}^{t+1}$. As $\mathfrak{C}_{t+1}(\mathcal{N}_\gamma) = \mathfrak{C}_e(\mathcal{N}_\gamma)$ by the maximality of t, $\rho_e^e = \rho_{t+1}^{t+1} < \rho_{k_i}^{t+1} = \rho_{k_i}^e$. But $\mathfrak{C}_e(\mathcal{N}_\gamma) = \mathcal{J}_{\beta_i}^{\mathcal{M}}$, so this contradicts the fact that $\langle\beta_i, k_i\rangle$ is the last term of this restriction of the λ-dropdown sequence of $\mathcal{M}$, so that $\rho_e(\mathcal{J}_{\beta_i}^{\mathcal{M}}) < \rho_{k_i}(\mathcal{J}_{\beta_i}^{\mathcal{M}})$ is impossible if $k_i < e \leq j$. If $\mathcal{J}_{\beta_i}^{\mathcal{M}} = \mathcal{M}$ then we must verify that $e \leq j$ in order to apply this fact. Now if $\gamma = \xi$, then $e \leq j$ by the choice of $\langle\gamma, e\rangle$, and if $\gamma < \xi$, then $\mathfrak{C}_e(\mathcal{N}_\gamma) = \mathfrak{C}_j(\mathcal{N}_\xi)$ and it is easy to see that this is impossible.

Next, suppose $\mathcal{J}_{\beta_i}^{\mathcal{M}}$ is a proper initial segment of $\mathfrak{C}_e(\mathcal{N}_\gamma)$. From Claim 2, we see that $e = 0$, so that $\mathcal{J}_{\beta_i}^{\mathcal{M}}$ is a proper initial segment of $\mathcal{N}_\gamma$.

If γ is a limit, then the definition of $\mathcal{N}_\gamma$ guarantees that $\mathcal{J}_{\beta_i}^{\mathcal{M}}$ is a proper initial segment of some $\mathfrak{C}_\omega(\mathcal{N}_\tau)$ for $\tau < \gamma$. But then Claim 2 implies $\mathcal{J}_{\beta_i}^{\mathcal{M}}$ is a proper initial segment of $\mathcal{N}_\tau$, a contradiction. Thus γ is a successor.

Let $\gamma = \tau + 1$. From the definition of $\mathcal{N}_{\tau+1}$ (either we add an extender predicate to $\mathcal{M}_\tau$ or extend the J-hierarchy for one more step), $\mathcal{J}_{\beta_i}^{\mathcal{M}}$ is an initial segment of $\mathcal{M}_\tau = \mathfrak{C}_\omega(\mathcal{N}_\tau)$. This contradicts the minimality of $\langle\gamma, e\rangle$.

Thus $\mathcal{J}_{\beta_i}^{\mathcal{M}} = \mathfrak{C}_{k_i}(\mathcal{N}_\gamma)$ for some $\gamma \leq \xi$. There is a unique such γ by the following easy fact, whose proof we omit: if $\gamma \neq \delta$ then $\mathfrak{C}_e(\mathcal{N}_\gamma) \neq \mathfrak{C}_k(\mathcal{N}_\delta)$, $\forall \gamma, \delta, e, k$. □

We can now define the (j, ξ) *resurrection sequence for E*.

CASE 1. $i = 0$. Notice that $\rho_1(\mathcal{J}_\lambda^{\mathcal{M}}) < \lambda$, since $\mathcal{J}_\lambda^{\mathcal{M}}$ is active. Since $\langle\beta_1, k_1\rangle = \langle\lambda, 1\rangle$ is not defined we must have $\mathcal{N}_\xi = \mathcal{M} = \mathcal{J}_\lambda^{\mathcal{M}}$ and $j = 0$. Then E is the

[1] By the way, it is also true, though not at all obvious, that $\rho_u^u < \rho_{u-1}^u \Rightarrow \rho_u^{u+1} < \rho_{u-1}^{u+1}$.

last extender of $\mathcal{N}_\xi$, and the (j,ξ) resurrection sequence for E is defined to be the empty sequence.

CASE 2. $i > 0$. Let $\gamma \leq \xi$ be such that $\mathcal{J}^{\mathcal{M}}_{\beta_i} = \mathfrak{C}_{k_i}(\mathcal{N}_\gamma)$. Notice that $k_i \geq 1$ as $\rho_{k_i}(\mathcal{N}_\gamma) < \lambda$ and $\omega\lambda \leq \mathrm{OR}^{\mathcal{N}_\gamma}$. Let $\pi : \mathfrak{C}_{k_i}(\mathcal{N}_\gamma) \to \mathfrak{C}_{k_i-1}(\mathcal{N}_\gamma)$ be the inverse of the collapse. Then the (j,ξ) resurrection sequence for E is $\langle\beta_i, k_i, \gamma, \pi\rangle^\frown S$, where S is the $(k_i - 1, \gamma)$ resurrection sequence for $\pi(E)$. (Here if E is the last extender of $\mathfrak{C}_{k_i}(\mathcal{N}_\gamma)$, then by $\pi(E)$ we mean the last extender of $\mathfrak{C}_{k_i-1}(\mathcal{N}_\gamma)$.)

This completes the recursive definition of the (j,ξ) resurrection sequence for E.

For any premouse $\mathcal{P}$ with $\omega\alpha = \mathrm{OR}^{\mathcal{P}}$ and $t < \omega$, and $\omega\lambda \leq \mathrm{OR}^{\mathcal{P}}$, the (t,λ) *dropdown sequence* of $\mathcal{P}$ is just that initial segment of the λ-dropdown sequence of $\mathcal{P}$ consisting of pairs (β,k) such that $(\beta,k) \leq_{\mathrm{lex}} (\alpha,t)$.

Now let us return to the situation of Case 2 of the definition of the (j,ξ) resurrection sequence for E, and adopt the notation there. Let us adopt our standard notational device by taking $\pi(\mathrm{OR}^{\mathfrak{C}_{k_i}(\mathcal{N}_\gamma)})$ to be $\mathrm{OR}^{\mathfrak{C}_{k_i-1}(\mathcal{N}_\gamma)}$. One can easily see from our results on preservation of projecta that the $(k_i - 1, \pi(\lambda))$ dropdown sequence for $\mathfrak{C}_{k_i-1}(\mathcal{N}_\gamma)$, which is what we use to resurrect $\pi(E)$, has the form

$$\langle\langle\pi(\beta_0), k_0\rangle, \ldots, \langle\pi(\beta_{i-1}), k_{i-1}\rangle\rangle^\frown u\,,$$

where

$$u = \varnothing \quad \text{or} \quad u = \langle\pi(\beta_i), k_i - 1\rangle\,.$$

We do not know whether it is possible that $u \neq \varnothing$. In order for this to happen we would need to have $\langle\beta_{i-1}, k_{i-1}\rangle \neq \langle\beta_i, k_i - 1\rangle$, $\rho_{k_i-1}(\mathcal{J}^{\mathcal{M}}_{\beta_i}) = \rho_{k_{i-1}}(\mathcal{J}^{\mathcal{M}}_{\beta_{i-1}})$, and $\rho_{k_i-1}(\mathfrak{C}_{k_i-1}(\mathcal{N}_\gamma)) < \pi(\rho_{k_i-1}(\mathcal{J}^{\mathcal{M}}_{\beta_i}))$. We only know that $\rho_{k_i-1}(\mathfrak{C}_{k_i-1}(\mathcal{N}_\gamma)) = \sup \pi''\rho_{k_i-1}(\mathcal{J}^{\mathcal{M}}_{\beta_i})$. It seems plausible that π preserves the k_i – 1st projectum, so that in fact $u = \varnothing$ must hold.

Notice that if $u \neq \varnothing$, then the last integer k_i in the dropdown sequence gets decreased by 1 at the next stage of resurrection. Thus there are cofinally many stages in the resurrection at which the u associated to the stage is $\varnothing$. These stages are important, so we now give a formal definition.

Let E be on the sequence of $\mathfrak{C}_j(\mathcal{N}_\xi)$, $\lambda = \mathrm{lh}\,E$, and let

$$\langle\langle\beta_0, k_0\rangle, \ldots, \langle\beta_i, k_i\rangle\rangle$$

be the (j,λ) dropdown sequence of $\mathfrak{C}_j(\mathcal{N}_\xi)$, and let

$$\langle\langle\delta_0, \ell_0, \gamma_0, \pi_0\rangle, \ldots, \langle\delta_t, \ell_t, \gamma_t, \pi_t\rangle\rangle$$

be the (j,ξ) resurrection sequence for E. (We suppose the resurrection to be non-empty. Thus $(\beta_1, k_1) = (\lambda, 1)$ is defined.) We have at once from the definitions that

$$\begin{aligned}
&(\delta_0, \ell_0) = (\beta_i, k_i)\,,\\
&\mathcal{J}_{\delta_0}^{\mathfrak{C}_j(\mathcal{N}_\xi)} = \mathfrak{C}_{\ell_0}(\mathcal{N}_{\gamma_0})\,,\\
&\pi_0 : \mathfrak{C}_{\ell_0}(\mathcal{N}_{\gamma_0}) \rightarrow \mathfrak{C}_{\ell_0-1}(\mathcal{N}_{\gamma_0})\,,
\end{aligned}$$

and for $1 \leq e \leq t$,

$$\begin{aligned}
(\delta_e, \ell_e) = \text{last term in the } &(\ell_{e-1} - 1, \pi_{e-1} \circ \cdots \circ \pi_0(\lambda))\\
&\text{dropdown sequence of } \mathfrak{C}_{\ell_{e-1}-1}(\mathcal{N}_{\gamma_{e-1}})\,,\\
&\mathcal{J}_{\delta_e}^{\mathfrak{C}_{\ell_{e-1}-1}(\mathcal{N}_{\gamma_{e-1}})} = \mathfrak{C}_{\ell_e}(\mathcal{N}_{\gamma_e})\,,
\end{aligned}$$

and

$$\pi_e : \mathfrak{C}_{\ell_e}(\mathcal{N}_{\gamma_e}) \rightarrow \mathfrak{C}_{\ell_e-1}(\mathcal{N}_{\gamma_e})\,.$$

From our earlier remarks on the new dropdown sequences, we can find stages

$$1 \leq e_1 < e_2 < \cdots < e_{i-1} = t$$

such that

$$\begin{aligned}
(\delta_{e_1}, \ell_{e_1}) &= \pi_{e_1-1} \circ \cdots \circ \pi_0((\beta_{i-1}, k_{i-1}))\\
(\delta_{e_2}, \ell_{e_2}) &= \pi_{e_2-1} \circ \cdots \circ \pi_0((\beta_{i-2}, k_{i-2}))\\
&\vdots\\
(\delta_{e_{i-1}}, \ell_{e_{i-1}}) &= \pi_{e_{i-1}-1} \circ \cdots \circ \pi_0((\beta_1, k_1))\,.
\end{aligned}$$

Here if $e_1 = 1$, the notation "$\pi_{e_1-1} \circ \cdots \circ \pi_0$" stands for π_0. We also set $e_0 = 0$, and interpret "$\pi_{e_0-1} \circ \cdots \circ \pi_0$" to stand for the identity embedding. We then have for $0 \leq n \leq i-1$

$$(\delta_{e_n}, \ell_{e_n}) = \pi_{e_n-1} \circ \cdots \circ \pi_0((\beta_{i-n}, k_{i-n}))\,.$$

This enables us to define embeddings and models resurrecting the various $\mathcal{J}_{\beta_e}^{\mathcal{M}}$, where $\mathcal{M} = \mathfrak{C}_j(\mathcal{N}_\xi)$. Set

$$\sigma_{i-n} = \pi_{e_n} \circ \pi_{e_n-1} \circ \cdots \circ \pi_1 \circ \pi_0$$

so that

$$\sigma_{i-n} : \mathcal{J}_{\beta_{i-n}}^{\mathcal{M}} \rightarrow \mathfrak{C}_{\ell_{e_n}-1}(\mathcal{N}_{\gamma_{e_n}})$$

is an $\ell_{e_n} - 1$ embedding, for $0 \leq n \leq i-1$. In order to simplify the indexing a bit set $\tau_{i-n} = \gamma_{e_n}$ for $0 \leq n \leq i-1$. Notice also that $k_{i-n} = \ell_{e_n}$. Thus, setting $p = i - n$, we have that for $1 \leq p \leq i$

$$\sigma_p : \mathcal{J}_{\beta_p}^{\mathcal{M}} \rightarrow \mathfrak{C}_{k_p-1}(\mathcal{N}_{\tau_p})$$

is a $k_p - 1$ embedding. Let us set

$$\mathrm{Res}_p = \mathfrak{C}_{k_p-1}(\mathcal{N}_{\tau_p})$$

and call $(\sigma_p, \mathrm{Res}_p)$ the pth *partial resurrection of E from stage (j,ξ)*. (Notice that if $p < q$, then Res_p represents "more resurrection" than Res_q in the sense that it goes back to an earlier model $\mathcal{N}_\eta$ and hence nearer to the first appearance of the prototype of E. On the other hand, Res_p resurrects less of $\mathcal{M}$ in the sense that the domain $\mathcal{J}^{\mathcal{M}}_{\beta_p}$ of σ_p is smaller than that of σ_q.

The partial resurrections of E agree with one another in the following way: For $1 \le p \le i$, let

$$\kappa_p = \rho_{k_p}(\mathcal{J}^{\mathcal{M}}_{\beta_p}).$$

Then one can check without too much difficulty that $\kappa_1 > \kappa_2 > \cdots > \kappa_i$, and that if $p < q$ then $\sigma_p \restriction \kappa_{q-1} = \sigma_q \restriction \kappa_{q-1}$ and the models Res_p and Res_q agree below $\sup \sigma_q'' \kappa_{q-1}$. For example, consider the case $q = i$. Then $\sigma_i = \pi_0 : \mathcal{J}^{\mathcal{M}}_{\beta_i} \to \mathfrak{C}_{k_i-1}(\mathcal{N}_{\gamma_0})$, and moreover, the last term of the $(k_i - 1, \pi_0(\lambda))$ dropdown sequence for $\mathfrak{C}_{k_i-1}(\mathcal{N}_{\gamma_0})$ corresponds to a projectum which is greater than or equal to $\sup(\pi_0'' \kappa_{i-1})$. This implies that $\pi_j \restriction \sup \pi_0'' \kappa_{i-1}$ is the identity, for all $j > 0$. So

$$\sigma_p \restriction \kappa_{i-1} = \pi_{e_{i-p}} \circ \cdots \circ \pi_1 \circ \pi_0 \restriction \kappa_{i-1} = \pi_0 \restriction \kappa_{i-1} = \sigma_i \restriction \kappa_{i-1}.$$

See figure 1 for a diagram of some of the relationships above.

Finally, the *complete resurrection* of E from (j,ξ) is the pair (identity, $\mathcal{N}_\xi$) if the (j,ξ) resurrection sequence for E is $\varnothing$ (so that $j = 0$ and E is the last extender of $\mathcal{N}_\xi$), and the pair $(\sigma_1, \mathrm{Res}_1)$ if the (j,ξ) resurrection sequence for E is nonempty.

Notice that in any case, $\mathrm{Res} = \mathcal{N}_\gamma$ for some $\gamma \le \xi$ and σ is a 0-embedding from $\mathcal{J}^{\mathfrak{C}_j(\mathcal{N}_\xi)}_\lambda$ into $\mathcal{N}_\gamma$.

Of course, the notions associated to resurrection can be interpreted not just in $V = R_0$, but in any model R_α of the tree $\mathcal{U}$ (using the construction $i^{\mathcal{U}}_{0\alpha}(\mathbb{C})$). We shall do this in what follows.

Definition of $\mathcal{U}$: Induction hypotheses. During the recursive definition of the tree $\mathcal{U}$ and the embeddings π_α we will be maintaining a number of induction hypotheses, which we have numbered H1 through H7. Recall that R_α is the αth model of the tree $\mathcal{U}$.

H1. There is an ordinal ξ such that the map π_α is a weak n-embedding from $\mathcal{P}_\alpha$ into Q_α, where $n = \deg^{\mathcal{T}} \alpha$ and $Q_\alpha = (\mathfrak{C}_n(\mathcal{N}_\xi))^{R_\alpha}$.

H2. (commutativity) If $\beta T \alpha$ and $(\beta, \alpha]_T \cap D^{\mathcal{T}} = \varnothing$ then $\pi_\alpha \circ i^{\mathcal{T}}_{\beta,\alpha} = i^{\mathcal{U}}_{\beta,\alpha} \circ \pi_\beta$.

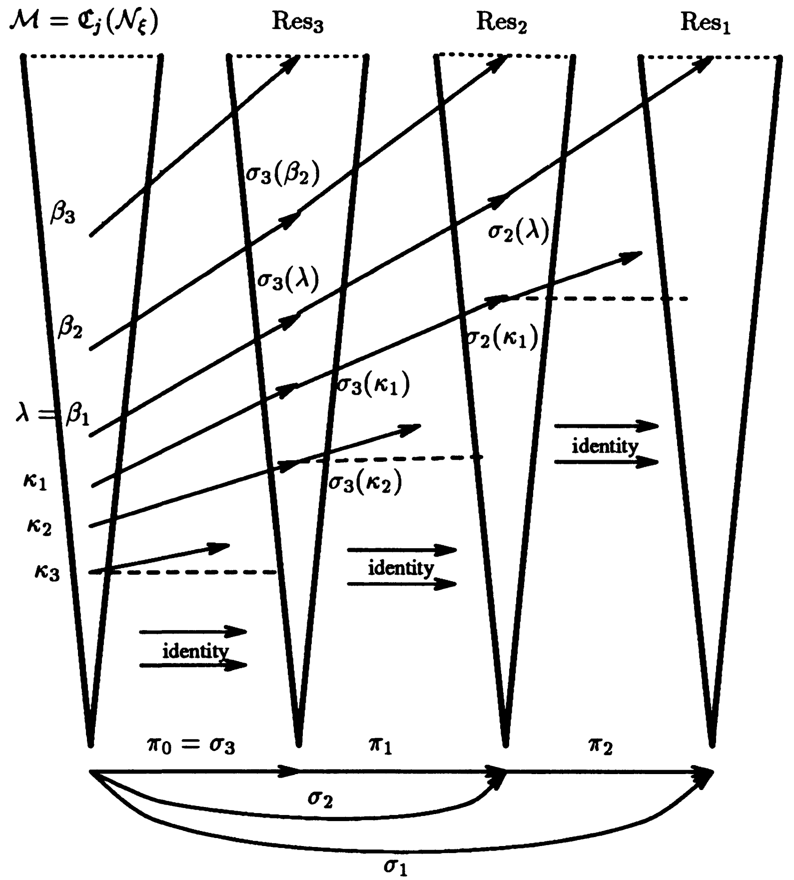

FIGURE 1. To simplify matters, this diagram assumes that $i = 3$ and $\beta_1 < \beta_2 < \beta_3$. It also assumes that the new dropdown sequence is just the image of the old minus its last term, that is, that "$u = \emptyset$" always holds. Thus $t = 2$, $e_0 = 0$, $e_1 = 1$, and $e_2 = 2$. Also, $\sigma_3 = \pi_0$, $\sigma_2 = \pi_1 \circ \pi_0$, and $\sigma_1 = \pi_2 \circ \pi_1 \circ \pi_0$. Finally, we assume that $\pi_e(\rho_{\ell_e - 1}(\mathfrak{C}_{\ell_e}(\mathcal{N}_{\gamma_e}))) = \rho_{\ell_e - 1}(\mathfrak{C}_{\ell_e - 1}(\mathcal{N}_{\gamma_e}))$ for $e = 1, 2$, which, together with a similar assumption on π_0, implies $u = \emptyset$.

Next, we have some agreement of models and embeddings to maintain. For each ordinal $\beta < \operatorname{lh} \mathcal{T}$, let ν_β be the natural length of $E^{\mathcal{T}}_\beta$ and let $(\sigma^\beta, \mathrm{Res}^\beta)$ be the complete resurrection of $\pi_\beta(E^{\mathcal{T}}_\beta)$ from stage (j, τ), where $j = \deg^{\mathcal{T}}(\beta)$ and $Q_\beta = (\mathfrak{C}_j(\mathcal{N}_\tau))^{R_\beta}$.

H3. For each $\beta < \alpha$, if Res^β is type I or III then Q_α agrees with Res^β below

ν^{Res^β}, moreover

$$\pi_\alpha \restriction \nu_\beta = \sigma^\beta \circ \pi_\beta \restriction \nu_\beta \quad \text{and} \quad \pi_\alpha(\nu_\beta) \geq \nu^{\mathrm{Res}^\beta}.$$

H4. For each $\beta < \alpha$, if Res^β is type II then Q_α agrees with Res^β below $\mathrm{OR}^{\mathrm{Res}^\beta}$, and moreover

$$\pi_\alpha \restriction \mathrm{lh}\, E^{\mathcal{T}}_\beta = \sigma^\beta \circ \pi_\beta \restriction \mathrm{lh}\, E^{\mathcal{T}}_\beta \quad \text{and} \quad \pi_\alpha(\mathrm{lh}\, E^{\mathcal{T}}_\beta) \geq \mathrm{OR}^{\mathrm{Res}^\beta}.$$

H5. For each $\beta < \alpha$, R_α agrees with R_β below $\nu^{\mathrm{Res}^\beta} + \omega$, that is $V_\gamma^{R_\alpha} = V_\gamma^{R_\beta}$ where $\gamma = \nu^{\mathrm{Res}^\beta} + \omega$.

In order to handle the limit case in the definition of $\mathcal{U}$, we will require two final induction hypotheses.

If $Q = \mathfrak{C}_k(\mathcal{N}_\gamma)$ and $Q' = \mathfrak{C}_j(\mathcal{N}_\xi)$ where $\mathcal{N}_\gamma$ and $\mathcal{N}_\xi$ are two models of the construction $\mathbb{C}$, then we write $Q \leq_{\mathbb{C}} Q'$ iff $(\gamma, k) \leq_{\mathrm{lex}} (\xi, j)$.

H6. Let $\beta = T\text{-Pred}(\alpha+1)$ and $\mathbb{C}^{\alpha+1} = i^{\mathcal{U}}_{0,\alpha+1}(\mathbb{C})$. Then
  (a) $Q_{\alpha+1} \leq_{\mathbb{C}^{\alpha+1}} i^{\mathcal{U}}_{\beta,\alpha+1}(Q_\beta)$, and
  (b) if $\alpha + 1 \in D^{\mathcal{T}}$, then $Q_{\alpha+1} <_{\mathbb{C}^{\alpha+1}} i^{\mathcal{U}}_{\beta,\alpha+1}(Q_\beta)$.

H7. If λ is a limit ordinal then $i^{\mathcal{U}}_{\alpha\lambda}(Q_\alpha) = Q_\lambda$ for all sufficiently large $\alpha T \lambda$.

We shall need to know that $\mathcal{U}$ is a tree in the "coarse structure" sense of [MS]. Set $\rho^{\mathcal{U}}_\beta = \nu^{\mathrm{Res}^\beta}$. Then it will be obvious from the construction that $E^{\mathcal{U}}_\beta$ is $\rho^{\mathcal{U}}_\beta + \omega$ strong in the model R_β. We shall show in the remark following claim 1 below that $\rho^{\mathcal{U}}_\beta < \rho^{\mathcal{U}}_\delta$ whenever $\beta < \delta$, and the agreement condition on the models R_β follows at once from this. This guarantees that $\mathcal{U}$ is a normal iteration tree in the sense of [MS], provided that no illfounded model appears in $\mathcal{U}$. Thus we need to know that we encounter no illfounded ultrapowers or direct limits in the formation of $\mathcal{U}$. This follows from the following theorem, which is proved by the methods of [MS].

Theorem. *If there is no ordinal $\gamma \leq \xi$ such that $L(V_\gamma) \models$ "γ is a Woodin cardinal" then every iteration tree on $L(V_\xi)$ has a unique cofinal wellfounded branch.*

Note that if theorem 12.1 holds for all $\eta' < \eta$, so that $\mathcal{N}_\eta$ exists, then $\mathcal{N}_\eta$ is constructed in V_ξ for some ordinal ξ smaller than the least cardinal δ such that $L[V_\delta]$ satisfies that δ is a Woodin cardinal. Thus we can apply this theorem to the trees derived from $\mathcal{U}$.

We now begin the recursive definition of the tree $\mathcal{U}$ and the embeddings π_α. For $\alpha = 0$ we take $Q_0 = \mathcal{P}_0$, $R_0 = L(V_\theta)$ where θ is the least ordinal γ such that $L(V_\gamma) \models$ "γ is a Woodin cardinal", and $\pi_0 = $ identity.

Definition of $\mathcal{U}$: The Successor Step. We assume that the tree has been defined through the αth model R_α, and we have the embeddings π_α mapping P_α into Q_α, where $j = \deg^{\mathcal{T}}(\alpha)$ and $Q_\alpha = (\mathfrak{C}_j(\mathcal{N}_\xi))^{R_\alpha}$, and we have (in R_α)

$$(\sigma^\alpha, \mathrm{Res}^\alpha) = \text{complete resurrection of } \pi_\alpha(E_\alpha^{\mathcal{T}}) \text{ from } (j, \xi)\,,$$

where $\pi_\alpha(E_\alpha^{\mathcal{T}})$ is the last extender predicate of Q_α in case $E_\alpha^{\mathcal{T}}$ is the last extender predicate of $\mathcal{P}_\alpha$.

CLAIM 1. If γ is strictly smaller than α then $\sigma^\alpha \restriction \pi_\alpha(\mathrm{lh}\, E_\gamma^{\mathcal{T}}) = \text{identity}$.

PROOF. Fix $\gamma < \alpha$. Then $\mathrm{lh}\, E_\gamma^{\mathcal{T}}$ is a cardinal of $\mathcal{P}_\alpha$, so $\pi_\alpha(\mathrm{lh}\, E_\gamma^{\mathcal{T}})$ is a cardinal of Q_α. Thus $\rho_\omega(\mathcal{J}_\beta^{Q_\alpha}) \geq \pi_\alpha(\mathrm{lh}\, E_\gamma^{\mathcal{T}})$ for all β such that $\pi_\alpha(\mathrm{lh}\, E_\alpha^{\mathcal{T}}) \leq \omega\beta < \mathrm{OR}^{Q_\alpha}$. We claim that also $\rho_j(Q_\alpha) \geq \pi_\alpha(\mathrm{lh}\, E_\gamma^{\mathcal{T}})$. (Recall that $j = \deg(\alpha)$.) Assume first that α is a successor ordinal. Then $\mathcal{P}_\alpha = \mathrm{Ult}_j(\mathcal{P}_\alpha^*, E_{\alpha-1}^{\mathcal{T}})$, and so $\mathrm{lh}\, E_{\alpha-1}^{\mathcal{T}} < \rho_j(\mathcal{P}_\alpha)$. Thus $\mathrm{lh}\, E_\gamma^{\mathcal{T}} < \rho_j(\mathcal{P}_\alpha)$, and as π_α is a weak j-embedding, $\pi_\alpha(\mathrm{lh}\, E_\gamma^{\mathcal{T}}) < \rho_j(Q_\alpha)$. Now our claim for the case α is a limit ordinal follows from the successor case applied to sufficiently large $\alpha' T \alpha$.

Thus no projectum associated to a term in the $(j, \pi_\alpha(\mathrm{lh}\, E_\alpha^{\mathcal{T}}))$ dropdown sequence for Q_α lies below $\pi_\alpha(\mathrm{lh}\, E_\gamma^{\mathcal{T}})$, and it follows that σ^α is the identity below $\pi_\alpha(\mathrm{lh}\, E_\gamma^{\mathcal{T}})$.

REMARK. The claim enables us to show that $\rho_\alpha^{\mathcal{U}} > \rho_\beta^{\mathcal{U}}$ for all $\beta < \alpha$. For

$$\rho_\alpha^{\mathcal{U}} = \nu^{\mathrm{Res}^\alpha} = \sigma^\alpha \circ \pi_\alpha(\nu_\alpha)\,.$$

But now, for $\beta < \alpha$, $\mathrm{lh}\, E_\beta^{\mathcal{T}}$ is a cardinal of $\mathcal{P}_\alpha$ and $\mathrm{lh}\, E_\beta^{\mathcal{T}} < \mathrm{lh}\, E_\alpha^{\mathcal{T}}$, so that $\mathrm{lh}\, E_\beta^{\mathcal{T}} \leq \nu_\alpha$. Thus $\nu_\beta < \nu_\alpha$ for $\beta < \alpha$. So

$$\rho_\alpha^{\mathcal{U}} > \sigma^\alpha \circ \pi_\alpha(\nu_\beta)\,.$$

But Claim 1 tells us $\sigma^\alpha \circ \pi_\alpha(\nu_\beta) = \pi_\alpha(\nu_\beta)$, and our induction hypotheses on agreement of embeddings say $\pi_\alpha(\nu_\beta) \geq \nu^{\mathrm{Res}^\beta}$. So

$$\rho_\alpha^{\mathcal{U}} > \sigma^\alpha \circ \pi_\alpha(\nu_\beta) = \pi_\alpha(\nu_\beta) \geq \nu^{\mathrm{Res}^\beta} = \rho_\beta^{\mathcal{U}}\,.$$

We can now define $E_\alpha^{\mathcal{U}}$ and $R_{\alpha+1}$. Set

$$F = \sigma^\alpha \circ \pi_\alpha(E_\alpha^{\mathcal{T}}) = \text{last extender of } \mathrm{Res}^\alpha\,.$$

Now Res^α is an "$\mathcal{N}$ model" in the universe R_α, so its last extender has a "background extender". Set $E_\alpha^{\mathcal{U}} = F^*$, the background extender for F in R_α. Let $\beta = T\text{-pred}(\alpha+1)$ and set

$$R_{\alpha+1} = \mathrm{Ult}(R_\beta, F^*)\,.$$

Notice that $\mathrm{Ult}_0 = \mathrm{Ult}_\omega$ since $R_\beta \models ZFC$.

Let us note that R_α and R_β are in sufficient agreement that this ultrapower makes sense. This is clear if $\beta = \alpha$, so we may suppose that $\beta < \alpha$. By our induction hypotheses, R_α agrees with R_β to $\nu^{\mathrm{Res}^\beta} + \omega$. Now crit $E_\alpha^{\mathcal{T}} < \nu_\beta$ because $\beta = \mathcal{T}\text{-pred}(\alpha + 1)$. As σ^α is the identity on $\pi_\alpha(\mathrm{lh}\, E_\beta^{\mathcal{T}})$, crit $F^* =$ crit $F =$ crit $\sigma^\alpha(\pi_\alpha(E_\alpha^{\mathcal{T}})) < \sup \pi_\alpha'' \nu_\beta = \sup \sigma^\beta \circ \pi_\beta'' \nu_\beta \le \nu^{\mathrm{Res}^\beta}$. Thus the ultrapower makes sense.

We now define $\pi_{\alpha+1}$ and $Q_{\alpha+1}$. Let $n = \deg^{\mathcal{T}}(\beta)$, and $\lambda = \mathrm{lh}\, E_\beta^{\mathcal{T}}$, let

$$\langle(\eta_0, k_0), \ldots, (\eta_e, k_e)\rangle \text{ be the } (n, \lambda) \text{ dropdown sequence of } \mathcal{P}_\beta,$$

and set $\kappa_i = \rho_{k_i}(\mathcal{J}_{\eta_i}^{\mathcal{P}_\beta})$ for $0 \le i \le e$.

The following claim relates these to the $(n, \pi_\beta(\lambda))$ dropdown sequence of Q_β. The claim is slightly complicated by the fact that π_β is not a full n-embedding. Notice that $\kappa_e \le \rho_n(\mathcal{P}_\beta)$.

CLAIM 2. The $(n, \pi_\beta(\lambda))$-dropdown sequence of Q_β is the sequence given by the appropriate clause below:

(a) If $\kappa_e < \rho_n(\mathcal{P}_\beta)$ then the dropdown sequence is

$$\langle(\pi_\beta(\eta_0), k_0), \ldots, (\pi_\beta(\eta_e), k_e)\rangle.$$

(b) If $\kappa_e = \rho_n(\mathcal{P}_\beta)$ but $(\omega\eta_e, k_e) \neq (\mathrm{OR}^{\mathcal{P}_\beta}, n)$ then the dropdown sequence is

$$\langle(\pi_\beta(\eta_0), k_0), \ldots, (\pi_\beta(\eta_e), k_e)\rangle^\frown u,$$

where $u = \varnothing$ or $u = (\eta, n)$ for $\omega\eta = \mathrm{OR}^{Q_\beta}$.

(c) If $(\omega\eta_e, k_e) = (\mathrm{OR}^{\mathcal{P}_\beta}, n)$ then the dropdown sequence is

$$\langle(\pi_\beta(\eta_0), k_0), \ldots, (\pi_\beta(\eta_{e-1}), k_{e-1})\rangle^\frown u,$$

where $u = \varnothing$ or $u = (\pi_\beta(\eta_e), k_e) = (\omega\eta, n)$, for $\omega\eta = \mathrm{OR}^{Q_\beta}$.

REMARK. Note that $\kappa_e = \rho_n(\mathcal{P}_\beta)$ in case (c). If $e = 0$, then $n = 0 = k_0$ and $\eta_0 = \lambda = \omega\lambda = \mathrm{OR}^{\mathcal{P}_\beta}$. The $(n, \pi_\beta(\lambda))$ dropdown sequence for Q_β is then $\langle(\mathrm{OR}^{Q_\beta}, 0)\rangle$, which falls under case (c).

REMARK. The $u = \varnothing$ case in (c) would not be necessary if π_β were a full n-embedding.

The claim follows easily from the fact that π_β is a weak n-embedding. For (a), notice that $\pi_\beta(\kappa_e) < \sup \pi_\beta'' \rho_n(\mathcal{P}_\beta) \le \rho_n(Q_\beta)$. Recall that π_β preserves cardinals, so that if for example $\omega\eta_e < \mathrm{OR}^{\mathcal{P}_\beta}$ then $\mathcal{P}_\beta \models \forall\gamma \ge \eta_e(\rho_\omega(\mathcal{J}_\gamma^{\dot{E}}) \ge \rho_{k_e}(\mathcal{J}_{\eta_e}^{\dot{E}}))$, and thus $Q_\beta \models \forall\gamma \ge \pi_\beta(\eta_e)(\rho_\omega(\mathcal{J}_\gamma^{\dot{E}}) \ge \pi_\beta(\kappa_e))$. □

Let $\mu = \operatorname{crit}(E^{\mathcal{T}}_\alpha)$, and let

$$i = \begin{cases} e+1 & \text{if } \mu < \kappa_e, \\ \text{least } j \text{ s.t. } \kappa_j \le \mu & \text{if } \kappa_e \le \mu. \end{cases}$$

Notice that $i > 0$ since $\kappa_0 = \lambda > \mu$. Because $\mathcal{T}$ is maximal

$$\mathcal{P}^*_{\alpha+1} = \begin{cases} \mathcal{J}^{\mathcal{P}_\beta}_{\eta_i} & \text{if } i \le e, \\ \mathcal{P}_\beta & \text{if } i = i+1, \end{cases}$$

and

$$\deg^{\mathcal{T}}(\alpha+1) = \begin{cases} k_i - 1 & \text{if } i \le e, \\ n & \text{if } i = e+1. \end{cases}$$

Let $(\sigma^\beta_i, \mathrm{Res}^\beta_i)$ be the ith partial resurrection of $\pi_\beta(E^{\mathcal{T}}_\beta)$ from stage (n,τ), where $Q_\beta = \mathfrak{C}_n(\mathcal{N}_\tau)^{R_\beta}$, if this resurrection is defined. The resurrection is undefined if $i = e+1$ and defined if $i < e$ by claim 2. If $i = e$ then $(\sigma^\beta_i, \mathrm{Res}^\beta_i)$ is undefined just in case $(\omega\eta_e, k_e) = (\mathrm{OR}^{\mathcal{P}_\beta}, n)$ and the conclusion of (c) of claim 2 holds with $u = \varnothing$.

Now let

$$Q^*_{\alpha+1} = \begin{cases} \mathrm{Res}^\beta_i & \text{if } \mathrm{Res}^\beta_i \text{ is defined} \\ Q_\beta & \text{otherwise,} \end{cases}$$

$$\sigma = \begin{cases} \sigma^\beta_i & \text{if } \mathrm{Res}^\beta_i \text{ is defined,} \\ \text{identity} & \text{otherwise.} \end{cases}$$

Then $\sigma \circ (\pi_\beta \restriction \mathcal{P}^*_{\alpha+1})$ is, in any case, a weak $\deg^{\mathcal{T}}(\alpha+1)$ embedding from $\mathcal{P}^*_{\alpha+1}$ into $Q^*_{\alpha+1}$. To see this, assume first that Res^β_i is defined, so that $i \le e$, $\deg^{\mathcal{T}}(\alpha+1) = k_i - 1$, and $\sigma = \sigma^\beta_i$ is a full $(k_i - 1)$ embedding. Looking at claim 2, we see that in all cases the domain of σ is $\mathcal{J}^{Q_\beta}_{\pi_\beta(\eta_i)}$ since we cannot have the situation in (c) with $i = e$ and $u = \varnothing$. But $\mathcal{P}^*_{\alpha+1} = \mathcal{J}^{\mathcal{P}_\beta}_{\eta_i}$, and $\pi_\beta \restriction \mathcal{P}^*_{\alpha+1}$ is a weak $(k_i - 1)$ embedding. In fact, if $\omega\eta_i < \mathrm{OR}^{\mathcal{P}_\beta}$ then $\pi_\beta \restriction \mathcal{P}^*_{\alpha+1}$ is fully elementary, and if $\omega\eta_i = \mathrm{OR}^{\mathcal{P}_\beta}$ then $k_i \le n$, so $\pi_\beta \restriction \mathcal{P}^*_{\alpha+1}$ is a weak k_i-embedding. It follows that $\sigma \circ (\pi_\beta \restriction \mathcal{P}^*_{\alpha+1})$ is a weak $k_i - 1$-embedding from $\mathcal{P}^*_{\alpha+1}$ into $Q^*_{\alpha+1}$. Assume next that Res^β_i is undefined. Then either $i = e+1$ or we have the situation in (c) of claim 2 with $u = \varnothing$. In either case, $\deg^{\mathcal{T}}(\alpha+1) \le n$. Also $\mathcal{P}^*_{\alpha+1} = \mathcal{P}_\beta$, $Q^*_{\alpha+1} = Q_\beta$, and σ is the identity. Since π_β is a weak n-embedding, $\sigma \circ \pi_\beta$ is a weak $\deg^{\mathcal{T}}(\alpha+1)$-embedding from $\mathcal{P}^*_{\alpha+1}$ into $Q^*_{\alpha+1}$.

Let $Q_\beta = \mathfrak{C}_n(\mathcal{N}_\tau)^{R_\beta}$, so that $(\sigma^\beta, \mathrm{Res}^\beta)$ is the complete resurrection of $\pi_\beta(E^{\mathcal{T}}_\beta)$ from stage (n,τ). Let ψ be the complete resurrection embedding for $\sigma \circ \pi_\beta(E^{\mathcal{T}}_\beta)$ from the appropriate stage, which is (n,τ) if Res^β_i is undefined and $(k_i - 1, \eta)$, where $\mathrm{Res}_i^\beta = \mathfrak{C}_{k_i-1}(\mathcal{N}_\eta)$, otherwise. Then $\psi\colon \mathcal{J}^{Q^*_{\alpha+1}}_{\sigma\circ\pi_\beta(\lambda)} \to \mathrm{Res}^\beta$ and $\sigma^\beta = \psi \circ (\sigma \restriction \mathcal{J}^{Q_\beta}_{\pi_\beta(\lambda)})$.

CLAIM 3. $\psi \restriction (\sup(\sigma \circ \pi_\beta{}''\kappa_{i-1})) = \text{identity}$.

PROOF. Suppose first that Res_i^β exists, so that $i \le e$ and $\sigma = \sigma_i^\beta$. From claim 2 and the fact that π_β is a weak n-embedding we see that $\pi_\beta(\kappa_{i-1})$ is the projectum associated to the $(i-1)$st element of the $(n, \pi_\beta(\lambda))$-dropdown sequence of Q_β. As we remarked earlier, ψ is therefore the identity on $\sup(\sigma_i^\beta{}''\pi_\beta(\kappa_{i-1}))$, and this implies the claim.

Suppose next that Res_i^β is undefined, so that either $i = e+1$ or else $i = e$ and (c) of claim 2 holds with $u = \varnothing$. In either case the projectum associated to the last term of the $(n, \pi_\beta(\lambda))$ dropdown sequence of Q_β is at least $\sup(\pi_\beta{}''\kappa_{i-1})$. Thus $\sigma^\beta \restriction \sup(\pi_\beta{}''\kappa_{i-1})$ is the identity, but $\psi = \sigma^\beta$ and σ is the identity, so this implies the claim. □

We can now define $Q_{\alpha+1} = i^{\mathcal{U}}_{\beta,\alpha+1}(Q^*_{\alpha+1})$. Before we define $\pi_{\alpha+1}$ and verify the induction hypotheses, however, we must describe the agreement between $Q^*_{\alpha+1}$ and Res^α. Set

$$\gamma = \begin{cases} (\mu^+)^{\mathcal{P}^*_{\alpha+1}} & \text{if } \mathcal{P}^*_{\alpha+1} \models \mu^+ \text{ exists} \\ \mathrm{OR}^{\mathcal{P}^*_{\alpha+1}} & \text{otherwise.} \end{cases}$$

CLAIM 4. $\gamma \le \lambda = \mathrm{lh}(E^{\mathcal{T}}_\beta)$, and if $\gamma = \mathrm{OR}^{\mathcal{P}^*{}_{\alpha+1}}$ then $\mathcal{P}^*_{\alpha+1} = \mathcal{J}^{\mathcal{P}_\beta}_\lambda$ and $\mathcal{P}^*_{\alpha+1}$ is type II.

PROOF. If $\beta = \alpha$, then $(\mu^+)^{\mathcal{J}^{\mathcal{P}_\alpha}_\lambda}$ exists and $\mathcal{P}^*_{\alpha+1}$ is the shortest initial segment of $\mathcal{P}_\alpha$ over which a subset of μ not in $\mathcal{J}^{\mathcal{P}_\alpha}_\lambda$ is definable. Thus $(\mu^+)^{\mathcal{P}^*_{\alpha+1}} = (\mu^+)^{\mathcal{J}^{\mathcal{P}_\alpha}_\lambda} < \lambda \le \mathrm{OR}^{\mathcal{P}^*_{\alpha+1}}$, so $\gamma < \lambda \le \mathrm{OR}^{\mathcal{P}^*_{\alpha+1}}$.

If $\beta < \alpha$ then the subsets of μ in $\mathcal{P}_\alpha$ are just those in $\mathcal{J}^{\mathcal{P}_\beta}_\lambda$ and $\mathcal{P}^*_{\alpha+1}$ is the shortest initial segment of $\mathcal{P}_\beta$ over which a subset of μ not in $\mathcal{J}^{\mathcal{P}_\beta}_\lambda$ is definable, so if $(\mu^+)^{\mathcal{J}^{\mathcal{P}_\beta}_\lambda}$ exists then $(\mu^+)^{\mathcal{P}^*_{\alpha+1}} = (\mu^+)^{\mathcal{J}^{\mathcal{P}_\beta}_\lambda} < \lambda$. Otherwise μ is the largest cardinal of $\mathcal{J}^{\mathcal{P}_\beta}_\lambda$, so $\mathcal{P}^*_{\alpha+1} = \mathcal{J}^{\mathcal{P}_\beta}_\lambda$ since λ is definably collapsed over the active ppm $\mathcal{J}^{\mathcal{P}_\beta}_\lambda$. In this case we see also that $\mathcal{P}^*_{\alpha+1}$ is type II, since otherwise $\mu < \nu_\beta < \lambda$ and ν_β is a cardinal of $\mathcal{J}^{\mathcal{P}_\beta}_\lambda$. □

Claim 4 implies $\gamma \le \kappa_{i-1}$. If $\kappa_{i-1} = \lambda$ then this is obvious. Otherwise κ_{i-1} is a cardinal of $\mathcal{J}^{\mathcal{P}_\beta}_\lambda$, since it is a projectum of some $\mathcal{J}^{\mathcal{P}_\beta}_\eta$ with $\eta \ge \lambda$. Since $\mu < \kappa_{i-1}$ by the choice of i, we have $\gamma \le \kappa_{i-1}$.

The next claim shows that Res^α and $Q^*_{\alpha+1}$ have the agreement required for the use of the shift lemma.

CLAIM 5. (a) Res^α agrees with $Q^*_{\alpha+1}$ below $\sup(\sigma \circ \pi_\beta{}''\gamma)$.

(b) $\sigma^\alpha \circ \pi_\alpha \restriction \gamma = \sigma \circ \pi_\beta \restriction \gamma$.

PROOF. The proof of claim 5 is divided up into three subclaims.

Subclaim A. $Q^*_{\alpha+1}$ and Res^β agree below $\sup(\sigma \circ \pi_\beta{}''\gamma)$, and $\sigma \circ \pi_\beta \restriction \gamma = \psi \circ \sigma \circ \pi_\beta \restriction \gamma$.

This follows at once from claim 3 and the fact that $\gamma \leq \kappa_{i-1}$.

Subclaim B. If $\beta < \alpha$ then Res^β and Q_α agree below $\sup(\sigma \circ \pi_\beta{}''\gamma)$, and $\psi \circ \sigma \circ \pi_\beta \restriction \gamma = \pi_\alpha \restriction \gamma$.

Recall that $\psi \circ \sigma \circ \pi_\beta = \sigma^\beta \circ \pi_\beta$. This subclaim is therefore just our induction hypotheses on agreement. If Res^β is type I or type III then claim 4 yields $\gamma \leq \nu_\beta$ and we can apply H3. If Res^β is type II then $\gamma \leq \mathrm{lh}\, E^{\mathcal{T}}_\beta$ by claim 4 so we can apply H4.

Subclaim C. If $\beta < \alpha$ then Q_α and Res^α agree below $\sup(\sigma \circ \pi_\beta{}''\gamma)$ and $\pi_\alpha \restriction \gamma = \sigma^\alpha \circ \pi_\alpha \restriction \gamma$.

We have $\gamma \leq \lambda$, and $\sigma \circ \pi_\beta = \pi_\alpha \restriction \gamma$, so $\sup(\sigma \circ \pi_\beta{}''\gamma) \leq \pi_\alpha(\lambda)$. By claim 1, Q_α and Res^α agree below $\pi_\alpha(\lambda)$ and σ^α is the identity there.

Together, subclaims A, B and C yield claim 5. □

Now define, for $a \in [\nu_\alpha]^{<\omega}$ and appropriate f

$$\pi_{\alpha+1}\left([a, f]^{\mathcal{P}^*_{\alpha+1}}_{E^{\mathcal{T}}_\alpha}\right) = [\sigma^\alpha \circ \pi_\alpha(a), \sigma \circ \pi_\beta(f)]^{R_\beta}_{F^*}.$$

If $f = f_{\tau,q}$ then by "$\sigma \circ \pi_\beta(f)$" we mean $f_{\tau,\sigma\circ\pi_\beta(q)}$, the later function being defined over the ppm $Q^*_{\alpha+1}$. In order to see that $\pi_{\alpha+1}$ has the desired properties, it is useful to factor it. Let $k = \deg^{\mathcal{T}}(\alpha+1)$ and $Q'_{\alpha+1} = \mathrm{Ult}(Q^*_{\alpha+1}, F)$. Let $i\colon Q^*_{\alpha+1} \to Q'_{\alpha+1}$ be the canonical embedding and let $\pi'_{\alpha+1}\colon \mathcal{P}_{\alpha+1} \to Q'_{\alpha+1}$ be the weak k-embedding given by the shift lemma. Finally, let $\tau\colon Q'_{\alpha+1} \to Q_{\alpha+1}$ be the natural map given by $\tau\left([a, f]^{Q^*_{\alpha+1}}_{\sigma^\alpha\circ\pi_\alpha(E^{\mathcal{T}}_\alpha)}\right) = [a, f]^{R_\beta}_{F^*}$. Then $\pi_{\alpha+1} = \tau \circ \pi'_{\alpha+1}$ and we have the commutative diagram

$$\begin{tikzcd}
\mathcal{P}_{\alpha+1} \arrow[r, "\pi'_{\alpha+1}"] & Q'_{\alpha+1} \arrow[r, "\tau"] & Q_{\alpha+1} \\
\mathcal{P}^*_{\alpha+1} \arrow[u, "(i^*_{\alpha+1})^{\mathcal{T}}"] \arrow[r, "\sigma\circ\pi_\beta"] & Q^*_{\alpha+1} \arrow[u, "i"] \arrow[ur, "i^{\mathcal{U}}_{\beta,\alpha+1}"'] &
\end{tikzcd}$$

In order to verify H1 we need to show that $\pi_{\alpha+1}$ is a weak k-embedding, where $k = \deg(\alpha+1)$, which means that we have to find a witness set X on which

$\pi_{\alpha+1}$ is $r\Sigma_{k+1}$ elementary. If $k = \deg(\alpha+1) = n$ and $\mathcal{P}^*_{\alpha+1} = \mathcal{P}_\beta$ then we can take the witnessing set to be $X = i^*_{\alpha+1}{}''X_\beta$, where X_β is a set witnessing that π_β is a weak k-embedding. Otherwise take $X = i^*_{\alpha+1}{}'' \left|\mathcal{P}^*_{\alpha+1}\right|$. In either case the shift lemma implies that $\pi'_{\alpha+1}$ is $r\Sigma_{k+1}$ elementary on parameters from X. On the other hand the Łos theorem 4.1 implies that τ is $r\Sigma_{k+1}$ elementary on parameters from $i'' \left|Q^*_{\alpha+1}\right|$, and since $\pi'_{\alpha+1}{}''X \subset i'' \left|Q^*_{\alpha+1}\right|$ it follows that $\pi_{\alpha+1}$ is $r\Sigma_{k+1}$ elementary on parameters from X. Thus X witnesses that $\pi_{\alpha+1}$ is a weak k-embedding and we have verified H1. Induction hypothesis H2 comes from the commutativity of the diagram above.

We now verify H3 and H4. Let $\eta < \alpha+1$. If Res^η is type I or III then we must show that $Q_{\alpha+1}$ agrees with Res^η below ν^{Res^η} and moreover that $\pi_{\alpha+1} \restriction \nu_\eta = \sigma^\eta \circ \pi_\eta \restriction \nu_\eta$ and $\pi_{\alpha+1}(\nu_\eta) \geq \nu^{\mathrm{Res}^\eta}$. If Res^η is of type II, on the other hand, then we must show that $Q_{\alpha+1}$ agrees with Res^η below $\mathrm{OR}^{\mathrm{Res}^\eta}$ and moreover that $\pi_{\alpha+1} \restriction \mathrm{lh}\, E^{\mathcal{T}}_\eta = \sigma^\eta \circ \pi_\eta \restriction \mathrm{lh}\, E^{\mathcal{T}}_\eta$ and $\pi_{\alpha+1}(\mathrm{lh}\, E^{\mathcal{T}}_\eta) \geq \mathrm{OR}^{\mathrm{Res}^\eta}$.

We consider first the case $\eta = \alpha$. Set $\mu' = \pi_\beta(\mu)$. By claim 3, $\mathcal{J}^{Q^*_{\alpha+1}}_{\mu'} = \mathcal{J}^{\mathrm{Res}^\alpha}_{\mu'}$ so that

$$\mathcal{J}^{Q_{\alpha+1}}_{i^{\mathcal{U}}_{\beta,\alpha+1}(\mu')} = \mathrm{Ult}(\mathcal{J}^{Q^*_{\alpha+1}}_{\mu'}, F^*) = \mathrm{Ult}(\mathcal{J}^{\mathrm{Res}^\alpha}_{\mu'}, F^*),$$

where the ultrapowers are computed using all functions which are members of R_β, or equivalently of R_α, and which map $[\mu']^i$ into $\mathcal{J}^{Q^*_{\alpha+1}}_{\mu'}$ for some integer i.

Now the canonical embedding

$$\psi : \mathrm{Ult}_0(\mathrm{Res}^\alpha, F) \to \mathrm{Ult}(\mathrm{Res}^\alpha, F^*)$$

(where the first ultrapower uses all functions belonging to Res^α, and the second uses all functions in R_α) has critical point $\geq \nu^{\mathrm{Res}^\alpha}$ if Res^α is type I or III, and $\geq \mathrm{OR}^{\mathrm{Res}^\alpha} = \mathrm{lh}\, F$ if Res^α is type II. Moreover, $\mathrm{Ult}_0(\mathrm{Res}^\alpha, F)$ agrees with Res^α below $\mathrm{lh}\, F = \mathrm{OR}^{\mathrm{Res}^\alpha}$. As $i^{\mathcal{U}}_{\beta,\alpha+1}(\mu') > \mathrm{lh}\, F$, $Q_{\alpha+1}$ agrees with Res^α below $\nu^{\mathrm{Res}^\alpha}$ in the type I or III case, and below $\mathrm{OR}^{\mathrm{Res}^\alpha}$ in the type II case.

Next we consider the agreement of embeddings. Suppose first Res^α is type I or III, and $\xi < \nu_\alpha$. Then $\xi = [\{\xi\}, \mathrm{id}]^{\mathcal{P}^*_{\alpha+1}}_{E^{\mathcal{T}}_\alpha}$, where id = identity function, so

$$\pi_{\alpha+1}(\eta) = \left[\{\sigma^\alpha \circ \pi_\alpha(\xi)\}, \mathrm{id}\right]^{R_\beta}_{F^*} = \sigma^\alpha \circ \pi_\alpha(\xi)$$

as desired. Also, let $f \in |\mathcal{P}_\alpha| \cap |\mathcal{P}^*_{\alpha+1}|$ and $a \in [\nu_\alpha]^{<\omega}$ be such that $\nu_\alpha = [a, f]^{\mathcal{P}_\alpha}_{E^{\mathcal{T}}_\alpha} = [a, f]^{\mathcal{P}^*_{\alpha+1}}_{E^{\mathcal{T}}_\alpha}$. Then

$$\begin{aligned}
\pi_{\alpha+1}(\nu_\alpha) &= [\sigma^\alpha \circ \pi_\alpha(a),\ \pi_\beta(f)]^{R_\beta}_{F^*} \\
&= [\sigma^\alpha \circ \pi_\alpha(a),\ \sigma^\alpha \circ \pi_\alpha(f)]^{R_\alpha}_{F^*} \\
&\geq [\sigma^\alpha \circ \pi_\alpha(a),\ \sigma^\alpha \circ \pi_\alpha(f)]^{\mathrm{Res}^\alpha}_{F}.
\end{aligned}$$

But for $(E_\alpha^{\mathcal{T}})_{a\cup\{\nu_\alpha\}}$ a.e. $(\bar{u}, v)$, $f(\bar{u}) = v$. Also $\sigma^\alpha \circ \pi_\alpha(\nu_\alpha) = \nu^{Res^\alpha}$, so $\sigma^\alpha \circ \pi_\alpha(f)(\bar{u}) = v$ for $(F)_{\sigma^\alpha \circ \pi_\alpha(a)\cup\{\nu^{\mathrm{Res}^\alpha}\}}$ a.e. $(\bar{u}, v)$. Thus

$$\nu^{\mathrm{Res}^\alpha} = [\sigma^\alpha \circ \pi_\alpha(a),\, \sigma^\alpha \circ \pi_\alpha(f)]_F^{\mathrm{Res}^\alpha}$$

and $\pi_{\alpha+1}(\nu_\alpha) \geq \nu^{\mathrm{Res}^\alpha}$, as desired.

These calculations carry over easily to the case Res^α is type II to give the agreement of embeddings facts in part (b) of the claim. We omit further detail.

We must now consider the case $\eta < \alpha$. Let's just prove (a), the proof of (b) being similar. So assume Res^η is type I or III.

From the $\eta = \alpha$ case we know that $Q_{\alpha+1}$ agrees with Res^α below $\nu^{\mathrm{Res}^\alpha}$. But we showed in the proof of claim 5 that Res^α agrees with Q_α below $\pi_\alpha(\mathrm{lh}\, E_\eta^{\mathcal{T}})$. Also, $\pi_\alpha(\mathrm{lh}\, E_\eta^{\mathcal{T}})$ is a cardinal of Res^α, hence $\pi_\alpha(\mathrm{lh}\, E_\eta^{\mathcal{T}}) \leq \nu^{\mathrm{Res}^\alpha}$. Thus $Q_{\alpha+1}$ agrees with Q_α below $\pi_\alpha(\mathrm{lh}\, E_\eta^{\mathcal{T}})$. But by induction hypothesis, Q_α agrees with Res^η below ν^{Res^η}, and $\pi_\alpha(\nu_\eta) \geq \nu^{\mathrm{Res}^\eta}$. Thus $Q_{\alpha+1}$ agrees with Res^η below ν^{Res^η}, as desired. For agreement of embeddings, we argue similarly that $\pi_{\alpha+1} \restriction \nu_\alpha = \sigma^\alpha \circ \pi_\alpha \restriction \nu_\alpha$. Furthermore since $\mathrm{lh}\, E_\eta^{\mathcal{T}}$ is a cardinal of $\mathcal{P}_\alpha$ and $\mathrm{lh}\, E_\eta^{\mathcal{T}} < \mathrm{lh}\, E_\alpha^{\mathcal{T}}$, we know that $\mathrm{lh}\, E_\eta^{\mathcal{T}} \leq \nu_\alpha$, and since σ^α is the identity on $\pi_\alpha(\mathrm{lh}\, E_\eta^{\mathcal{T}})$ we get that $\pi_{\alpha+1} \restriction \mathrm{lh}\, E_\eta^{\mathcal{T}} = \pi_\alpha \restriction \mathrm{lh}\, E_\eta^{\mathcal{T}}$. But then since $\pi_\alpha \restriction \nu_\eta = \sigma^\eta \circ \pi_\eta \restriction \nu_\eta$ by the induction hypothesis, $\pi_{\alpha+1} \restriction \nu_\eta = \sigma^\eta \circ \pi_\eta \restriction \nu_\eta$, as desired. Notice also that we get $\pi_{\alpha+1}(\nu_\eta) = \pi_\alpha(\nu_\eta) > v^{\mathrm{Res}^\eta}$ by induction.

This verifies H3 and H4. A much simpler coarse structural argument along the same lines gives H5. Finally, H6 is easy to check and H7 is vacuous in the successor case.

Now let λ be a limit ordinal with $\lambda < \theta = \mathrm{lh}\, \mathcal{T}$. We are given sequences $\mathcal{U} \restriction \lambda$, $\langle Q_\alpha \mid \alpha < \lambda\rangle$, and $\langle \pi_\alpha \mid \alpha < \lambda\rangle$ satisfying our inductive hypothesis, and must define $\mathcal{U} \restriction \lambda + 1$, Q_λ, and π_λ.

Let $c = [0, \lambda)_T = \{\alpha \mid \alpha T \lambda\}$. We claim that $\lim_{\alpha \in c} R_\alpha$ is wellfounded, where the limit is taken along the maps $i_{\alpha\beta}^{\mathcal{U}}$ for $\alpha, \beta \in c$.

For this it suffices, using results of [MS] asserting that $\mathcal{T}$ has at least one well founded branch, to show that if b is a branch of $T \restriction \lambda$ which is cofinal in λ, and $b \neq c$, then $\lim_{\alpha \in b} R_\alpha$ is illfounded. So let b be such a branch.

We may assume $i_{\alpha\beta}^{\mathcal{U}}(Q_\alpha) = Q_\beta$ for all sufficiently large α and β in b, $\alpha < \beta$, as otherwise our last induction hypothesis 6(a) implies that $i_{0b}^{\mathcal{U}}(<_{\mathrm{C}})$ is illfounded, so $\lim_{\alpha \in b} R_\alpha$ is illfounded. (Here $i_{\alpha b}^{\mathcal{U}}$ is the canonical embedding from R_α, $\alpha \in b$, into $\lim_{\alpha \in b} R_\alpha$.) This in turn implies $D^{\mathcal{T}} \cap b$ is finite via 6(b).

Let $\mathcal{P}_b = \lim_{\alpha \in b} \mathcal{P}_\alpha$, and $Q_b = \lim_{\alpha \in b} Q_\alpha$, which is the common value of $i_{\alpha b}^{\mathcal{U}}(Q_\alpha)$ for $\alpha \in b$ sufficiently large. Then $\mathcal{P}_b$ exists as $D^{\mathcal{T}} \cap b$ is finite, and $\mathcal{P}_b$ is illfounded as $\mathcal{T} \restriction \lambda$ is simple and $b \neq c$. There is a natural $\pi : \mathcal{P}_b \to Q_b$ given by our commutativity hypothesis: $\pi(i^{\mathcal{T}}_{\alpha b}(x)) = i^{\mathcal{U}}_{\alpha,b}(\pi_\alpha(x))$, for $\alpha \in b$ sufficiently large. Thus Q_b is illfounded, and hence $\mathcal{R}_b$ is illfounded since $\mathcal{R}_b = \lim_{\alpha\in b} R_\alpha \models$ "Q_b is wellfounded".

We set $R_\lambda = \lim_{\alpha\in c} R_\alpha$, and this gives us $\mathcal{U} \restriction \lambda + 1$. Notice that $i^{\mathcal{U}}_{\alpha\lambda}(Q_\alpha)$ is constant on all sufficiently large $\alpha T \lambda$, as otherwise $i^{\mathcal{U}}_{0\lambda}(<_C)$ is illfounded. Set Q_λ equal to the eventual value of $i^{\mathcal{U}}_{\alpha\lambda}(Q_\alpha)$ for sufficiently large $\alpha T\lambda$. Set

$$\pi_\lambda(i^{\mathcal{T}}_{\alpha\lambda}(x)) = i^{\mathcal{U}}_{\alpha\lambda}(\pi_\alpha(x))$$

for $\alpha < \lambda$ sufficiently large, $\alpha T \lambda$.

Let $n = \deg^{\mathcal{T}}(\lambda) = \deg^{\mathcal{T}}(\alpha)$ for $\alpha T\lambda$ sufficiently large. It is easy to check that π_λ is a weak n-embedding which is $r\Sigma_{n+1}$ elementary on the appropriate set, and that π_λ commutes properly. Our last induction hypothesis is just the definition of Q_λ so we need only check that Q_λ and π_λ agree properly with Res^β and $\sigma^\beta \circ \pi_\beta$ for $\beta < \lambda$.

Let $\beta < \lambda$. We have already shown that if $\gamma > \beta$, then $\nu^{\mathrm{Res}^\gamma} > \nu^{\mathrm{Res}^\beta}$. But $\nu^{\mathrm{Res}^\gamma} \leq \mathrm{lh}\, E^{\mathcal{U}}_\gamma$, and thus $\beta < \gamma \Rightarrow \nu^{\mathrm{Res}^\beta} < i^{\mathcal{U}}_{\eta,\gamma+1}(\mathrm{crit}\ E^{\mathcal{U}}_\gamma)$ where $\eta = T\text{-pred}\,(\gamma + 1)$. As R_λ is wellfounded, we must have $\nu^{\mathrm{Res}^\beta} < \mathrm{crit}\ E^{\mathcal{U}}_\gamma$, for all sufficiently large $\gamma + 1\ T\,\lambda$. We can then find $\gamma + 1\ T\,\lambda$ sufficiently large that $\nu^{\mathrm{Res}^\beta} < \mathrm{crit}\ i^{\mathcal{U}}_{\gamma+1,\lambda}$ and $i^{\mathcal{U}}_{\gamma+1,\lambda}(Q_{\gamma+1}) = Q_\lambda$. By induction, $Q_{\gamma+1}$ agrees with Res^β below ν^{Res^β}. Q_λ agrees with $Q_{\gamma+1}$ below $\mathrm{crit}\ i^{\mathcal{U}}_{\gamma+1,\lambda}$. So Q_λ agrees with Res^β below ν^{Res^β}. For the embeddings, notice that $\beta < \gamma \Rightarrow \nu_\beta < \nu_\gamma < i^{\mathcal{T}}_{\eta,\gamma+1}(\mathrm{crit}\ E^{\mathcal{T}}_\gamma)$, where $\eta = T\ \mathrm{pred}(\gamma + 1)$. So we can assume the ordinal $\gamma + 1$ of the last paragraph is such that $i^{\mathcal{T}}_{\gamma+1,\lambda}$ is defined and $\nu_\beta < \mathrm{crit}\ i^{\mathcal{T}}_{\gamma+1,\lambda}$.

But then, for $\alpha < \nu_\beta$,

$$\pi_\lambda(\alpha) = \pi_\lambda(i^{\mathcal{T}}_{\gamma+1,\lambda}(\alpha)) = i^{\mathcal{U}}_{\gamma+1,\lambda}(\pi_{\gamma+1}(\alpha)) = \pi_{\gamma+1}(\alpha)\,.$$

Since $\pi_{\gamma+1} \restriction \nu_\beta = \sigma^\beta \circ \pi_\beta \restriction \nu_\beta$ by induction, $\pi_\lambda \restriction \nu_\beta = \sigma^\beta \circ \pi_\beta \restriction \nu_\beta$, as desired. This proves the agreement hypothesis in the case Res^β is type I or III. The type II case is almost the same.

We have completed the definition of $\mathcal{U} \restriction \theta = \mathcal{U}$. Assuming that θ is a limit ordinal, methods of [MS] yield a cofinal, wellfounded branch b of $\mathcal{U}$. It is easy to see (cf. the limit case above) that b is a wellfounded branch of $\mathcal{T}$. This was what we needed.

In the case θ is a successor, the fact that $\mathcal{U}$ can be extended freely one more step guarantees the same for $\mathcal{T}$, as desired.

The remaining clauses of k-iterability can be proved similarly, using that the corresponding operations on $L(V_\theta)$ yield wellfoundedness.

This completes the proof of 12.1. □

The 0-iterability of the bicephali and psuedo-premice arising in the construction of §11 can be proved similarly.