§10. Closure under Initial Segment

We prove a result, theorem 10.1, which implies that certain structures arising in the construction done in §11 satisfy the initial segment condition on premice. As was pointed out in the last section, this will be used when bicephali cannot be used because one of the extenders being compared is of type II and one is not.

Definition 10.0.1. A psuedo-premouse is a structure $\mathcal{M} = (J_\alpha^{\vec{E}}, \in, \vec{E}, \tilde{F})$ such that

(1) $(J_\alpha^{\vec{E}}, \in, \vec{E})$ is a passive premouse,
(2) F satisfies conditions 1 through 4 in the definition of "good at α" (i.e. everything but the initial segment condition), and
(3) There is a $\delta < \alpha$ s.t. (i) $\mathcal{M} \models \delta$ is the largest cardinal and (ii) for some γ s.t. $\delta < \gamma < \alpha$, $\gamma \in \operatorname{dom} \vec{E}$ and $E_\gamma =$ trivial completion of $F \restriction \delta$.

Any psuedo-premouse $\mathcal{M}$ is weakly amenable with respect to its predicate $\dot{F}^{\mathcal{M}}$ for the last extender. Consequently, if E is an extender from the sequence of some psuedo-premouse $\mathcal{N}$, then we can define $\mathrm{Ult}_0(\mathcal{M}, E)$ in the natural way, as for premice. If $\mathrm{Ult}_0(\mathcal{M}, E)$ is wellfounded, we identify it with its transitive collapse. Łos' theorem holds for $r\Sigma_0$ formulae and so if $i : \mathcal{M} \to \mathrm{Ult}_0(\mathcal{M}, E)$ is the canonical embedding, i is $r\Sigma_1$ elementary. The calculations of §2 show that, if transitive, $\mathrm{Ult}_0(\mathcal{M}, E)$ is a psuedo-premouse. (If δ, γ witness 3 for $\mathcal{M}$, then $i(\delta)$, $i(\gamma)$ witness 3 for $\mathrm{Ult}_0(\mathcal{M}, E)$.) We can thus construct 0-maximal iteration trees on a psuedo-premouse $\mathcal{M}$. We define the notions of simplicity and iterability for psuedo-premice just as for premice. (We only consider 0-maximal trees.) The notion of 1-smallness also generalizes in an obvious way.

Theorem 10.1. *Let $\mathcal{M}$ be an iterable, 1-small psuedo-premouse. Then $\mathcal{M}$ is a premouse.*

Proof. We must show that the initial segment condition holds. Let $\mathcal{M} = (J_\alpha^{\vec{E}}, \in, \vec{E}, \tilde{F})$, and suppose toward a contradiction that the initial segment condition fails for $F \restriction \rho$. Thus ρ is the natural length of $F \restriction \rho$ and if G is the trivial completion of $F \restriction \rho$ then G is not on the $\dot{\vec{E}}^{\mathcal{M}}$ sequence, and if $\rho \in \operatorname{dom} \dot{\vec{E}}^{\mathcal{M}}$ then G is not on the $\dot{E}^{\mathrm{Ult}(\mathcal{M}, \dot{E}_\rho^{\mathcal{M}})}$ sequence.

Notice that if ρ is a successor ordinal then $\rho - 1$ is a generator of F, and if ρ is a limit ordinal then either ρ is a limit of generators of F or else ρ is equal to κ^+ of $\mathcal{M}$ where $\kappa = \mathrm{crit}(F)$. Also, ρ is smaller than natural length of F and as $\mathcal{M}$ is a psuedo-premouse ρ is larger than any cardinal of $\mathcal{M}$.

We obtain a contradiction by comparing $\mathcal{M}$ with $\mathrm{Ult}_0(\mathcal{M}, G)$. That is, we define 0-maximal iteration trees $\mathcal{T}$ and $\mathcal{U}$ on $\mathcal{M}$ with models $\mathcal{P}_\alpha$ and $\mathcal{Q}_\alpha$ respectively as follows:

$$\mathcal{P}_0 = Q_0 = \mathcal{M}$$
$$\mathcal{P}_1 = \mathrm{Ult}_0(\mathcal{M}, G)$$
$$Q_1 = \begin{cases} \mathrm{Ult}_0(\mathcal{M}), \dot{E}^{\mathcal{M}}_\rho & \text{if } \rho \in \operatorname{dom} \dot{E}^{\mathcal{M}} \\ \mathcal{M} & \text{if } \rho \notin \operatorname{dom} \dot{E}^{\mathcal{M}} \end{cases}$$

Thus $E^{\mathcal{T}}_0$ is always equal to G, and $E^{\mathcal{U}}_0$ either does not exist or is equal to $\dot{E}^{\mathcal{M}}_\rho$.

The remainder of the trees $\mathcal{T}$ and $\mathcal{U}$ is determined by the comparison process. At successor steps we pick an extender, or two extenders, representing the least disagreement, and apply these to possibly earlier models in their respective trees so as not to move generators along branches of $\mathcal{T}$ and $\mathcal{U}$. At limit steps we use the unique cofinal wellfounded branches of $\mathcal{T}$ and $\mathcal{U}$ given by the iterability and 1-smallness of $\mathcal{M}$.

First we will verify that the iteration stops, that is, that there is an ordinal θ such that the θth model $\mathcal{P}_\theta$ of $\mathcal{T}$ is an initial segment of the θth model Q_θ of $\mathcal{U}$ or vice-versa. There is a slight wrinkle here because the proof that the comparison process terminates uses the initial segment condition on premice, and we don't yet know that this holds for the final extender F of $\mathcal{M}$.

Suppose that the iteration never stops. As in the proof of the comparison lemma (7.1) we have ordinals $1 \leq \alpha < \beta$ such that $E^{\mathcal{T}}_\alpha$ is the trivial completion of $E^{\mathcal{U}}_\beta \restriction \rho^{\mathcal{T}}_\alpha$ (where $\rho^{\mathcal{T}}_\alpha$ is the sup of the generators of $E^{\mathcal{T}}_\alpha$) or, symmetrically, $E^{\mathcal{U}}_\alpha$ is the trivial completion of $E^{\mathcal{T}}_\beta \restriction \rho^{\mathcal{U}}_\alpha$. We may as well assume the former. This is a contradiction as in the proof of the comparison lemma unless $[0,\beta]_U \cap D^{\mathcal{U}} = \varnothing$ and $E^{\mathcal{U}}_\beta = \dot{F}^{Q_\beta}$; that is, $E^{\mathcal{U}}_\beta$ is the unique extender from the Q_β sequence for which we don't have the initial segment condition. But then Q_β is a psuedo-premouse, and thus obeys the initial segment condition on $\dot{F}^{Q_\beta}$ somewhere past its largest cardinal. It follows that $\rho^{\mathcal{T}}_\alpha \geq$ largest cardinal of Q_β. Thus $\mathrm{lh}\, E^{\mathcal{T}}_\alpha$ is not a cardinal of Q_β. On the other hand, $\mathrm{lh}\, E^{\mathcal{T}}_\alpha$ is a cardinal of $\mathcal{P}_{\alpha+1}$, hence of $\mathcal{P}_\beta$. This contradicts the fact that $\dot{F}^{Q_\beta}$ is part of the least disagreement between $\mathcal{P}_\beta$ and Q_β, and hence the comparison must terminate.

So let θ be such that $\mathcal{P}_\theta$ is an initial segment of Q_θ or vice-versa. The Dodd-Jensen lemma, adapted to our present situation, implies that $\mathcal{P}_\theta = Q_\theta$, $D^{\mathcal{T}} \cap [0,\theta]_T = \varnothing = D^{\mathcal{U}} \cap [0,\theta]_U$, and $i^{\mathcal{T}}_{0\theta} = i^{\mathcal{U}}_{0\theta}$. The trees involving the extender G must have well founded branches since they can be embedding into trees using F instead of G. Thus we can apply the Dodd-Jensen lemma to a tree involving G even though G is not a member of $\mathcal{M}$.

Now let α be least such that $\alpha + 1 \in (0,\theta)_T$, and β be least such that $\beta + 1 \in (0,\theta)_U$. As $i^{\mathcal{T}}_{0\theta} = i^{\mathcal{U}}_{0,\theta}$ we have that $E^{\mathcal{T}}_\alpha$ and $E^{\mathcal{U}}_\beta$ are compatible up to $\inf(\rho^{\mathcal{T}}_\alpha, \rho^{\mathcal{U}}_\beta)$, that is, either $E^{\mathcal{T}}_\alpha$ is the trivial completion of $E^{\mathcal{U}}_\beta \restriction \rho^{\mathcal{T}}_\alpha$ or $E^{\mathcal{U}}_\beta$ is the trivial completion of $E_\alpha^\mathcal{T} \restriction \rho_\beta^\mathcal{U}$. When we proved that this comparison terminates we derived a contradiction from this situation using the assumption that both of the ordinals α and β are greater than 0. It follows that at least one of the ordinals α and β must be equal to 0, but on the other hand α and β cannot both be 0, for $E_\alpha^\mathcal{T}$ is always G while if $E_\beta^\mathcal{U}$ exists then it is equal to $\dot{E}_\rho^\mathcal{M}$. If $E_\beta^\mathcal{U}$ exists then ρ is a limit ordinal and ρ is a limit of generators of G, but not a limit of generators of $\dot{E}_\rho^\mathcal{M}$ and hence G and $\dot{E}_\rho^\mathcal{M}$ are not compatible with each other.

Thus we have two cases, depending on which of α and β is equal to 0. Suppose first $\beta = 0$, so that $\alpha \neq 0$ and $E_\beta^\mathcal{U} = \dot{E}_\rho^\mathcal{M}$ (and $\rho \in \operatorname{dom} \dot{E}^\mathcal{M}$). Then, ρ is a limit ordinal, hence a limit of generators of F, and hence a cardinal of $\mathcal{P}_1 = \mathrm{Ult}(\mathcal{M}, G)$. So ρ is a cardinal of $\mathcal{P}_\alpha$. As $\mathcal{P}_\alpha$ is a psuedo-premouse or a premouse, $E_\alpha^\mathcal{T}$ satisfies the initial segment condition in $\mathcal{P}_\alpha$ somewhere past ρ. But $\dot{E}_\rho^\mathcal{M}$ is the trivial completion of $E_\alpha^\mathcal{T} \restriction \nu$, where $\nu < \rho \leq \rho_\alpha^\mathcal{T}$ is the sup of the generators of $\dot{E}_\rho^\mathcal{M}$. Thus $\dot{E}_\rho^\mathcal{M} \in \mathcal{P}_\alpha$, so that ρ is not a cardinal of $\mathcal{P}_\alpha$. This is a contradiction, and hence $\beta \neq 0$.

Now suppose $\alpha = 0$, so that $E_\alpha^\mathcal{T} = G$.

First suppose that $\rho - 1$ exists. Then $\rho \notin \operatorname{dom} \dot{E}^\mathcal{M}$ so $\mathcal{Q}_1 = \mathcal{Q}_0$. Also, letting $\gamma = \operatorname{lh} G$, $\mathrm{Ult}_0(\mathcal{M}, G)$ and $\mathrm{Ult}_0(\mathcal{M}, F)$ agree below γ, so that $\mathcal{M}$ and $\mathrm{Ult}_0(\mathcal{M}, G)$ agree below γ. That is, $\mathcal{P}_1$ and $\mathcal{Q}_1$ agree below γ. Since γ is a cardinal of $\mathcal{P}_1$, γ is a cardinal of $\mathcal{P}_\beta$. As $\mathcal{Q}_\beta$ is either a premouse or a psuedo-premouse, and $E_\beta^\mathcal{U}$ is part of the least disagreement between $\mathcal{P}_\beta$ and $\mathcal{Q}_\beta$ (so that γ is a cardinal in $J_\eta^{\mathcal{Q}_\beta}$, where $\eta = \operatorname{lh} E_\beta^\mathcal{U}$), $E_\beta^\mathcal{U} \restriction \gamma = G$ is on the sequence of $\mathcal{Q}_\beta$. (One must also consider the $\eta = \gamma$ case. Then $E_\beta^\mathcal{U} = G$, and we have the same contradiction.) Thus G is on the sequence of $\mathcal{Q}_1 = \mathcal{Q}_0 = \mathcal{M}$. This contradicts our choice of G, so that ρ is not a successor ordinal.

Suppose next ρ is a limit ordinal, but $\rho \notin \operatorname{dom} \dot{E}^\mathcal{M}$. If ρ is not itself a generator of F, then again $\mathrm{Ult}_0(\mathcal{M}, G)$ agrees with $\mathrm{Ult}_0(\mathcal{M}, F)$ below $\gamma = \operatorname{lh} G$, and the argument from the last paragraph yields a contradiction. If ρ is a generator of F, then the natural embedding $\pi : \mathrm{Ult}(\mathcal{M}, G) \to \mathrm{Ult}(\mathcal{M}, F)$ has critical point ρ, so the agreement is not obvious. Nevertheless, Theorem 8.2 easily implies that $\mathrm{Ult}_0(\mathcal{M}, G)$ and $\mathrm{Ult}_0(\mathcal{M}, F)$ do agree below $\gamma = \operatorname{lh} G = (\rho^+)^{\mathrm{Ult}_0(\mathcal{M}, G)}$. So again we reach a contradiction as in the last paragraph.

Finally, suppose ρ is a limit ordinal and $\rho \in \operatorname{dom} \dot{E}^\mathcal{M}$. From Theorem 8.2 we get that $\mathrm{Ult}_0(\mathcal{M}, G)$ agrees with $\mathrm{Ult}_0(\mathrm{Ult}_0(\mathcal{M}, F), \dot{E}_\rho^\mathcal{M})$ below $\gamma = \operatorname{lh} G$, which implies $\mathcal{P}_1$ agrees with $\mathcal{Q}_1 = \mathrm{Ult}_0(\mathcal{M}, \dot{E}_\rho^\mathcal{M})$ below γ. As γ is a cardinal of $\mathcal{P}_1$, γ is a cardinal of $\mathcal{P}_\beta$, hence of $J_\eta^{\mathcal{Q}_\beta}$ where $\eta = \operatorname{lh} E_\beta^\mathcal{U}$. Since $\rho \notin \operatorname{dom} \dot{E}^{\mathcal{Q}_\beta}$, and $\mathcal{Q}_\beta$ satisfies at worst the initial segment condition on psuedo-premice, G is on the $\mathcal{Q}_\beta$ sequence. So G is on the sequence of $\mathcal{Q}_1 = \mathrm{Ult}_0(\mathcal{M}, \dot{E}_\rho^\mathcal{M})$. This contradicts our choice of G. $\square$