§5. Iteration Trees

We generalize the key tool of Martin-Steel [MS] to the fine structure context.

DEFINITION 5.0.1. A *tree order on α* (for $\alpha \in \mathrm{OR}$) is a strict partial order T of α such that

(1) $\beta \neq 0 \Rightarrow 0T_\beta$,
(2) $\beta T\gamma \Rightarrow \beta < \gamma$,
(3) $\{\beta \mid \beta T\gamma\}$ is wellordered by T,
(4) γ limit $\Rightarrow \{\beta \mid \beta T\gamma\}$ is cofinal in γ (i.e. $\in$ cofinal) and
(5) γ successor $\Leftrightarrow \gamma$ is a T-successor.

DEFINITION 5.0.2. If T is a tree order then

$$[\beta,\gamma]_T = \{\eta \mid \eta = \beta \vee \beta T\eta T\gamma \vee \eta = \gamma\}$$

and similarly for $(\beta,\gamma]_T$, $[\beta,\gamma)_T$, and $(\beta,\gamma)_T$.

DEFINITION 5.0.3. T-Pred$(\gamma+1)$ is the unique ordinal $\eta T\gamma$ such that $(\eta,\gamma)_T = \varnothing$.

DEFINITION 5.0.4. Let $\mathcal{M} = \mathcal{J}^{\vec{E}}_\beta$ be a ppm. Then for $\gamma \leq \beta$, $\mathcal{J}^{\mathcal{M}}_\gamma = \mathcal{J}^{\vec{E}}_\gamma$. For $\gamma > \beta$, $\mathcal{J}^{\mathcal{M}}_\gamma$ is undefined.

DEFINITION 5.0.5. Let $\mathcal{M}$ and $\mathcal{N}$ be ppm's. Then $\mathcal{M}$ is an *initial segment* of $\mathcal{N}$ iff $\exists\gamma(\mathcal{M} = \mathcal{J}^{\mathcal{N}}_\gamma)$. $\mathcal{M}$ is a *proper initial segment* of $\mathcal{N}$ iff $\mathcal{M}$ is an initial segment of $\mathcal{N}$ and $\mathcal{N}$ is not an initial segment of $\mathcal{M}$.

Notice that if $\beta \in \operatorname{dom} \vec{E}$, then $(J^{\vec{E}}_\beta, \in, \vec{E} \restriction \beta)$ is not an initial segment of $\mathcal{J}^{\vec{E}}_\beta$ according to our definition, although we might reasonably have regarded it as such.

DEFINITION 5.0.6. Let $\mathcal{M}$ and $\mathcal{N}$ be ppm's. Then $\mathcal{M}$ and $\mathcal{N}$ *agree below* γ iff $\mathcal{J}^{\mathcal{M}}_\beta = \mathcal{J}^{\mathcal{N}}_\beta$ for all $\beta < \gamma$. (In particular, $\mathcal{J}^{\mathcal{M}}_\beta$ is defined iff $\mathcal{J}^{\mathcal{N}}_\beta$ is defined, for all $\beta < \gamma$.)

If $\mathcal{M}$ is a ppm then a *iteration tree of length θ on $\mathcal{M}$* is a 4-tuple

$$\mathcal{T} = \langle T, \deg, D, \langle E_\alpha, \mathcal{M}^*_{\alpha+1} \mid \alpha+1 < \theta\rangle\rangle,$$

where T is a tree order, which satisfies conditions (1–8) below. We write ρ_α for the natural length of E_α. We will also define ppm $\mathcal{M}_\alpha$ for $\alpha < \theta$ and embeddings $i_{\alpha,\beta}\colon \mathcal{M}_\alpha \to \mathcal{M}_\beta$ for ordinals α and β less than θ such that $\alpha T\beta$ and $D \cap (\alpha\beta]_T = \varnothing$.

(1) $\mathcal{M}_0 = \mathcal{M}$, and each $\mathcal{M}_\alpha$ is a ppm.

(2) E_α is the extender coded by $\dot{F}^{\mathcal{N}}$, for some active ppm $\mathcal{N}$ which is an initial segment of $\mathcal{M}_\alpha$.

(3) $\alpha < \beta \Rightarrow \mathrm{lh}(E_\alpha) < \mathrm{lh}(E_\beta)$.

(4) If T-Pred$(\alpha+1) = \beta$ then $\kappa = \mathrm{crit}\, E_\alpha < \rho_\beta$, and $\mathcal{M}^*_{\alpha+1}$ is an initial segment $\mathcal{J}_\gamma^{\mathcal{M}_\beta}$ of $\mathcal{M}_\beta$ such that $P(\kappa) \cap \mathcal{M}^*_{\alpha+1} = P(\kappa) \cap \mathcal{N}$. Moreover

$$\alpha + 1 \in D \Leftrightarrow \mathcal{J}_\gamma^{\mathcal{M}_\beta} \text{ is a proper initial segment of } \mathcal{M}_\beta\,.$$

If we take $n = \deg(\alpha+1)$ then $\kappa < \rho_n^{\mathcal{M}^*_{\alpha+1}}$ and

$$\mathcal{M}_{\alpha+1} = \mathrm{Ult}_n(\mathcal{M}^*_{\alpha+1}, E_\alpha)$$

and if $\alpha + 1 \notin D$, then

$$i_{\beta,\alpha+1} = \text{canonical embedding of } \mathcal{M}_\beta \text{ into } \mathrm{Ult}_n(\mathcal{M}_\beta, E_\alpha)\,,$$

and $i_{\gamma,\alpha+1} = i_{\beta,\alpha+1} \circ i_{\gamma,\beta}$ for all $\gamma T \beta$ such that $(\gamma,\beta]_T \cap D = \varnothing$.

(5) If $\lambda < \theta$ is a limit, then $D \cap [0,\lambda)_T$ is finite, and letting γ be the largest element of $D \cap [0,\lambda)_T$,

$$\begin{aligned} \mathcal{M}_\lambda &= \text{direct limit of } \mathcal{M}_\alpha, \alpha \in [\gamma,\lambda)_T, \text{ under the } i_{\alpha\beta}\text{'s} \\ i_{\eta\lambda} &= \text{canonical embedding of } \mathcal{M}_\eta \text{ into } \mathcal{M}_\lambda\,, \text{ for } \eta \in [\gamma,\lambda)_T\,. \end{aligned}$$

(6) $\mathcal{M}^*_{\alpha+1}$ is $\deg(\alpha+1)$-sound.

(7) If $\gamma + 1 T \alpha + 1$ and $D \cap (\gamma+1, \alpha+1]_T = \varnothing$, then $\deg(\gamma+1) \geq \deg(\alpha+1)$.

(8) For $\lambda \leq \theta$ a limit, $\deg(\lambda) = \deg(\alpha+1)$, for all sufficiently large $\alpha + 1 T \lambda$.

Notice that $\mathcal{T}$ determines the ordinals ρ_α's, the embeddings $i_{\alpha\beta}$'s, and the ppm $\mathcal{M}_\alpha$.

Conditions (6–8) can be dropped in some contexts. Condition (6) guarantees that $i^*_{\alpha+1}$ is a $\deg(\alpha+1)$-embedding. Condition (7) says that the ultrapowers taken along branches of $\mathcal{T}$ are of decreasing elementarity; it allows us to "copy $\mathcal{T}$" via certain embeddings.

Lemma 5.1. *Let $\mathcal{T} = \langle T, \deg, D, \langle E_\alpha, \mathcal{M}^*_{\alpha+1} \mid \alpha+1 < \theta\rangle\rangle$ be an iteration tree. Then if $\alpha < \beta < \theta$*

1. *$\mathcal{M}_\alpha$, and $\mathcal{M}_\beta$ agree below $\mathrm{lh}\, E_\alpha$, and*
2. *$\mathrm{lh}(E_\alpha)$ is a cardinal of $\mathcal{M}_\beta$, and in particular $\mathcal{M}_\alpha$ and $\mathcal{M}_\beta$ do not agree below $\mathrm{lh}(E_\alpha) + 1$.*

Proof. By induction on β. Let $\beta = \gamma + 1$. Since $\alpha \leq \gamma \Rightarrow \mathrm{lh}\, E_\alpha \leq \mathrm{lh}\, E_\gamma$, it is enough for (1) to show that $\mathcal{M}_{\gamma+1}$ and $\mathcal{M}_\gamma$ agree below $\mathrm{lh}\, E_\gamma$. Let $E_\gamma = \dot{F}^{\mathcal{N}}$

where $\mathcal{N}$ is an initial segment of $\mathcal{M}_\gamma$. Now $\mathcal{M}_\gamma$ agrees with $\mathrm{Ult}_0(\mathcal{N}, E_\gamma)$ below $\mathrm{lh}\, E_\gamma$ by coherence. But $M_{\gamma+1} = \mathrm{Ult}_n(\mathcal{M}^*_{\gamma+1}, E_\gamma)$, where $\mathcal{M}^*_{\gamma+1}$ is an initial segment of $\mathcal{M}_\delta$, some $\delta \leq \gamma$, with crit $E_\gamma < \min(\mathrm{OR}^{M^*_{\gamma+1}}, \mathrm{lh}\, E_\delta)$. By induction, $\mathcal{M}_\delta$ agrees with $\mathcal{M}_\gamma$ below $\mathrm{lh}\, E_\delta$, hence below crit E_γ. Thus $\mathcal{M}^*_{\gamma+1}$ agrees with $\mathcal{M}_\gamma$ below crit E_γ. So $\mathcal{M}_{\gamma+1}$ agrees with $\mathrm{Ult}_0(\mathcal{N}, E_\gamma)$ below $\mathrm{lh}\, E_\gamma$, hence with $\mathcal{M}_\gamma$ below $\mathrm{lh}\, E_\gamma$. (Notice here that if $\eta < \mathrm{lh}\, E_\gamma$, then the function representing $\mathcal{J}_\eta^{\mathcal{M}_{\gamma+1}}$ is in both $\mathcal{M}^*_{\gamma+1}$ and $\mathcal{N}$. In fact, $P(\text{crit } E_\gamma) \cap \mathcal{M}^*_{\gamma+1} = P(\text{crit } E_\gamma) \cap \mathcal{N}$. For $\subseteq$ is true by fiat and $\supseteq$ by our induction hypotheses.)

For the second assertion it is enough to show $\mathrm{lh}\, E_\gamma$ is a cardinal in $\mathcal{M}_{\gamma+1}$ (using (1) and strong acceptability). Let us adopt the notation of the last paragraph. The definition of good extender sequence guarantees $\mathrm{lh}\, E_\gamma$ is a cardinal in $\mathrm{Ult}_0(\mathcal{N}, E_\gamma)$. But if $A \subseteq \mathrm{lh}\, E_\gamma$ and $A \in \mathcal{M}_{\gamma+1}$ then $A = [a, f]$ for some function $f : [\mathrm{crit}(E_\gamma)]^n \to \mathcal{J}^{\mathcal{M}^*_{\gamma+1}}_{\mathrm{crit}(E_\gamma)}$ in $\mathcal{M}^*_{\gamma+1}$. But then $f \in \mathcal{N}$, so $A \in \mathrm{Ult}_0(\mathcal{N}, E_\gamma)$, so A doesn't collapse $\mathrm{lh}\, E_\gamma$.

We leave the case β is a limit to the reader. □

Let H_λ be the set of sets hereditarily of cardinality $< \lambda$. From 5.1 we get, using the notation there, that if $\alpha < \beta$ and $\lambda = \mathrm{lh}\, E_\alpha$, then $H_\lambda^{\mathcal{M}_\beta} = |\mathcal{J}_\lambda^{\mathcal{M}_\alpha}|$.

A few miscellaneous remarks on the definition of an iteration tree:

(a) It is easy to see from the above that if $\mathcal{T}$ is an iteration tree of length θ, $\alpha < \beta < \theta$, and F is an extender from the $\mathcal{M}_\beta$ sequence (i.e. F on $\dot{E}^{\mathcal{M}_\beta}$ or $F = \dot{F}^{\mathcal{M}_\beta}$), then $E_\alpha \restriction \rho_\alpha \neq F \restriction \rho_\alpha$. For suppose $E_\alpha \restriction \rho_\alpha = F \restriction \rho_\alpha$. If F is on $\dot{E}^{\mathcal{M}_\beta}$, this implies $E_\alpha \restriction \rho_\alpha \in \mathcal{M}_\beta$, and therefore that $\mathrm{lh}\, E_\alpha$ is not a cardinal of $\mathcal{M}_\beta$, contrary to 5.1. If $F = \dot{F}^{\mathcal{M}_\beta}$, then $\dot{\nu}^{\mathcal{M}_\beta} = \nu \geq \mathrm{lh}\, E_\alpha$ since $\mathrm{lh}\, E_\alpha$ is a cardinal of $\mathcal{M}_\beta$, and $\rho_\alpha < \nu$. By the initial segment condition on good extender sequences, $F \restriction \rho_\alpha \in \mathcal{M}_\beta$. Since $E_\alpha \restriction \rho_\alpha$ collapses $\mathrm{lh}\, E_\alpha$, we again have a contradiction.

(b) The demand in (4) that crit $E_\alpha < \rho_\beta$, rather than just crit $E_\alpha < \mathrm{lh}\, E_\beta$, makes a difference only when $E_\beta = \dot{F}^{\mathcal{P}}$ for some $\mathcal{P}$ of type III, so that $\rho_\beta = \nu^{\mathcal{P}}$, and crit $E_\alpha = \rho_\beta = \nu^{\mathcal{P}}$. In this case our official definition won't allow us to apply E_α to an initial segment of $\mathcal{M}_\beta$ to form $\mathcal{M}_{\alpha+1}$.

(c) Suppose we have an iteration tree

$$\mathcal{T} = \langle T, \mathrm{deg}, D, \langle E_\gamma, \mathcal{M}^*_{\gamma+1} \mid \gamma + 1 < \alpha + 1 \rangle \rangle,$$

so that the last model $\mathcal{M}_\alpha$ of $\mathcal{T}$ is determined. Suppose $F = \dot{F}^{\mathcal{P}}$ for some initial segment $\mathcal{P}$ of $\mathcal{M}_\alpha$. How may we extend $\mathcal{T}$ one step further so that $F = E_\alpha$? Let us assume all ultrapowers to follow are wellfounded. Assume also that $\mathrm{lh}\, F > \mathrm{lh}\, E_\gamma$ for all $\gamma < \alpha$. Let $\kappa = \text{crit } F$.

(i) We may set $\alpha T \alpha + 1$ and take $\mathcal{M}^*_{\alpha+1}$ to be any initial segment of $\mathcal{M}_\alpha$ such that $\mathcal{P}$ is an initial segment of $\mathcal{M}^*_{\alpha+1}$ and $P(\kappa) \cap |\mathcal{P}| = P(\kappa) \cap |\mathcal{M}^*_{\alpha+1}|$. Notice that if Q is a type III initial segment of $\mathcal{M}_\alpha$, $\mathcal{P}$ an initial segment of Q, and $P(\kappa)^{\mathcal{P}} = P(\kappa)^Q$, then $\kappa < \nu^Q$ since $(\kappa^+)^{\mathcal{P}} = (\kappa^+)^Q$, whereas ν^Q is the largest cardinal of Q. Thus we can form $\mathrm{Ult}(Q^{\mathrm{sq}}, F)$.

(ii) Suppose $\beta < \alpha$ and $\kappa < \rho_\beta$. Then we may set $\beta = T\text{-pred}(\alpha+1)$. The candidates for $\mathcal{M}^*_{\alpha+1}$ are precisely those structures $\mathcal{J}_\gamma^{\mathcal{M}_\beta}$ such that $\gamma \geq \mathrm{lh}\, E_\beta$ and $P(\kappa) \cap |\mathcal{J}_\gamma^{\mathcal{M}_\beta}| = P(\kappa) \cap |\mathcal{J}_{\mathrm{lh}\, E_\beta}^{\mathcal{M}_\beta}|$. Any of these candidates will do for $\mathcal{M}^*_{\alpha+1}$. Notice again that if $Q = \mathcal{J}_\gamma^{\mathcal{M}_\beta}$ for such a γ, then $\kappa < \nu^Q$ as $(\kappa^+)^{\mathcal{P}}$ is a cardinal of Q. So we can squash Q if necessary and still apply F.

In almost all of the iteration trees used in this paper, the extension of $\mathcal{T}$ to $\alpha+1$ will be determined by the choice of E_α. We take T-Pred$(\alpha+1)$ to be the least ordinal α^*, if there is one, such that $\rho_{\alpha^*} > \mathrm{crit}(E_\alpha)$ and $\alpha^* = T\text{-Pred}(\alpha+1) = \alpha$ otherwise. Then we take $\mathcal{M}^*_{\alpha+1}$ to be the largest initial segment of $\mathcal{M}_{\alpha^*}$ which does not contain any subset of $\mathrm{crit}(E_\alpha)$ other than those measured by E_α. Finally we take $\deg(\alpha+1)$ to be the the largest ordinal such that $\mathrm{crit}(E_\alpha) < \rho_n^{\mathcal{M}^*_{\alpha+1}}$. See the definition of n-maximal, definition 6.1.2, for details.

Iterability. If T is a tree order on θ, then a branch of T is a set $b \subset \theta$ such that b is wellordered by $\mathcal{T}$ with limit order type, and $\forall\alpha \in b\, \forall\beta(\beta T\alpha \Rightarrow \beta \in b)$. We call b *cofinal* iff $\sup b = \theta$. We call b *maximal* iff $b \neq [0,\lambda)_T$ for all $\lambda < \theta$. If $\mathcal{T} = \langle T, \deg, D, \langle E_\alpha, \mathcal{M}^*_{\alpha+1} \mid \alpha+1 < \theta\rangle\rangle$ is an iteration tree, then a (maximal, cofinal) branch of $\mathcal{T}$ is a (maximal, cofinal) branch of $\mathcal{T}$. If b is a branch of $\mathcal{T}$ such that $D \cap b$ is finite, with largest element γ, then we set

$$\mathcal{M}_b = \text{direct limit of } \mathcal{M}_\alpha,\ \alpha \in b - \gamma, \quad \text{under the} \quad i_{\alpha_\beta}\text{'s}.$$

We say a ***branch** b of $\mathcal{T}$ **is wellfounded*** iff $D \cap b$ is finite and $\mathcal{M}_b$ is wellfounded.

We now state the iterability property which qualifies premice having no more than one Woodin cardinal as mice. We shall eventually show that all levels of the model we construct have this property by quoting results of Martin-Steel [MS].

DEFINITION 5.1.1. If $\mathcal{T} = \langle T, \deg, D, \langle E_\alpha, \mathcal{M}^*_{\alpha+1} \mid \alpha+1 < \theta\rangle\rangle$ then for $\beta \leq \theta$

$$\mathcal{T} \restriction \beta = \langle T \cap (\beta \times \beta), \deg \restriction \beta, D \cap \beta, \langle E_\alpha, \mathcal{M}^*_{\alpha+1} \mid \alpha+1 < \beta\rangle\rangle.$$

DEFINITION 5.1.2. Let $\mathcal{T}$ be an iteration tree of length θ. $\mathcal{T}$ is *simple* if and only if every maximal wellfounded branch of $\mathcal{T}$ is cofinal in θ, and $\mathcal{T}$ has at most one cofinal in θ wellfounded branch.

Notice that by definition 5.0.1(4) it follows that $\mathcal{T}$ is simple iff for every limit $\lambda \leq \theta$, $\mathcal{T} \restriction \lambda$ has at most one cofinal wellfounded branch.

We shall deal almost exclusively with simple iteration trees. The fact that it suffices to do so is one of the key things we must prove. (c.f. Theorem 6.2.)

DEFINITION 5.1.3. Let $\kappa \le \omega$. Then an iteration tree $\mathcal{T}$ is ***k*-bounded** iff $\deg^{\mathcal{T}}(\alpha+1) \le k$ whenever α is such that $[0, \alpha+1]_T \cap D^{\mathcal{T}} = \emptyset$.

Notice that by clause (7) in the definition of "iteration tree", if $\deg(\alpha+1) \le k$ whenever $\alpha+1 \notin D$ and T-pred$(\alpha+1) = 0$, then $\mathcal{T}$ is k-bounded.

DEFINITION 5.1.4. Let $\mathcal{M}$ be a ppm, and let $k \le \omega$. (1) $\mathcal{M}$ is ***singly k-iterable*** if any k-bounded iteration tree

$$\mathcal{T} = \langle T, \deg, D, \langle E_\alpha, \mathcal{M}^*_{\alpha+1} \mid \alpha+1 < \theta\rangle\rangle$$

such that $\mathcal{T} \restriction \lambda$ is simple for all $\lambda < \theta$ satisfies conditions (a) and (b) below:

(a) If θ is a limit ordinal, then $\mathcal{T}$ has a cofinal wellfounded branch.
(b) Suppose $\alpha < \theta = \beta + 1$ and $\mathcal{N}$ is an active initial segment of $\mathcal{M}_\beta$, such that $\mathrm{crit}(\dot{F}^{\mathcal{N}}) < \rho_\alpha$, and suppose that $\mathcal{P} = \mathcal{J}^{\mathcal{M}_\alpha}_\gamma$ for some $\gamma \ge \mathrm{lh}\, E_\alpha$, with $\kappa = \mathrm{crit}(\dot{F}^{\mathcal{N}}) < \rho^{\mathcal{P}}_n$ and $P(\kappa) \cap |\mathcal{P}| \subseteq \mathcal{N}$. Then

$$\mathrm{Ult}_n(\mathcal{P}, \dot{F}^{\mathcal{N}}) \text{ is wellfounded}$$

(provided also $n \le k$ when $[0, \alpha]_T \cap D \ne \emptyset$ and $\mathcal{P} = \mathcal{M}_\alpha$).

(2)We say $\mathcal{M}$ is ***k-iterable*** if it is singly k-iterable and satisfies conditions (a) and (b) below:

(a) If $n < \omega$, and $(\mathcal{T}_i : i \le n)$ is a sequence of iteration trees such that $\mathcal{T}_0$ is a k-bounded simple iteration tree on $\mathcal{M}$, and for $i > 0$ $\mathcal{T}_i$ is a simple iteration tree on the last model $\mathcal{M}^{\mathcal{T}_{i-1}}_{\theta_{i-1}}$ of $\mathcal{T}_{i-1}$, and $\mathcal{T}_i$ is k-bounded whenever $D^{\mathcal{T}_j} \cap [0, \theta_j]_{T_j} = \emptyset$ for all $j < i$, then the last model $\mathcal{M}^{\mathcal{T}_n}_{\theta_n}$ of $\mathcal{T}_n$ is singly k-iterable.
(b) Suppose that $(\mathcal{T}_i : i < \omega)$ is as in (a). Then $[0, \theta_i]_{T_i} \cap D_i = \emptyset$ for all but finitely many i, so that we have a canonical embedding $\tau_i : \mathcal{M}^i_0 \to \mathcal{M}^{i+1}_0 = \mathcal{M}^i_{\theta_i}$ defined for sufficiently large $i < \omega$. Moreover, the direct limit of the $\mathcal{M}^i_0$'s under the τ_i's is wellfounded.

It is easy to see that if $\mathcal{M}$ is k-iterable, $\mathcal{T}$ is a k-bounded simple tree on $\mathcal{M}$, and $\mathcal{P}$ is a model on $\mathcal{T}$, then $\mathcal{P}$ is k-iterable.

It may seem that we can derive (2) and (3) from (1). Given $\mathcal{T}_i$'s as in (2) or (3), we can lay the $\mathcal{T}_i$'s "end-to-end" and produce a tree $\mathcal{S}$ to which we can then apply (1). The problem is that $\mathcal{S}$ may not be, formally speaking, an iteration tree: we may have $\alpha < \beta$ such that $\mathrm{lh}\, E^{\mathcal{S}}_\alpha \not< \mathrm{lh}\, E^{\mathcal{S}}_\beta$. This can definitely occur in the proof of the Dodd-Jensen lemma on the minimality of iteration maps, which is our application of (2) and (3). Rather than generalize the definition of "iteration tree" we prefer to complicate the definition of iterability.

The k-iterability of $\mathcal{M}$ allows us to build k-bounded iteration trees on $\mathcal{M}$ freely as long as the tree built so far is simple. For then (1b) guarantees we can proceed at successor steps without fear of illfoundedness. Clause (1a) guarantees that at a limit ordinal λ we have a cofinal in λ wellfounded branch. Thus we can choose this branch to be $[0,\lambda)_{\mathcal{T}}$.

It should be remarked that a theorem of Woodin asserts that the model $L[\vec{E}]$ which we are constructing is not fully iterable, in the sense that there is a tree which is a member of $L[\vec{E}]$ but which has no well founded branch which is a member of $L[\vec{E}]$. If we make the additional assumption that every set has a sharp then we can prove that $V \models L[\vec{E}]$ is iterable: that is, every tree on $L[\vec{E}]$ has a well founded branch, with both the tree and the branch being in V. It is a theorem of ZFC that every iteration tree which involves only extenders from a proper initial segment of the sequence $\vec{E}$ has a well founded branch, so that this much iterablity is true in both V and $L[\vec{E}]$. The proof that our construction works will depend on this iterability in V of initial segments of $\vec{E}$. It is important for this that $L[\vec{E}]$ has no more than the one Woodin cardinal, which is the supremum of $\mathrm{dom}(\vec{E})$.

DEFINITION 5.1.5. Let $\mathcal{M}$ be a ppm. Then $\mathcal{M}$ is 1-*small* iff whenever $\kappa =$ crit $\dot{F}^{\mathcal{N}}$ for some initial segment $\mathcal{N}$ of $\mathcal{M}$, then $\mathcal{J}^{\mathcal{M}}_{\kappa} \models$ "There are no Woodin cardinals".

It is possible for a 1-small ppm $\mathcal{M}$ to satisfy "there is a Woodin cardinal"; however, such an $\mathcal{M}$ cannot satisfy "there is a sharp for an inner model with a Woodin cardinal".

DEFINITION 5.1.6. A 1-*small mouse* is a 1-small, ω-iterable premouse.

DEFINITION 5.1.7. A 1-*small coremouse* is a 1-small mouse which is completely sound.

In general (for models with more than a Woodin cardinal) ω-iterability will not convert a premouse into a mouse.

Since all the mice we shall deal with in the moderately near future will be 1-small, we make the temporary convention:

$$\text{mouse} = \text{1- small mouse}$$

$$\text{coremouse} = \text{1- small coremouse}$$

Embeddings of Iteration Trees. We now head toward the Dodd-Jensen lemma on the minimality of iteration maps. For that we must show, given an embedding $\pi\colon \mathcal{M} \to \mathcal{N}$ and a iteration tree $\mathcal{T}$ on $\mathcal{M}$, how to extend π to an embedding from $\mathcal{T}$ into an iteration tree $\mathcal{U}$ on $\mathcal{N}$. Since not all of the embeddings involved will be full n-embeddings we need a new definition:

DEFINITION. We say $\pi\colon \mathcal{M} \to \mathcal{N}$ is a *weak n-embedding* if $\mathcal{M}$ and $\mathcal{N}$ are premice of types I or II or sppm's, and there is a set $X \subset \mathcal{M}$ such that the following four conditions hold:

(i) The models $\mathcal{M}$ and $\mathcal{N}$ are n-sound, and X is a subset of $\mathcal{M}$ such that

$\{p_n^{\mathcal{M}}, \rho_n^{\mathcal{M}}\} \subset X$, and X is cofinal in $\rho_n^{\mathcal{M}}$.

(ii) π is $r\Sigma_n$ (respectively $q\Sigma_n$) elementary, and π is $r\Sigma_{n+1}$ (respectively $q\Sigma_{n+1}$) elementary on parameters from X.

(iii) $\pi(p_i(\mathcal{M})) = p_i(\mathcal{N})$ for $i \leq n$

(iv) $\pi(\rho_i(\mathcal{M})) = \rho_i(\mathcal{N})$ for $i < n$, and $\sup \pi''\rho_n(\mathcal{M}) \leq \rho_n(\mathcal{N})$.

If $\mathcal{M}$ and $\mathcal{N}$ are type III, a weak n-embedding from $\mathcal{M}$ to $\mathcal{N}$ is a weak n-embedding from $\mathcal{M}^{sq}$ to $\mathcal{N}^{sq}$.)

Note that this definition is obtained from the definition of a n-embedding by weakening clause (ii) from $r\Sigma_{n+1}$ to $r\Sigma_n$ except for parameters from X, and weakening clause (iv) by eliminating the requirement that $\pi''\rho_n(\mathcal{M})$ be cofinal in $\rho_n(\mathcal{N})$. Normally it is the existence of a set X which is important, rather than the choice of the set X.

The following is a useful fact about (n, X)-embeddings:

Proposition. *Suppose that $\pi\colon \mathcal{P} \to \mathcal{Q}$ is a weak n-embedding and κ is an ordinal in $\mathrm{OR}^{\mathcal{P}}$. Then $\mathcal{P} \models \kappa$ is a cardinal if and only if $\mathcal{Q} \models \pi(\kappa)$ is a cardinal.*

Proof. This a is obvious if $n \geq 1$, so let $n = 0$. Recall $\rho_0(\mathcal{P}) = \mathrm{OR}^{\mathcal{P}}$, so that the set X on which π is $r\Sigma_1$ elementary is cofinal in $\mathrm{OR}^{\mathcal{P}}$. Fix κ s.t. $\mathcal{P} \models \kappa$ is a cardinal, and let $\mu \in X$ be such that $\kappa < \mu$. Let $\xi \in X$, $\mu < \xi$, be such that

$$\mathrm{card}^{\mathcal{P}}(\mu) = \mathrm{card}^{S_\xi^{\mathcal{P}}}(\mu),$$

where "$S_\xi^{\mathcal{P}}$" refers to the ξth level of the Jensen S-hierarchy. Then

$$\mathcal{P} \models \mathrm{card}^{S_\xi^{\mathcal{P}}}(\mu) \text{ is a cardinal}$$

and as $\xi, \mu \in X$

$$\mathcal{Q} \models \mathrm{card}^{S_{\pi(\xi)}^{\mathcal{Q}}}(\pi(\mu)) \text{ is a cardinal.}$$

So, setting $\nu = \mathrm{card}^{S_\xi^{\mathcal{P}}}(\mu)$, we know that $\kappa \leq \nu$ and $\pi(\nu)$ is a cardinal of $\mathcal{Q}$. If $\kappa = \nu$ we're done. If $\kappa < \nu$, then $\mathcal{J}_\nu^{\mathcal{P}} \models \kappa$ is a cardinal, so since the relation $R(z, x) \Leftrightarrow$ "x is a cardinal relative to z" is Σ_0-in-$\mathcal{L} \setminus \{\dot{F}\}$ we know that $\mathcal{J}_{\pi(\nu)}^{\mathcal{Q}} \models \pi(\kappa)$ is a cardinal, and hence $\mathcal{Q} \models \pi(\kappa)$ is a cardinal. □

Lemma 5.2 (Shift lemma). *Let $\bar{\mathcal{M}}$ and $\bar{\mathcal{N}}$ be ppm's, let $\bar{\kappa} = \mathrm{crit}(\dot{F}^{\mathcal{N}})$, and let*

$$\pi : \bar{\mathcal{M}} \to \mathcal{M} \quad \text{be an weak } n\text{-embedding } (n \leq \omega) \text{ and}$$
$$\psi : \bar{\mathcal{N}} \to \mathcal{N} \quad \text{be a weak } 0\text{-embedding}$$

such that $\bar{\mathcal{M}}$ and $\bar{\mathcal{N}}$ agree below $(\bar{\kappa}^+)^{\bar{\mathcal{M}}} \leq (\bar{\kappa}^+)^{\bar{\mathcal{N}}}$, while $\mathcal{M}$ and $\mathcal{N}$ agree below $(\kappa^+)^{\mathcal{M}} \leq (\kappa^+)^{\mathcal{N}}$, and $\pi \restriction (\bar{\kappa}^+)^{\bar{\mathcal{M}}} = \psi \restriction (\bar{\kappa}^+)^{\bar{\mathcal{M}}}$. Suppose $\bar{\kappa} < \rho_n^{\bar{\mathcal{M}}}$, so *that* $\mathrm{Ult}_n(\bar{\mathcal{M}}, \dot{F}^{\bar{\mathcal{N}}})$ *makes sense, as does* $\mathrm{Ult}_n(\mathcal{M}, \dot{F}^{\mathcal{N}})$, *and that both of these ultrapowers are wellfounded. Then there is an embedding* $\sigma\colon \mathrm{Ult}_n(\bar{\mathcal{M}}, \dot{F}^{\bar{\mathcal{N}}}) \to \mathrm{Ult}_n(\mathcal{M}, \dot{F}^{\mathcal{N}})$ *satisfying the following four conditions:*

(a) *The map* σ *is an weak* n*-embedding, and if* π *is an* n*-embedding then so is* σ.

(b) $\mathrm{Ult}_n(\bar{\mathcal{M}}, \dot{F}^{\bar{\mathcal{N}}})$ *agrees with* $\bar{\mathcal{N}}$ *below* $\mathrm{lh}(\dot{F}^{\bar{\mathcal{N}}})$, *while* $\mathrm{Ult}_n(\mathcal{M}, \dot{F}^{\mathcal{N}})$ *agrees with* $\mathcal{N}$ *below* $\mathrm{lh}(\dot{F}^{\mathcal{N}})$.

(c) $\sigma \restriction \mathrm{lh}(\dot{F}^{\bar{\mathcal{N}}}) + 1 = \psi \restriction \mathrm{lh}(\dot{F}^{\bar{\mathcal{N}}}) + 1$.

(d) *The diagram*

$$\begin{array}{ccc} \bar{\mathcal{M}} & \xrightarrow{\pi} & \mathcal{M} \\ \big\downarrow i & & \big\downarrow j \\ \mathrm{Ult}_n(\bar{\mathcal{M}}, \dot{F}^{\bar{\mathcal{N}}}) & \xrightarrow{\sigma} & \mathrm{Ult}_n(\mathcal{M}, \dot{F}^{\mathcal{N}}) \end{array}$$

commutes, where i *and* j *are the canonical* n*-embeddings.*

Remark. We want to allow the possibility $(\bar{\kappa}^+)^{\mathcal{M}} = \mathrm{OR}^{\bar{\mathcal{M}}}$. In this case, we make our standard convention: $\pi(\mathrm{OR}^{\bar{\mathcal{M}}}) = \mathrm{OR}^{\mathcal{M}}$. We allow $\mathrm{lh}(\dot{F}^{\bar{\mathcal{N}}}) = \mathrm{OR}^{\bar{\mathcal{N}}}$ as well, and make a similar convention in (c) of the conclusion.

PROOF. The map σ is defined by

$$\sigma\left([a, f]^{\bar{\mathcal{M}}}_{\dot{F}^{\bar{\mathcal{N}}}}\right) = [\psi(a), \pi(f)]^{\mathcal{M}}_{\dot{F}^{\mathcal{N}}} \qquad \text{if } n = 0, \text{ and}$$

$$\sigma\left([a, f_{\tau,q}]^{\bar{\mathcal{M}}}_{\dot{F}^{\bar{\mathcal{N}}}}\right) = [\psi(a), f_{\tau,\pi(q)}]^{\mathcal{M}}_{\dot{F}^{\mathcal{N}}} \qquad \text{if } n > 0.$$

If X is the set used to show that π is a weak n-embedding then the set $i''X$ will show that σ is a weak n-embedding. It is straightforward to verify that this works. □

DEFINITION. If $\mathcal{T}$ and $\mathcal{U}$ are iteration trees then we say that $\vec{\pi} = \langle \pi_\alpha : \alpha < \mathrm{lh}(\mathcal{T}) \rangle$ is a weak n-embedding from $\mathcal{T}$ to $\mathcal{U}$ if the following 6 conditions are satisfied.

(1) $T^{\mathcal{T}} = T^{\mathcal{U}}$, $\deg^{\mathcal{T}} = \deg^{\mathcal{U}}$ and $D^{\mathcal{T}} = D^{\mathcal{U}}$.

(2) $\pi_0\colon \mathcal{M}_0 \to \mathcal{N}_0$ is a weak n-embedding.

(3) For each ordinal α with $0 \le \alpha < \mathrm{lh}\,\mathcal{T}$ there is a set Y such that $\pi_\alpha\colon \mathcal{M}_\alpha \to \mathcal{N}_\alpha$ is a $(\deg^{\mathcal{T}}(\alpha), Y)$-embedding, where $\mathcal{M}_\alpha$ and $\mathcal{N}_\alpha$ are the αth models of $\mathcal{T}$ and $\mathcal{U}$ respectively.

(4) $\pi_\alpha \restriction \mathrm{lh}\,E_\alpha + 1 = \pi_\delta \restriction \mathrm{lh}\,E_\alpha + 1$ whenever $\alpha < \delta < \theta'$.

(5) $\pi_\gamma \circ i^{\mathcal{T}}_{\alpha\gamma} = i^{\mathcal{U}}_{\alpha\gamma} \circ \pi_\alpha$ whenever $\alpha T \gamma$ and $(\alpha, \gamma]_T \cap D = \varnothing$.

DEFINITION. We say that $\vec{\pi}$ is a *tree embedding* if it is a weak n-embedding for some $n \le \omega$ such that $\mathcal{T}$ is n-bounded if $n < \omega$.

Lemma. *Suppose that*

$$\mathcal{T} = \langle T, \deg, D, \langle E_\alpha, \mathcal{M}^*_{\alpha+1} \mid \alpha + 1 < \theta \rangle \rangle$$

is a n-maximal, n-bounded iteration tree on $\mathcal{M}$, where $n \leq \omega$, and that $\pi\colon \mathcal{M} \to \mathcal{N}$ is an weak n-embedding, where $\mathcal{N}$ is a n-iterable premouse. Then there is a tree $\pi\mathcal{T}$ on $\mathcal{N}$ and a tree embedding $\vec{\pi}\colon \mathcal{T} \to \pi\mathcal{T}$ such that $\pi_0 = \sigma$.

PROOF. We define $\pi\mathcal{T} \restriction \alpha + 1$ and π_α by recursion on $\alpha < \theta$. For $\beta = 0$ we have $\mathcal{N}_0 = \mathcal{N}$ and $\pi_0 = \pi$. Now suppose we have defined $\pi\mathcal{T} \restriction \beta$, together with sets Y_α such that π_α is a $(\deg^{\mathcal{T}}(\alpha), Y_\alpha)$-embedding for each ordinal $\alpha < \beta$.

If $\beta > 0$ is a limit ordinal then we set

$$\mathcal{N}_\beta = \operatorname{dir\,lim}\{\, \mathcal{N}_\alpha : \alpha T \beta \text{ and } D \cap [\alpha, \beta]_T = \varnothing \,\},$$

where the direct limit is taken along the maps $j_{\alpha\gamma}$, and we define π_β by setting $\pi_\beta(i_{\alpha\beta}(x)) = j_{\alpha,\beta}(\pi_\alpha(x))$ for $\alpha T \beta$ such that $[\alpha, \beta)_T \cap D = \varnothing$. Finally we set $Y_\alpha = i^{\mathcal{T}}_{\beta,\alpha}{}''Y_\beta$ for any $\beta T \alpha$ large enough that $i^{\mathcal{T}}_{\beta,\alpha}$ is defined.

For successor ordinals $\beta = \delta + 1$, let $E_\delta = \dot{F}^{\mathcal{P}}$, where $\mathcal{P} = \mathcal{J}^{\mathcal{M}_\delta}_\eta$. Set $Q = \mathcal{J}^{\mathcal{N}_\delta}_{\pi_\delta(\eta)}$, (with the usual convention if $\eta = \operatorname{dom} \pi_\delta$) and $F_\delta = \dot{F}^{Q}$. Let T-Pred$(\delta+1) = \alpha$, let $\mathcal{M}^*_{\delta+1} = \mathcal{J}^{\mathcal{M}_\alpha}_\gamma$, and set $\mathcal{N}^*_{\delta+1} = \mathcal{J}^{\mathcal{N}_\alpha}_{\pi_\alpha(\gamma)}$, again with the usual convention if $\gamma = \operatorname{dom} \pi_\alpha$.

We will use the shift lemma, to define $\pi_{\delta+1}$. Let σ be the natural embedding of $\mathcal{M}^*_{\delta+1}$ into $\mathcal{N}^*_{\delta+1}$. Let $\bar{\kappa} = \operatorname{crit} E_\gamma$. Then $(\bar{\kappa}^+)^{\mathcal{M}^*_{\delta+1}} \leq \operatorname{lh} E_\alpha$ (possibly with $(\bar{\kappa}^+)^{\mathcal{M}^*_{\delta+1}} = \mathrm{OR}^{\mathcal{M}^*_{\delta+1}}$), so σ and π_γ agree up to and at $(\bar{\kappa}^+)^{\mathcal{M}^*_{\delta+1}}$. Thus we can apply the shift lemma to get $\pi_{\delta+1}\colon \mathcal{M}_{\delta+1} \to \mathcal{N}_{\delta+1}$ satisfying our inductive hypotheses on commutativity and agreement. If $\mathcal{M}^*_{\delta+1} = \mathcal{M}_\alpha$ and $\deg^{\mathcal{T}}(\delta+1) = \deg^{\mathcal{T}}(\alpha)$ then set $Y_{\delta+1} = i^{\mathcal{T}}_{\alpha,\delta+1}{}''Y_\alpha$. Otherwise take $Y_{\delta+1} = i^{*\mathcal{T}}_\beta{}''\mathcal{M}^*_\beta$. To see $\pi_{\delta+1}$ is a $\deg(\delta + 1, Y_{\delta+1})$-embedding when T-Pred$(\delta + 1) = 0$, use n-boundedness, and for T-Pred$(\delta + 1) > 0$. Note that σ is fully elementary if $\mathcal{M}^*_{\delta+1} \neq \mathcal{M}_\alpha$, and that $\deg^{\mathcal{T}}(\delta + 1) < \deg^{\mathcal{T}}(\alpha)$ if the degrees are not equal.

This finishes the recursive definition of $\pi\mathcal{T}$, and it only remains to verify that each $\mathcal{N}_\beta$ is well founded. Suppose that it is not. Since $\mathcal{N}$ is n-iterable, it follows that there is another branch b in T, cofinal in β, such that if $\mathcal{N}_b$ is the limit along the branch b in $\mathcal{U}$ then $\mathcal{N}_b$ is well founded. This is impossible since there is an embedding $\pi_b\colon \mathcal{M}_b \to \mathcal{N}_b$, where $\mathcal{M}_b$ is the limit in $\mathcal{T}$ along the branch b, and $\mathcal{M}_b$ is ill founded since $\mathcal{T}$ is simple and $\mathcal{M}_\beta$ is well founded. □

The Dodd-Jensen Lemma. We are now ready to prove the Dodd-Jensen lemma on the minimality of iteration maps. This is a powerful tool which will be crucial in what follows. We shall call it simply the Dodd-Jensen lemma, though without meaning to suggest that this is the most important of the lemmas which they have proved. Our proof is just the obvious generalization of the original proof of Dodd and Jensen.

Lemma 5.3 (Dodd-Jensen Lemma). *Let $\mathcal{T} = \langle T, deg, D, \langle E_\alpha, \mathcal{M}^*_{\alpha+1} \mid \alpha + 1 < \theta + 1 \rangle\rangle$ be an n-bounded, simple iteration tree of length $\theta + 1$ on a n-iterable* *premouse $\mathcal{M}_0$. Suppose*

$$\sigma : \mathcal{M}_0 \to Q$$

is an weak n-embedding, where $n \le \omega$ and Q is an initial segment of $\mathcal{M}_\theta$. Then

(1) $Q = \mathcal{M}_\theta$.

Moreover, if there is an ordinal $\gamma \in [0,\theta]_T$ such that $\deg(\gamma') \geq n$ *whenever $\gamma' \in [\gamma,\theta]_T$ then the following two clauses hold in addition:*

(2) $D \cap [0,\theta]_T = \varnothing$, *so that* $\deg(\gamma) = n$ *for all $\gamma \in [0,\theta]_T$,*

(3) $i_{0,\theta}(\eta) \le \sigma(\eta)$, *for all $\eta \in OR \cap \mathcal{M}_0$.*

Remark. Notice that the additional precondition for clauses (2) and (3) is equivalent to the condition that $\mathcal{M}_\theta$ is n-sound. This equivalence will be used in many of our applications: We will know from the construction of σ that Q is n-sound, so that clause (1) implies that $\mathcal{M}_\theta = Q$ and hence $\mathcal{M}_\theta$ is n-sound so that that clauses (2) and (3) of the lemma must are valid as well.

PROOF. We will define a sequence $\langle \mathcal{T}_i : i < \omega \rangle$ of iteration trees as in clause (b) of the definition of k-iterable, together with maps $\sigma_i : \mathcal{M}_0^i \to \mathcal{M}_\theta^i$ where $\mathcal{M}_\gamma^i$ is the γth model of $\mathcal{T}_i$. For each integer i the pair $(\mathcal{T}_i, \sigma_i)$ will satisfy the same conditions as the pair $(\mathcal{T}_0, \sigma_0) = (\mathcal{T}, \sigma)$, and it will follow that any failure of the lemma will imply that $\langle \mathcal{T}_i : i < \omega \rangle$ violates condition (b) of the definition of k-iterable.

We first give the definition under the assumption $Q = \mathcal{M}_\theta$. We will then modify the definition slightly to prove that $Q = \mathcal{M}_\theta$.

We have $\mathcal{T}_0 = \mathcal{T}$ and $\sigma_0 = \sigma$. Now suppose we are given a simple, n-bounded tree $\mathcal{T}_i$ on the n-iterable model M_0^i, together with a (n, X_i)-embedding $\sigma_i \colon \mathcal{M}_0^i \to \mathcal{M}_\theta^i$. Let

$$\mathcal{T}_{i+1} = \sigma_i \mathcal{T}_i \,.$$

Since $\mathcal{M}_0^i$ is n-iterable and $\mathcal{T}_i$ is simple, $\mathcal{M}_\theta^i = \mathcal{M}_0^{i+1}$ is n-iterable. Thus $\mathcal{T}_{i+1}$ has length $\theta + 1$ and is simple and n-bounded. Let $\vec{\pi}^i \colon \mathcal{T}_i \to \sigma_i \mathcal{T}_i = \mathcal{T}_{i+1}$ be the tree embedding given by the copying procedure, and set

$$\sigma_{i+1} = \pi_\theta^i \colon \mathcal{M}_\theta^i \to \mathcal{M}_\theta^{i+1}.$$

Since $\deg(\gamma + 1) \ge n$ for all sufficiently large $\gamma + 1 \in [0,\theta]_T$, σ_{i+1} is a (n, X_{i+1})-embedding, where X_{i+1} is given by the copying procedure. Thus we are ready for the next stage of the construction.

This completes the definition of the $\mathcal{T}_i$'s and σ_i's. We must have $D \cap [0,\theta]^{\mathcal{T}} \neq \varnothing$, since otherwise $D_i \cap [0,\theta]_{\mathcal{T}_i} \neq \varnothing$ for all $i < \omega$, contradicting clause (b) of the definition of k-iterable. Thus there are canonical n-embeddings

$$\tau_i : \mathcal{M}_0^i \to \mathcal{M}_\theta^i$$

given by composing the embeddings along the branch $[0,\theta]_T$ of $\mathcal{T}_i$. We have the commutative diagram

$$\begin{tikzcd}
\mathcal{M}_0^0 \arrow[r, "\tau_0"] & \mathcal{M}_\theta^0 = \mathcal{M}_0^1 \arrow[r, "\tau_1"] & \mathcal{M}_\theta^1 = \mathcal{M}_0^2 \arrow[r, "\tau_2"] & \cdots \\
& \mathcal{M}_0^0 \arrow[u, "\sigma_0"] \arrow[r, "\tau_0"] & \mathcal{M}_0^1 \arrow[u, "\sigma_1"] \arrow[r, "\tau_1"] & \cdots
\end{tikzcd}$$

Suppose toward a contradiction that $i_{0,\theta}(\eta_0) = \tau_0(\eta_0) > \sigma(\eta_0)$. Set $\eta_{i+1} = \sigma_i(\eta_i)$. It is routine to check that $\tau_i(\eta_i) > \sigma_i(\eta_i) = \eta_{i+1}$ for all i and it follows that $\operatorname{dir\,lim}(\mathcal{M}_i : i < \omega)$ is not well founded, contradicting clause (b) in the definition of n-iterability of $\mathcal{M}_0^\theta$.

To show $Q = \mathcal{M}_\theta$ we proceed essentially as above. If $Q \neq \mathcal{M}_\theta$ we will have $\sigma_i \colon \mathcal{M}_i^0 \to Q_i$, with $Q_0 = Q$ and Q_i a proper initial segment of $\mathcal{M}_i^\theta$. In this case $\sigma_i \mathcal{T}_i$ is a tree on Q_i rather than $\mathcal{M}_\theta^i$, but it can be modified slightly to make it a tree on $\mathcal{M}_\theta$ which immediately drops to Q_i at all $\mathcal{T}_0$ successors of 0. That is, $\mathcal{T}_{i+1}$ is the same as $\sigma_i \mathcal{T}_i$ except that we put $\gamma+1$ into D_1 whenever T-pred$(\gamma+1) = 0$, and we set $(\mathcal{M}_{\gamma+1}^1)^* = Q_i$ or the appropriate initial segment thereof. With this modification the construction works as before, giving a sequence of trees $(\mathcal{T}_i : i < \omega)$ such that $D_i \cap [0,\theta]_{T_i} \neq \varnothing$ for every $i > 0$ and thus contradicting clause (b) of the definition of n-iterability. Notice that in this case we don't need the hypothesis that $\deg(\gamma+1) \geq n$ for all sufficiently large $\gamma + 1 \in [0,\theta]$, since for example it is not σ_i but $\sigma_i \restriction Q_i$ which will be used to produce $\mathcal{T}_{i+1}$, and $\sigma_i \restriction Q_i$ is fully elementary. $\square$