§4. ULTRAPOWERS

Let $\mathcal{M}$ be either a ppm or an sppm, and $\kappa < \rho_n^{\mathcal{M}}$. Let E be a (κ, λ) pre-extender over $\mathcal{M}$. (We are interested in the case that for some ppm or sppm $\mathcal{N}$ such that $P(\kappa)^{\mathcal{N}} = P(\kappa)^{M}$ we have $E = \dot{F}^{\mathcal{N}}$ or $\exists\gamma(E = \dot{E}_\gamma^{\mathcal{N}})$. It is easy to check that E is a pre-extender over $\mathcal{M}$ in this case.) We wish to define $\mathrm{Ult}_n(\mathcal{M}, E)$.

We begin with the universe of $\mathrm{Ult}_n(\mathcal{M}, E)$ and the $\in$ relation on it.

If $n = 0$, then the elements of $\mathrm{Ult}_0(\mathcal{M}, E)$ are equivalence classes $[a, f]_E^{\mathcal{M}}$, where $a \subseteq \lambda$ is finite and $f \in |\mathcal{M}|$ has domain $[\kappa]^{\mathrm{card}\, a}$. The equivalence relation is as usual: $(a, f) \sim (b, g)$ iff for $E_{a\cup b}$ a.e. $\bar{x}$, $\tilde{f}(\bar{x}) = \tilde{g}(\bar{x})$ where $\tilde{f}$ and $\tilde{g}$ come from f and g by adding the appropriate dummy variables. E measures enough sets that the definition makes sense. The $\in$ relation on equivalence classes is as usual.

If $n > 0$, then let $\tau = \tau(v_0 \cdots v_i)$ be a term in Sk_n if $\mathcal{M}$ is a ppm or in SK_n if $\mathcal{M}$ is an sppm. Let $q \in |\mathcal{M}|$. Then for $\bar{\alpha} \in [\kappa]^i$

$$f_{\tau,q}(\bar{\alpha}) = \tau^{\mathcal{M}}[\bar{\alpha}, q].$$

The elements of $\mathrm{Ult}_n(\mathcal{M}, E)$ are equivalence classes $[a, f]_E^{\mathcal{M}}$ where $a \subseteq \lambda$ is finite and $f = f_{\tau,q}$ for some $q \in |\mathcal{M}|$ and $\tau \in \mathrm{Sk}_n$ (resp. SK_n). The equivalence relation is as usual. E measures enough sets that the definition makes sense because $\kappa < \rho_n^{\mathcal{M}}$. Again, the $\in$ relation is as usual.

$\mathrm{Ult}_n(\mathcal{M}, E)$ may be illfounded; however, if it is wellfounded we shall identify it with the transitive set to which it is isomorphic.

We must define $\dot{E}^{\mathrm{Ult}_n(\mathcal{M},E)}$ and $\dot{F}^{\mathrm{Ult}_n(\mathcal{M},E)}$ to complete the definition of the structure $\mathrm{Ult}_n(\mathcal{M}, E)$. Let

$$[\langle a, f\rangle]_E^{\mathcal{M}} \in \dot{E}^{\mathrm{Ult}_n(\mathcal{M},E)} \text{ iff } \{\, \bar{\alpha} \mid f(\bar{\alpha}) \in \dot{E}^{\mathcal{M}} \,\} \in E_a .$$

It is easy to see that E_a measures the set in question, using the amenability of $\mathcal{M}$ with resp. to $\dot{E}^{\mathcal{M}}$ in case $n = 0$.

In case $\mathcal{M}$ is squashed or $n > 0$ we can set

$$[\langle a, f\rangle]_E^{\mathcal{M}} \in \dot{F}^{\mathrm{Ult}(\mathcal{M},E)} \text{ iff } \{\, \bar{\alpha} \mid f(\bar{\alpha}) \in \dot{F}^{\mathcal{M}} \,\} \in E_a ,$$

using amenability in the squashed $n = 0$ case. We are left with the case $\mathcal{M}$ is active and $n = 0$. Let $\mu = \mathrm{crit}\, \dot{F}^{\mathcal{M}}$. Let also $\eta = [\langle b, f\rangle]_E^{\mathcal{M}} \in \mathrm{OR} \cap \mathrm{Ult}_0(\mathcal{M}, E)$, and $h = [\langle b, g\rangle]_E^{\mathcal{M}}$, where h is a function with domain $i_E(\mu)$.

We want to put (a, h, η) into $\dot{F}^{\mathrm{Ult}(\mathcal{M},E)}$ for exactly one a. We may assume without loss of generality that $\mathrm{ran}\, h \subseteq \bigcup_n P([i_E(\mu)]^n)$.

Case 1. $\mu < \kappa$. So g is constant a.e.; in fact $g(\bar{x}) = h$ for almost every $\bar{x}$. Let $\gamma = \sup(\mathrm{ran}\, f \cap \mathrm{OR}^{\mathcal{M}})$, and let c be such that $\dot{F}^{\mathcal{M}}(c, h, \gamma)$. Using c we can compute k inside of $\mathcal{M}$:

$$k(\bar{x}) = \text{the unique } d \text{ such that } \dot{F}^{\mathcal{M}}(d, h, f(\bar{x})) .$$

Thus $k \in |\mathcal{M}|$. We then put

$$([\langle b, k\rangle]_E^{\mathcal{M}}, h, \eta) \in \dot{F}^{\mathrm{Ult}_0(\mathcal{M},E)}.$$

Case 2. $\kappa \leq \mu$. Let ℓ be a function in $|\mathcal{M}|$ with domain $= \mu$ and

$$\operatorname{ran} \ell = \bigcup_{\bar{x} \in [\kappa]^{\operatorname{card} b}} (\operatorname{ran} g(\bar{x}) \cap \bigcup_{n<\omega} P([\mu]^n)).$$

(We may assume $\operatorname{dom} g(\bar{x}) = \mu$ all $\bar{x}$.) Let $\gamma = \sup(\operatorname{ran} f \cap \mathrm{OR}^{\mathcal{M}})$. Let c be such that $\dot{F}^{\mathcal{M}}(c, \ell, \gamma)$. Using c we can compute in $\mathcal{M}$

$$k(\bar{x}) = \text{the unique } d \text{ such that } \dot{F}^{\mathcal{M}}(d, g(\bar{x}), f(\bar{x})).$$

So $k \in |\mathcal{M}|$. We then put

$$([\langle b, k\rangle]_E^{\mathcal{M}}, h, \eta) \in \dot{F}^{\mathrm{Ult}_0(\mathcal{M},E)}.$$

This completes the definition of $\mathrm{Ult}_n(\mathcal{M}, E)$. Notice the definition guarantees Los' Theorem holds for atomic formulae of $\mathcal{L} - \{\dot{v}, \dot{\gamma}\}$ (resp. $\mathcal{L}^*$).

Theorem 4.1. (*Los' Theorem*). *Let $n \geq 0$, let $\mathcal{M}$ be a ppm or sppm, and let E be a (κ, λ) pre-extender over $\mathcal{M}$, where $\kappa < \rho_n^{\mathcal{M}}$. Let $[a_i, f_i]_E^{\mathcal{M}}$ be an element of* $\mathrm{Ult}_n(\mathcal{M}, E)$ *for each $i \leq k$, and let $b = \bigcup_{i \leq k} a_i$. Then*

$$\mathrm{Ult}_n(\mathcal{M}, E) \models \varphi[[a_0, f_0]_E^{\mathcal{M}}, \ldots, [a_k, f_k]_E^{\mathcal{M}}]$$
$$\text{iff } \exists B \in E_b \forall \bar{u} \in B\, \mathcal{M} \models \varphi[\tilde{f}_0(\bar{u}), \ldots, \tilde{f}_k(\bar{u})]$$

for any generalized $r\Sigma_n$(resp. $q\Sigma_n$) *formula φ. Here $\tilde{f}_i$ comes from f_i by adding the appropriate dummy variables.*

Remark. Assume $\mathcal{M}$, etc., are as in the hypotheses. If $n > 0$, then

$$A = \{\bar{u} \in [\kappa]^{\operatorname{card} b} \mid \mathcal{M} \models \varphi[\tilde{f}_0(\bar{u}) \cdots \tilde{f}_k(\bar{u})]\}$$

is in $\mathcal{M}$ as $\kappa < \rho_n^{\mathcal{M}}$.

If $n = 0$, then $A \notin |\mathcal{M}|$ is possible. However, our proof will show there is a $B \in E_b$ (so $B \in |\mathcal{M}|$) such that $B \subseteq A$ or $B \cap A = \varnothing$.

PROOF. We consider only the case that $\mathcal{M}$ is a ppm (passive or active type I or II) as sppm behave exactly like passive ppm here.

Suppose first that φ is $r\Sigma_0$. If $n > 0$ we get the desired conclusion easily as there are enough functions defined by terms in Sk_n. So suppose $n = 0$.

For any $r\Sigma_0$ formula $\varphi = \varphi(v_0 \cdots v_k)$ and functions $f_0 \cdots f_k \in |\mathcal{M}|$ such that $\operatorname{dom} f_i = [\kappa]^{\operatorname{card} b}$ for all $i \le k$ (where $b \subset \lambda$ is finite), we let

$$\bar{u} \in A_{\varphi,\vec{f}} \text{ iff } \mathcal{M} \models \varphi[f_0(\bar{u}) \cdots f_k(\bar{u})]\,.$$

We show by induction on φ that there is a set $B \in E_b$ (so $B \in |\mathcal{M}|$) such that

$$B \subseteq A_{\varphi,\vec{f}} \text{ or } B \cap A_{\varphi,\vec{f}} = \varnothing$$

and

$$B \subseteq A_{\varphi,\vec{f}} \quad \text{iff} \quad \mathrm{Ult}_0(\mathcal{M}, E) \models \varphi[[\langle b, f_0\rangle]_E^{\mathcal{M}} \cdots [\langle b, f_k\rangle]_E^{\mathcal{M}}]\,.$$

For formulas φ which are Σ_0 in $\mathcal{L}$, a subinduction on Σ_0 in $\mathcal{L}$ formulas (using amenability) gives the result as usual. For $\varphi = \dot{F}(v_0, v_1, v_2)$, the construction of $\dot{F}^{\mathrm{Ult}_0(\mathcal{M},E)}$ guarantees the desired result. If φ is built from simpler $r\Sigma_0$ formulae by $\wedge$, $\vee$, or $\neg$ the inductive step is easy. Suppose $\varphi(v_0 \cdots v_k) =$ "v_0 is finite $\wedge$ $(\exists v_{k+1} \in v_0)\, \theta(v_0 \cdots v_{k+1})$". We may assume $f_0(\bar{u})$ is finite E_b a.e. as otherwise $B = \{\bar{u} \mid f_0(\bar{u}) \text{ infinite}\}$ does the job. But then we can fix $\ell \in \omega$ such that $\operatorname{card} f_0(\bar{u}) = \ell$ for E_b a.e. $\bar{u}$, and functions $g_1 \cdots g_\ell$ with $\operatorname{dom} = [\kappa]^{\operatorname{card} b}$ such that $f_0(\bar{u}) = \{g_1(\bar{u}) \cdots g_\ell(\bar{u})\}$ for E_b a.e. $\bar{u}$, say for $\bar{u} \in C$ where $C \in E_b$. Let B_i satisfy the induction hypothesis for $A_{\theta,\vec{f}^\frown g_i}$, and let $B = C \cap \bigcap_{i \le \ell} B_i$. Then B works for $A_{\varphi,\vec{f}}$.

This completes the proof of 4.1 in the case that φ is $r\Sigma_0$.

We now show by induction on $i \le n$ that 4.1 holds when φ is $r\Sigma_i$. We have done the case $i = 0$. The case φ is $r\Sigma_1$ and does not involve $\dot{\mu}$, $\dot{\nu}$, or $\dot{\gamma}$ now follows by the usual argument as there are enough functions defined by terms in Sk_n. But then 2.6 (b) implies that $\mathcal{P} = \mathrm{Ult}_n(\mathcal{M}, E)$ is a ppm of the same type as $\mathcal{M}$, and that $i_E(\mu^{\mathcal{M}}) = \mu^{\mathcal{P}}$, $i_E(\nu^{\mathcal{M}}) = \nu^{P}$, $i_E(\gamma^{\mathcal{M}}) = \gamma^{\mathcal{P}}$. This gives 4.1 for arbitrary $r\Sigma_1$ formulae φ.

So now let $i > 1$. Notice first that as the relation $\mathrm{Th}_{i-1}^{\mathcal{Q}}(a) = b$ is $\prod_1$ over $r\Sigma_{i-1}$ definable over $\mathcal{Q}$, uniformly over all ppm $\mathcal{Q}$, and as we have 4.1 for $\prod_1$ over $r\Sigma_{i-1}$ formulae by induction hypothesis and the fact that there are enough functions given by terms in Sk_n, we have (for $\mathrm{Ult} = \mathrm{Ult}_n(\mathcal{M}, E)$.),

$$(*) \qquad \mathrm{Th}_{i-1}^{\mathrm{Ult}}([a, f]) = [b, g] \text{ iff for } E_{a \cup b} \text{ a.e. } \bar{x},\ \mathrm{Th}_{i-1}^{\mathcal{M}}(\tilde{f}(\bar{x})) = \tilde{g}(\bar{x})\,.$$

Let $\pi : \mathcal{M} \to \mathrm{Ult}_n(\mathcal{M}, E)$ be the canonical embedding. It follows that

$$(**) \qquad \rho_{i-1}^{\mathrm{Ult}} = \begin{cases} \mathrm{OR}^{\mathcal{M}} & \text{if } \rho_{i-1}^{\mathcal{M}} = \mathrm{OR}^{\mathrm{Ult}} \\ \pi(\rho_{i-1}^{\mathcal{M}}) & \text{otherwise.} \end{cases}$$

We prove the case $\rho_{i-1}^{\mathrm{Ult}} = \pi(\rho_{i-1}^{\mathcal{M}})$. Suppose $\rho_{i-1}^{\mathcal{M}} < \mathrm{OR}^{\mathcal{M}}$. We show first that $\pi(\rho_{i-1}^{\mathcal{M}}) \le \rho_{i-1}^{\mathrm{Ult}}$. For let $\alpha = [a, f]_E^{\mathcal{M}} < \pi(\rho_{i-1}^{\mathcal{M}})$, and let $q = [a, g]_E^{\mathcal{M}}$. We may assume $f(\bar{x}) < \rho_{i-1}^{\mathcal{M}}$ for all $\bar{x}$. Define

$$\begin{aligned} h(\bar{x}) &= \mathrm{Th}_{i-1}^{\mathcal{M}}(f(\bar{x}) \cup \{g(\bar{x})\}) \\ &= \text{least } b \text{ such that } \dot{T}_{i=1}^{\mathcal{M}}(\langle f(\bar{x}), g(\bar{x})\rangle, b)\,. \end{aligned}$$

Then h is one of the functions used in forming $\mathrm{Ult}_n(\mathcal{M}, E)$, and as we observed above in (*)

$$\mathrm{Th}_{i-1}^{\mathrm{Ult}}(\alpha \cup \{q\}) = [a, h]_E^{\mathcal{M}} .$$

Thus $\alpha < \rho_{i-1}^{\mathrm{Ult}}$. Thus $\pi(\rho_{i-1}^{\mathcal{M}}) \leq \rho_{i-1}^{\mathrm{Ult}}$.

On the other hand, pick $q \in |\mathcal{M}|$ such that $\mathrm{Th}_{i-1}^{\mathcal{M}}(\rho_{i-1}^{\mathcal{M}} \cup \{q\}) \notin |\mathcal{M}|$. Then by (*) $\mathrm{Th}_{i-1}^{\mathrm{Ult}}(\pi(\rho_{i-1}^{\mathcal{M}}) \cup \{\pi(q)\}) \notin \mathrm{Ult}$. Thus $\rho_{i-1}^{\mathrm{Ult}} \leq \pi(\rho_{i-1}^{\mathcal{M}})$.

Putting (*) and (**) together, we have

$$\dot{T}_{i-1}^{\mathrm{Ult}}([a, f], [b, g]) \text{ iff for } E_{a\cup b} \quad \text{a.e. } \bar{x},\ \dot{T}_{i-1}^{\mathcal{M}}(\tilde{f}(\bar{x}), \tilde{g}(\bar{x})) .$$

Now suppose

$$\varphi(\bar{v}) = \exists a \exists b (\dot{T}_{i-1}(a, b) \wedge \psi(a, b, \bar{v}))$$

where ψ is $r\Sigma_1$. We check one direction of the conclusion of 4.1. Suppose that for E_b a.e. $\bar{u}$, $\mathcal{M} \models \varphi[f_0(\bar{u}) \cdots f_k(\bar{u})]$. Let $f_i = f_{\tau_i q_i}$ where $\tau_i \in \mathrm{Sk}_n$ and $q_i \in |\mathcal{M}|$. We can translate "$\psi(a, b, \bar{v}) \wedge \dot{T}_{i-1}(a, b)$" into an $r\Sigma_n$ formula; this gives us terms σ_0 and σ_1 in Sk_n which Skolemize the result, i.e., such that for E_b a.e. $\bar{u}$

$$\mathcal{M} \models \psi(\sigma_0(\bar{u}, \bar{q}), \sigma_1(\bar{u}, \bar{q}), \tau_0(\bar{u}, q_0) \cdots \tau_k(\bar{u}, q_k)) \wedge \dot{T}_{i-1}(\sigma_0(\bar{u}, \bar{q}), \sigma_1(\bar{u}, \bar{q})) ,$$

where $\bar{q} = \langle q_1 \cdots q_k \rangle$. But then, letting $g_0 = f_{\sigma_0, \bar{q}}$ and $g_1 = f_{\sigma_1, \bar{q}}$,

$$\mathrm{Ult}_n(\mathcal{M}, E) \models \dot{T}_{i-1}([b, g_0], [b, g_1]) \wedge \psi([b, g_0], [b, g_1][b, f_0] \cdots [b, f_k])$$

as desired.

Finally, we prove 4.1 in the case φ is generalized $r\Sigma_n$ with $n > 0$. Notice first that if $\tau(v_0 \cdots v_k) \in \mathrm{Sk}_n$, then

$$\tau^{\mathrm{Ult}}[[a, f_0], \ldots, [a, f_k]] = [a, \lambda \bar{u} \cdot \tau^{\mathcal{M}}[f_0(\bar{u}) \cdots f_k(\bar{u})]]_E^{\mathcal{M}}$$

for any $[a, f_0] \cdots [a, f_k] \in \mathrm{Ult} = \mathrm{Ult}_n(\mathcal{M}, E)$. To see this, it is enough to consider the basic terms $\tau_\theta \in \mathrm{Sk}_n$. But the graph of such a term is definable by a Boolean combination of $r\Sigma_n$ formulae, uniformly over all ppm, so we can use the term-free case of 4.1 just proved.

But now if $\varphi(v)$ is $r\Sigma_n$ and $\tau(v) \in \mathrm{Sk}_n$ then

$$\begin{aligned}
\mathrm{Ult} \models \varphi(\tau(v))[[a, f]_E^{\mathcal{M}}] &\text{ iff } \mathrm{Ult} \models \varphi[\tau^{\mathrm{Ult}}[[a, f]_E^{\mathcal{M}}]] \\
&\text{ iff } \mathrm{Ult} \models \varphi[[a, \lambda u \cdot \tau^{\mathcal{M}}[f(\bar{u})]]_E^{\mathcal{M}}] \\
&\text{ iff for } E_a \text{ a.e. } \bar{u}\, , \mathcal{M} \models \varphi[\tau^m[f(\bar{u})]] \\
&\text{ iff for } E_a \text{ a.e. } \bar{u}\, , \mathcal{M} \models \varphi(\tau(v))[f(\bar{u})]
\end{aligned}$$

as desired. Of course, the case φ or τ having more variables involves only more notation. □

In the course of proving 4.1 we have shown

Corollary 4.2. *Let $\mathcal{M}$, etc., be as in the hypotheses of* 4.1, *and let $\pi : \mathcal{M} \to \mathrm{Ult}_n(\mathcal{M}, E)$ be the canonical embedding. Then for $i < n$*

$$\rho_i^{\mathcal{M}} < \mathrm{OR}^{\mathcal{M}} \quad \text{iff } \rho_i^{\mathrm{Ult}} < \mathrm{OR}^{\mathrm{Ult}}$$

and

$$\rho_i^{\mathcal{M}} < \mathrm{OR}^{\mathcal{M}} \Rightarrow \pi(\rho_i^{\mathcal{M}}) = \rho_i^{\mathrm{Ult}}$$

(*where* $\mathrm{Ult} = \mathrm{Ult}_n(\mathcal{M}, E)$).

We would like to show that under the hypotheses of 4.1, the canonical $\pi : \mathcal{M} \to \mathrm{Ult}_n(\mathcal{M}, E)$ is generalized $r\Sigma_{n+1}$ elementary. For this we seem to need (essentially) that $\mathcal{M}$ be n-sound. Fortunately, we shall never want to form $\mathrm{Ult}_n(\mathcal{M}, E)$ unless $\mathcal{M}$ is n-sound.

Corollary 4.3. *Let $\mathcal{M}$, etc., be as in the hypotheses of* 4.1, *and let $\pi : \mathcal{M} \to \mathrm{Ult}_n(\mathcal{M}, E)$ be the canonical embedding. Suppose that for some $p \in |\mathcal{M}|$, $\mathcal{M} = \mathcal{H}_n^{\mathcal{M}}(\rho_n^{\mathcal{M}} \cup \{p\})$. Then π is generalized $r\Sigma_{n+1}$ (resp. $q\Sigma_{n+1}$) elementary; moreover $\rho_n^{\mathrm{Ult}(\mathcal{M},E)} = \sup \pi''\rho_n^{\mathcal{M}}$.*

Proof. Let $\mathrm{Ult} = \mathrm{Ult}_n(\mathcal{M}, E)$. We show first that $\sup \pi''\rho_n^{\mathcal{M}} \geq \rho_n^{\mathrm{Ult}}$; for this it is enough to show that if $\mathcal{H}_n^{\mathcal{M}}(\rho_n^{\mathcal{M}} \cup \{p\}) = \mathcal{M}$, then

$$\mathcal{H}_n^{\mathrm{Ult}}(\sup \pi''\rho_n^{\mathcal{M}} \cup \{\pi(p)\}) = \mathrm{Ult}.$$

(For then $\mathrm{Th}_n^{\mathrm{Ult}}(\sup \pi''\rho_n^{\mathcal{M}} \cup \{\pi(p)\}) \notin \mathrm{Ult}$ by a diagonal argument.) So let $\mathcal{H}_n^{\mathcal{M}}(\rho_n^{\mathcal{M}} \cup \{p\}) = \mathcal{M}$, and let $[a, f] \in \mathrm{Ult}$. Then there is a term $\tau \in \mathrm{Sk}_n$ (resp. SK_n) and parameters $\bar{b} \in [\rho_n^{\mathcal{M}} \cup \{p\}]^{<\omega}$ such that for all $\bar{u}$, $f(\bar{u}) = \tau^{\mathcal{M}}[\bar{u}, \bar{b}]$. Let $\mathrm{id}(\bar{u}) = \bar{u}$, $c_{\bar{b}}(\bar{u}) = \bar{b}$. By the Los Theorem, $[a, f]_E^{\mathcal{M}} = \tau^{\mathrm{Ult}}[[a, \mathrm{id}]_E^{\mathcal{M}}, [a, c_{\bar{b}}]_E^{\mathcal{M}}] = \tau^{\mathrm{Ult}}[a, \pi(\bar{b})]$. Since $a \in [\pi(\kappa)]^{<\omega}$ and $\kappa < \rho_n^{\mathcal{M}}$, and since $\pi(\bar{b}) \in [\sup \pi''\rho_n^{\mathcal{M}} \cup \{\pi(p)\}]^{<\omega}$, we're done.

We claim next that $\rho_n^{\mathrm{Ult}} \geq \sup \pi''\rho_n^{\mathcal{M}}$. For by the Los Theorem we have easily that for $a, b \in |\mathcal{M}|$

$$\mathrm{Th}_n^{\mathcal{M}}(a) = b \quad \text{iff} \quad \mathrm{Th}_n^{\mathrm{Ult}}(\pi(a)) = \pi(b)$$

[For the "only if" direction, let $\bar{c} \in \pi(a)^{<\omega}$ and φ be generalized $r\Sigma_n$. Let $\bar{c} = [d, f]_E^{\mathcal{M}}$. Then $(\varphi, \bar{c}) \in \mathrm{Th}_n^{\mathrm{Ult}}(\pi(a))$ iff (for E_d a.e. $\bar{u}$ $(\varphi, f(\bar{u})) \in \mathrm{Th}_n^{\mathcal{M}}(a)$) iff $(\varphi, \bar{c}) \in \pi(b)$.]

It follows that $\forall \gamma < \sup \pi''\rho_n^{\mathcal{M}}$, $\forall p \in |\mathcal{M}|$, $\mathrm{Th}_n^{\mathrm{Ult}}(\gamma \cup \{\pi(p)\}) \in \mathrm{Ult}$. Now fix p such that $\mathcal{H}_n^{\mathcal{M}}(\rho_n^{\mathcal{M}} \cup \{p\}) = \mathcal{M}$. Let $\alpha < \sup \pi''\rho_n^{\mathcal{M}}$ and $r \in \mathrm{Ult}$; we must see that $\mathrm{Th}_n^{\mathrm{Ult}}(\alpha \cup \{r\}) \in \mathrm{Ult}$. Fix $\gamma < \sup \pi''\rho_n^{\mathcal{M}}$ such that $\alpha < \gamma$ and for some $\bar{\beta} \in \gamma^{<\omega}$, and some $\tau \in \mathrm{Sk}_n$, $r = \tau^{\mathrm{Ult}}[\bar{\beta}, \pi(p)]$. Then $\mathrm{Th}_n^{\mathrm{Ult}}(\gamma \cup \{\pi(p)\})$ is in Ult, and from it we can compute $\mathrm{Th}_n^{\mathrm{Ult}}(\alpha \cup \{r\})$ inside Ult.

Thus $\rho_n^{\mathrm{Ult}} = \sup \pi'' \rho_n^{\mathcal{M}}$.

Now let $[a, f]_E^{\mathcal{M}} = \langle \alpha, q \rangle$, where $\alpha < \sup \pi'' \rho_n^{\mathcal{M}}$. We claim that for any g

$$\dot{T}_n^{\mathrm{Ult}}([a, f], [a, g]) \quad \text{iff for } E_a \text{ a.e. } \bar{u}\,,\ \dot{T}_n^{\mathcal{M}}(f(\bar{u}),\ g(\bar{u}))\,.$$

$\Leftarrow$ is easy since $\mathrm{Th}_n^{\mathcal{P}}(a) = b$ is uniformly Π_1 over $r\Sigma_n^{\mathcal{P}}$, and we have the Łos Theorem for $r\Sigma_n$ formulae. So suppose $\dot{T}_n^{\mathrm{Ult}}([a, f], [a, g])$. Let $q = \tau^{\mathrm{Ult}}[\bar{\beta}, \pi(p)]$ where $\bar{\beta} \in [\sup \pi'' \rho_n^{\mathcal{M}}]^{<\omega}$ and pick $\gamma < \rho_n^{\mathcal{M}}$ such that $\bar{\beta}, \alpha < \pi(\gamma)$. Let $b = \mathrm{Th}_n^{\mathcal{M}}(\gamma \cup \{p\})$, so that

$$\pi(b) = \mathrm{Th}_n^{\mathrm{Ult}}(\pi(\gamma) \cup \{\pi(p)\})\,.$$

Then we have

(1) $(\varphi, \bar{c}) \in [a, g]$ iff

$$\varphi \text{ is generalized } r\Sigma_n \text{ and } \bar{c} \in [\alpha \cup \{q\}]^{<\omega} \text{ and } (\varphi^*, \bar{c}^*) \in \pi(b),$$

where $\varphi^*(\bar{c}^*)$ is the obvious way of rewriting $\varphi(\bar{c})$ so that the parameters $\bar{c}^*$ come from $\pi(\gamma) \cup \{\pi(p)\}$. Thus the map $(\varphi, \bar{c}) \mapsto (\varphi^*, \bar{c}^*)$ is $r\Delta_1$ over Ult in the parameters α, q, and $\bar{\beta}$. Let $\bar{\beta} = [a, h]_E^{\mathcal{M}}$, where we assume for notational convenience that the support is a (otherwise enlarge all supports). Then the fact that (1) holds in Ult is a $r\Pi_1$ fact about $[a, f]$, $[a, g]$, and $[a, h]$. It follows that for E_a a.e. $\bar{u}$,

$(\varphi, \bar{c}) \in g(\bar{u})$ iff

$$\varphi \text{ is generalized } r\Sigma_n \text{ and } \bar{c} \in [\alpha_{\bar{u}} \cup \{q_{\bar{u}}\}]^{<\omega} \text{ and } (\varphi^*, \bar{c}^*) \in b\,,$$

where $f(\bar{u}) = \langle \alpha_{\bar{u}}, q_{\bar{u}} \rangle$ and $(\varphi^*, \bar{c}^*)$ comes from rewriting $(\varphi, \bar{c})$ by substituting $\tau(h(\bar{u}), p)$ for occurrences of $q_{\bar{u}}$. But now

$$q_{\bar{u}} = \tau^{\mathcal{M}}[h(\bar{u}),\, p]$$

for E_a a.e. $\bar{u}$, by the Łos Theorem. As $b = \mathrm{Th}_n^{\mathcal{M}}(\gamma \cup \{p\})$, we see

$$g(\bar{u}) = \mathrm{Th}_n^{\mathcal{M}}(\alpha_{\bar{u}} \cup \{q_{\bar{u}}\})$$

for E_a a.e. $\bar{u}$. As $\alpha_{\bar{u}} < \rho_n^{\mathcal{M}}$ a.e., we get

$$T_n^{\mathcal{M}}(f(\bar{u}),\ g(\bar{u}))$$

for E_a a.e. $\bar{u}$, as desired.

Finally, let

$$\varphi(\bar{v}) = \exists a \exists b\, (\dot{T}_n(a, b) \wedge \psi(a, b, \bar{v}))$$

be an $r\Sigma_{n+1}$ formulae. If $\mathcal{M} \models \varphi[\bar{x}]$, then we have a, b such that $\dot{T}_n^{\mathcal{M}}(a,b) \wedge \psi^{\mathcal{M}}(a,b,\bar{x})$, so $\dot{T}_n^{\mathrm{Ult}}(\pi(a),\pi(b))$ and $\psi^{\mathrm{Ult}}(\pi(a),\pi(b),\bar{x})$, so Ult $\models \varphi[\pi(\bar{x})]$. On the other hand if Ult $\models \varphi[\pi(\bar{x})]$, then we have a, f, g such that $[a,f] = \langle \alpha, q\rangle$ for some $\alpha < \sup \pi'' \rho_n^{\mathcal{M}}$, and

$$\mathrm{Ult} \models \dot{T}_n([a,f],[a,g]) \wedge \psi([a,f],[a,g],\pi(\bar{x}))\,.$$

By our claim, for E_a a.e. $\bar{u}$

$$\mathcal{M} \models \dot{T}_n(f(\bar{u}),\, g(\bar{u})) \wedge \psi(f(\bar{u}),\, g(\bar{u}),\, \bar{x})\,.$$

Thus $\mathcal{M} \models \varphi[\bar{x}]$, as desired.

We can now show $\pi(\tau^{\mathcal{M}}(x)) = \tau^{\mathrm{Ult}}(\pi(x))$ for all $\tau \in \mathrm{Sk}_{n+1}$, since the graphs of basic terms in Sk_{n+1} are definable by boolean combinations of $r\Sigma_{n+1}$ formulae. It follows that π is generalized $r\Sigma_{n+1}$ elementary.

Relations to Dodd-Jensen.

It is easy to see that if $\mathcal{M}$ is n-sound, $\mathrm{Ult}_n(\mathcal{M},E)$ is exactly what is obtained by the Dodd-Jensen procedure of coding $\mathcal{M}$ onto $\rho_n^{\mathcal{M}}$, taking a Σ_0 ultrapower of the coded structure, and then decoding.

For let $\mathcal{M}$ be a ppm or sppm, $n \geq 1$, and $\mathcal{M} = \mathcal{H}_n^{\mathcal{M}}(\rho_n^{\mathcal{M}} \cup \{q\})$. Let

$$\pi : \mathcal{M} \to \mathrm{Ult}_n(\mathcal{M},E) = \mathcal{N}$$

be the canonical embedding. Now let

$$A^{\mathcal{M}} = \mathrm{Th}_n^{\mathcal{M}}(\rho_n^{\mathcal{M}} \cup \{q\}),\ \text{coded as a subset of } \rho_n^{\mathcal{M}}\,,$$
$$A^{\mathcal{N}} = \mathrm{Th}_n^{\mathcal{N}}(\rho_n^{\mathcal{N}} \cup \{\pi(q)\}),\ \text{similarly coded}\,.$$

Let

$$\mathcal{P} = (J_{\rho_n^{\mathcal{M}}}^{\dot{E}^{\mathcal{M}}}, \in, \dot{E}^{\mathcal{M}} \restriction \rho_n^{\mathcal{M}}, A^{\mathcal{M}})$$
$$\mathcal{Q} = (J_{\rho_n^{\mathcal{N}}}^{\dot{E}^{\mathcal{N}}}, \in, \dot{E}^{\mathcal{N}} \restriction \rho_n^{\mathcal{N}}, A^{\mathcal{N}})$$

be the master code structures associated to $\mathcal{M}$ and $\mathcal{N}$. Then

$$\pi : \mathcal{P} \xrightarrow{\Sigma_0} \mathcal{Q}$$

cofinally; this is contained in 4.3. Note also that if $[a,f]_E^{\mathcal{M}} \in |\mathcal{Q}|$, then $\exists_\beta < \rho_n^{\mathcal{M}}$ such that $f(\bar{u}) < \beta$ E_a a.e., so since f is given by a term in Sk_n, in fact $f \in |\mathcal{M}|$ and hence $f \in |\mathcal{P}|$. So in fact

$$\mathcal{Q} = \mathrm{Ult}_0(\mathcal{P},E)$$

and $\pi \restriction |\mathcal{P}|$ is the canonical embedding for this Σ_0 ultrapower. Notice finally that all of $\mathcal{N}$ can be decoded from $\mathcal{Q}$, since $\mathcal{N} = \mathcal{H}_n^{\mathcal{N}}(\rho_n^{\mathcal{N}} \cup \{\pi(q)\})$.

Although we can make sense of $\mathrm{Ult}_n(\mathcal{M},E)$ in the case $\mathcal{M}$ is not n-sound, in practice we shall never need to form such an ultrapower. Thus our construction of $\mathrm{Ult}_n(\mathcal{M},E)$ does not go beyond Dodd-Jensen in any important way.

We describe now the preservation of the core parameters $p_i(\mathcal{M})$, for $i \leq n$, in the case that $\mathcal{M}$ is n-sound.

Lemma 4.4. *Let $\mathcal{M}$ be n-sound, let E be an extender over $\mathcal{M}$ with* $\mathrm{crit}(E) < \rho_n^{\mathcal{M}}$*, and let $\pi : \mathcal{M} \to \mathrm{Ult}_n(\mathcal{M}, E)$ be the canonical embedding. Then*

(a) $\mathrm{Ult}_n(\mathcal{M}, E)$ *is n-sound, and*

(b) π *is an n-embedding.*

PROOF. Let $\mathcal{N} = \mathrm{Ult}_n(\mathcal{M}, E)$. It is enough to show that for all $i \leq n$

$$\mathfrak{C}_i(\mathcal{N}) = \mathcal{N},$$

and

$$p_i(\mathcal{N}) = \pi(p_i(\mathcal{M})).$$

For then by soundness $\rho_i(\mathcal{N}) = \rho_i^{\mathcal{N}}$ for all $i \leq n$, and similarly for $\mathcal{M}$, so that π maps the core projecta properly by 4.2 and 4.3.

We proceed by induction on $i \leq n$. For $i = 0$ there is nothing to prove. Now let $i = 1$ and Let r be the first standard parameter of $\mathcal{M}$. Thus as $\mathcal{M}$ is 1-sound, $p_1(\mathcal{M}) = \langle r, \varnothing\rangle$ and r is 1-solid and 1-universal over $\mathcal{M}$.

Let $r = \langle \alpha_0 \cdots \alpha_\ell \rangle$, and

$$b_j = \mathrm{Th}_1^{\mathcal{M}}(\alpha_j \cup \{\alpha_0 \cdots \alpha_{j-1}\}), \quad 0 \leq j \leq \ell$$

so that $b_j \in |\mathcal{M}|$ by solidity. By 4.3, π is at least $r\Sigma_2$ elementary, so

$$\pi(b_j) = \mathrm{Th}_1^{\mathcal{N}}(\pi(\alpha_j) \cup \{\pi(\alpha_0) \cdots \pi(\alpha_{j-1})\}).$$

It follows that no $s <_{\mathrm{lex}} \pi(r)$ can serve as the 1st standard parameter of $\mathcal{N}$. On the other hand, $\mathrm{Th}_1^{\mathcal{N}}(\rho_1^N \cup \{\pi(r)\}) \notin |\mathcal{N}|$ and in fact $\mathcal{H}_1^{\mathcal{N}}(\rho_1^{\mathcal{N}} \cup \{\pi(r)\}) = |\mathcal{N}|$. [If $n = 1$ this is implicit in the proof of 4.3. Suppose $n > 1$. Let $[a, f]$ by an arbitrary element of $|\mathcal{N}|$. Notice that if we let, for $x \in |\mathcal{M}|$,

$$\begin{gathered} h(x) = 1 \text{ st } \ \text{(in order of construction)} \ (\tau, \bar{\beta}) \text{ such that} \\ \tau \in \mathrm{Sk}_1 \ \wedge \bar{\beta} \in (\rho_1^{\mathcal{M}})^{<\omega} \wedge \tau^{\mathcal{M}}[\bar{\beta}, r] = x \end{gathered}$$

then $h(x) = \sigma^{\mathcal{M}}[x, r]$ for some term $\sigma \in \mathrm{Sk}_2$. So if we let

$$g(\bar{u}) = h(f(\bar{u})),$$

then g is one of the functions used to form $\mathcal{N}$, and if $[a, g] = (\tau, \bar{\beta})$, then $\tau \in \mathrm{Sk}_1$ and $\bar{\beta} \in (\rho_1^{\mathcal{N}})^{<\omega}$ and $\tau^{\mathcal{N}}[\bar{\beta}, \pi(r)] = [a, f]$, as desired.]

So $\pi(r)$ is the 1st standard parameter of $\mathcal{N}$, is 1-solid and 1-universal and $\mathcal{N}$ is 1-sound. As $\mathcal{N}$ is 1-sound, $p_1(\mathcal{N}) = \langle \pi(r), \varnothing \rangle = \pi(p_1(\mathcal{M}))$, as desired.

The case $i > 1$ of the induction involves a bit more notation but no new ideas, so we omit it. □

The next question, clearly, is how π moves $\rho_{n+1}^{\mathcal{M}}$ and the $n+1$st standard parameter of $(\mathcal{M}, p_n(\mathcal{M}))$. We answer this under a solidity hypothesis. Our proof is in essence drawn from Mitchell [M?]; it was recently re-discovered by S. Friedman.

We must also, it seems, impose an additional condition on E.

DEFINITION 4.4.1. Let $\mathcal{M}$ be a ppm or an sppm, and E a (κ, λ) extender over $\mathcal{M}$. Then E is *close to* $\mathcal{M}$ iff for every $a \in [\lambda]^{<\omega}$

(1) E_a is $r\Sigma_1^{\mathcal{M}}$ (resp. $q\Sigma_1^{\mathcal{M}}$) and
(2) if $\mathcal{A} \in |\mathcal{M}|$ and $\mathcal{M} \models \text{card } \mathcal{A} \leq \kappa$, then $E_a \cap \mathcal{A} \in |\mathcal{M}|$.

The purpose of this restriction on E is explained by the following lemma.

Lemma 4.5. *Suppose E is a (κ, λ) extender which is close to $\mathcal{M}$, and $\kappa < \rho_n^{\mathcal{M}}$. Then*

$$P(\kappa) \cap |\mathcal{M}| = P(\kappa) \cap |\operatorname{Ult}_n(\mathcal{M}, E)|\,.$$

If, in addition, $\mathcal{M} = \mathcal{H}_n^{\mathcal{M}}(\rho_n^{\mathcal{M}} \cup \{q\})$ for some q and $\rho_{n+1}^{\mathcal{M}} \leq \kappa$, then

$$\rho_{n+1}^{\mathcal{M}} = \rho_{n+1}^{\operatorname{Ult}_n(\mathcal{M},E)}\,.$$

PROOF. The nontrivial part of the first sentence is the assertion that $P(\kappa) \cap |\operatorname{Ult}_n(\mathcal{M}, E)| \subseteq |\mathcal{M}|$. So let $[a, f]_E^{\mathcal{M}} \subseteq \kappa$, where $f : [\kappa]^i \to P(\kappa)$. Since $\kappa < \rho_n^{\mathcal{M}}$ and f is defined from a parameter by a term in Sk_n, in fact $f \in |\mathcal{M}|$. For $\alpha < \kappa$, let $A_\alpha = \{\bar{u} \in [\kappa]^i \mid \alpha \in f(\bar{u})\}$. From $E_a \cap \{A_\alpha \mid \alpha < \kappa\}$ we can compute $[a, f]_E^{\mathcal{M}}$. Since E is close to $\mathcal{M}$, we get $[a, f]_E^{\mathcal{M}} \in |\mathcal{M}|$.

For the second assertion it is convenient to use the master code structures. Let $\mathcal{N} = \operatorname{Ult}_n(\mathcal{M}, E)$ and fix q such that $\mathcal{M} = \mathcal{H}_n^{\mathcal{M}}(\rho_n^{\mathcal{M}} \cup \{q\})$. Set

$$\begin{aligned}
A^{\mathcal{M}} &= \mathrm{Th}_n^{\mathcal{M}}(\rho_n^{\mathcal{M}} \cup \{q\})\,\text{, coded as a subset of } \rho_n^{\mathcal{M}}\,, \\
A^{\mathcal{N}} &= \mathrm{Th}_n^{\mathcal{N}}(\rho_n^{\mathcal{N}}(\rho_n^{\mathcal{N}} \cup \{\pi(q)\})\,\text{, coded as a subset of } \rho_n^{\mathcal{N}}\,, \\
\mathcal{P} &= (J_{\rho_n^{\mathcal{M}}}^{\dot{E}^{\mathcal{M}}}, \in, \dot{E}^{\mathcal{M}} \restriction \rho_n^{\mathcal{M}}, A^{\mathcal{M}})\,, \\
\mathcal{Q} &= (J_{\rho_n^{\mathcal{N}}}^{\dot{E}^{\mathcal{N}}}, \in, \dot{E}^{\mathcal{N}} \restriction \rho_n^{\mathcal{N}}, A^{\mathcal{N}})\,,
\end{aligned}$$

so that $\mathcal{Q} = \operatorname{Ult}_0(\mathcal{P}, E)$ with canonical embedding π, which is cofinal and Σ_1 elementary. If $n = 0$ then we take $\mathcal{P} = \mathcal{M}$ and $\mathcal{Q} = \mathcal{N}$.

By Lemmas 2.10 and 2.11, $\rho_{n+1}^{\mathcal{M}}$ is the least α such that some Σ_1 over $\mathcal{P}$ set $B \subseteq \alpha$ is such that $B \notin |\mathcal{P}|$, and similarly for $\rho_{n+1}^{\mathcal{N}}$ and $\mathcal{Q}$.

To see that $\rho_{n+1}^{\mathcal{N}} \leq \rho_{n+1}^{\mathcal{M}}$, let $B \subseteq \rho_{n+1}^{\mathcal{M}}$ be Σ_1 over $\mathcal{P}$ and $B \notin |\mathcal{P}|$. Since $\rho_{n+1}^{\mathcal{M}} \leq \kappa$ and π is Σ_1 elementary, B is Σ_1 over $\mathcal{Q}$. But $P(\kappa) = P(\kappa)^{\mathcal{M}} = P(\kappa)^{\mathcal{N}} = P(\kappa)^{\mathcal{Q}}$ using the first assertion of 4.5 and strong acceptability. Thus $B \notin |\mathcal{Q}|$.

To see that $\rho_{n+1}^{\mathcal{M}} \le \rho_{n+1}^{\mathcal{N}}$, let $\alpha \le \rho_{n+1}^{\mathcal{M}}$ and $B \subseteq \alpha$ be $\mathbf{\Sigma}_1$ over $\mathcal{Q}$. It is enough to show that B is $\mathbf{\Sigma}_1$ over $\mathcal{P}$. Let

$$\eta \in B \Leftrightarrow \mathcal{Q} \models \psi[\eta, [a, f]]$$

where ψ is Σ_1 in the language of $\mathcal{Q}$. For $\delta < \mathrm{OR}^{\mathcal{Q}}$ let

$$\mathcal{Q}_\delta = (J_\delta^{\dot{E}^{\mathcal{N}}}, \in, \dot{E}^{\mathcal{N}} \restriction \omega\delta, A^{\mathcal{N}} \cap \omega\delta)$$

and similarly define $\mathcal{P}_\delta$ for $\delta < \mathrm{OR}^{\mathcal{P}}$. So

$$\eta \in B \Leftrightarrow \exists \delta < \mathrm{OR}^{\mathcal{Q}} \quad \mathcal{Q}_\delta \models \psi[\eta, [a, f]]$$

so

$$\begin{aligned}\eta \in B &\Leftrightarrow \exists \delta < \mathrm{OR}^{\mathcal{P}}\ \mathcal{Q}_{\pi(\delta)} \models \psi[\eta, [a, f]] \\ &\Leftrightarrow \exists \delta < \mathrm{OR}^{\mathcal{P}} \exists X \in E_a\ \forall \bar{u} \in X \mathcal{P}_\delta \models \psi[\eta, f(\bar{u})]\,.\end{aligned}$$

Now as E is close to $\mathcal{M}$, the E_a is an $r\Sigma_1^{\mathcal{M}}$ subset of $|\mathcal{P}|$. By Lemma 2.11, E_a is $\mathbf{\Sigma}_1$ over $\mathcal{P}$. Thus B is $\mathbf{\Sigma}_1$ over $\mathcal{P}$, as desired. □

We now consider preservation of the $n+1$st standard parameter.

Lemma 4.6. *Let $\mathcal{M}$ be a ppm or sppm, $n \ge 0$, and $\mathcal{M} = \mathcal{H}_n^{\mathcal{M}}(\rho_n^{\mathcal{M}} \cup \{q\})$ if $n \ge 1$. Let E be an extender close to $\mathcal{M}$ such that $\rho_{n+1}^{\mathcal{M}} \le \mathrm{crit}\ E < \rho_n^{\mathcal{M}}$. Let*

$$\pi : \mathcal{M} \to \mathrm{Ult}_n(\mathcal{M}, E) = \mathcal{N}$$

be the canonical embedding. Suppose that r is the $n+1$st standard parameter of $(\mathcal{M}, q)$ and that r is $n+1$-solid over $(\mathcal{M}, q)$.

Then $\pi(r)$ is the $n+1$st standard parameter of $(\mathcal{N}, \pi(q))$, and $\pi(r)$ is $n+1$-solid over $(\mathcal{N}, \pi(q))$.

Proof. We will give the proof for the case $n = 0$ with a passive premouse of limit length. The general proof is the same as this, using the fact that $r\Sigma_{n+1}$ is equivalent to Σ_1 over the appropriate master code structure. See lemma 2.11 for the case of $n > 0$ and the remark following corollary 2.2 for the case of $n = 0$ with an active premouse. For successor ordinals $\lambda = \gamma + 1$ write $\mathcal{M}_\lambda = \bigcup_{n \in \omega} S_{\omega\gamma+n}^{\mathcal{M}_\lambda}$, where $(S_\nu^{\mathcal{M}_\lambda} : \nu < \omega\lambda)$ is Jensen's S sequence, and use the same proof as below.

Let us consider first the case $n = 0$, $\mathcal{M}$ is passive, and

$$\mathcal{M} = (J_\lambda^{\dot{E}^{\mathcal{M}}}, \in, \dot{E}^{\mathcal{M}}) \quad (\lambda \text{ limit})\,.$$

Now by 4.5, $\rho_1^{\mathcal{M}} = \rho_1^{\mathcal{N}}$ and

$$\mathrm{Th}_1^{\mathcal{N}}(\rho_1^{\mathcal{N}} \cup \{\pi(\sigma), \pi(q)\}) = \mathrm{Th}_1^{\mathcal{M}}(\rho_1^{\mathcal{M}} \cup \{r, q\}) \notin |\mathcal{N}|\,,$$

so it is enough to show that $\pi(r)$ is 1-solid over $(\mathcal{N}, \pi(q))$. Let

$$r = \langle \alpha_0 \cdots \alpha_\ell \rangle$$
$$b_i = \mathrm{Th}_1^{\mathcal{M}}(\alpha_i \cup \{\alpha_0 \cdots \alpha_{i-1}, q\}) \cap \{(\varphi, \vec{c}) \mid \varphi \text{ is pure } r\Sigma_1\}$$
$$c_i = \mathrm{Th}_1^{\mathcal{N}}(\pi(\alpha_i) \cup \{\pi(\alpha_0) \cdots \pi(\alpha_{i-1}), \pi(q)\}) \cap \{(\varphi, \vec{c}) \mid \varphi \text{ is pure } r\Sigma_1\}$$

for $0 \leq i \leq \ell$. By 2.10, it will be enough to show $c_i \in |\mathcal{N}|$ for $0 \leq i \leq \ell$. So fix i such that $0 \leq i \leq \ell$.

For $\gamma < \lambda$ let

$$R_\gamma = b_i \cap \{(\varphi, \vec{c}) \mid \mathcal{M}_\gamma \models \varphi[\vec{c}]\}$$

so that $b_i = \bigcup_{\gamma<\lambda} R_\gamma$. Similarly, let

$$S_\gamma = c_i \cap \{(\varphi, \vec{c}) \mid \mathcal{N}_\gamma \models \varphi[\vec{c}]\}$$

so that $c_i = \bigcup_{\omega\gamma<\mathrm{OR}^{\mathcal{M}}} S_\gamma$. It is easy to see that $\pi(R_\gamma) = S_{\pi(\gamma)}$, and thus

$$c_i = \bigcup_{\gamma<\lambda} \pi(R_\gamma).$$

Case 1. $\exists\gamma < \lambda(b_i = R_\gamma)$; i.e. R_γ is eventually constant as $\gamma \to \lambda$.

PROOF. Let $b_i = R_\gamma = \bigcup_{\eta<\lambda} R_\eta$. Then $c_i = \bigcup_{\eta<\lambda} \pi(R_\eta) = \pi(R_\gamma) = \pi(b_i)$. Thus $c_i \in |\eta|$.

Case 2. Otherwise.

PROOF. For $x \in b_i$, let $\gamma_x =$ least γ such that $x \in R_\gamma$ and if $y \in b_i$, $x \leq y$ iff $\gamma_x \leq \gamma_y$. Thus $\leq$ is a prewellorder of b_i of limit order type. Notice $\leq$ is computable within $\mathcal{M}$ from b_i, so that our solidity hypothesis on r means $b_i \in |\mathcal{M}|$ and $\leq \in |\mathcal{M}|$. Now clearly, for $x \in b_i$

$$R_{\gamma_x} = \{y \in b_i \mid y \leq x\}$$

so

$$\pi(R_{\gamma_x}) = \{y \in \pi(b_i) \mid y \leq^* \pi(x)\}$$

where $\leq^* = \pi(\leq)$. By case hypothesis

$$c_i = \{y \in \pi(b_i) \mid \exists x \in b_i(y \leq^* \pi(x))\},$$

so that c_i is an $\leq^*$ initial segment of $\pi(b_i)$.

Subcase 2A. $\mathrm{cof}^{\mathcal{M}}(\leq) \neq \mathrm{crit}(E)$. In this case, ran π is $\leq^*$ cofinal in $\pi(b_i)$, so that $c_i = \pi(b_i)$, and $c_i \in |\mathcal{N}|$.

Subcase 2B. $\mathrm{cof}^{\mathcal{M}}(\leq) = \mathrm{crit}(E)$. Let $\kappa = \mathrm{crit}(E)$, $f \in |\mathcal{M}|$, $f : \kappa \to b_i$ such that ran f is $\leq$-cofinal. Then

$$\begin{aligned} y \in c_i &\Leftrightarrow \exists \alpha < \kappa (y \leq^* \pi(f(\alpha))) \\ &\Leftrightarrow \exists \alpha < \kappa (y \leq^* \pi(f)(\alpha)) \end{aligned}$$

so that $c_i \in |\mathcal{N}|$, as desired.

□

Of course, we shall need to know that fine structure "up to level n" is preserved not just under passage to Ult_n, but under iteration of this process. The following lemma summarizes the important facts.

Lemma 4.7. *Let $\mathcal{M} = \mathcal{M}_0$ be n-sound, where $n \leq \omega$. Suppose that for $\alpha < \theta$,*

$$\mathcal{M}_{\alpha+1} = \mathrm{Ult}_n(\mathcal{M}_\alpha, E_\alpha),$$

where E_α is close to $\mathcal{M}_\alpha$, and

$$\mathcal{M}_\lambda = \mathrm{dir}\lim_{\beta<\lambda} \mathcal{M}_\beta$$

for $\lambda \leq \theta$ a limit. (We assume each $\mathcal{M}_\alpha$ is wellfounded.) Let $\pi_{0\theta} : \mathcal{M} \to \mathcal{M}_\theta$ be the canonical embedding. Then

(a) $\pi_{0\theta}$ is an n-embedding.

If, in addition, $\mathcal{M}$ is $n+1$-sound (so $n < \omega$) and $\rho_{n+1}^{\mathcal{M}} \leq \mathrm{crit}\ \pi_{0\theta}$, then

(b) $\rho_{n+1}^{\mathcal{M}} = \rho_{n+1}^{\mathcal{M}_\theta}$.
(c) $\pi_{0\theta}(p_{n+1}(\mathcal{M})) = p_{n+1}(\mathcal{M}_\theta$.
(d) $\mathcal{M}_\theta$ is $n+1$-solid, and in fact $\mathfrak{C}_{n+1}(\mathcal{M}_\theta) = \mathcal{M}$, and letting

$$\sigma : \mathfrak{C}_{n+1}(\mathcal{M}_\theta) \to \mathfrak{C}_n(\mathcal{M}_\theta) = \mathcal{M}_\theta$$

be the inverse of the collapse, $\sigma = \pi_{0\theta}$.

Proof. This is a fairly routine induction on θ, using Lemmas 4.4, 4.5, and 4.6. The successor case is immediate from these lemmas, so let θ be a limit. Then (a), (b) are obvious, and (d) follows easily from (c). We sketch a proof of (c): let $p_{n+1}(\mathcal{M}) = \langle \bar{r}, \bar{u} \rangle$, where $\bar{r} = \langle \alpha_0 \cdots \alpha_\ell \rangle$. For $\gamma < \theta$ let

$$b_i^\gamma = \mathrm{Th}_{n+1}^{\mathcal{M}_\gamma}(\pi_{0\gamma}(\alpha_i) \cup \{\pi_{0\gamma}(\alpha_1), \ldots, \pi_{0\gamma}(\alpha_{i-1}), \pi_{0\gamma}(\bar{u})\}).$$

Part of our induction hypothesis should be that $b_i^\gamma \in |\mathcal{M}_\gamma|$ for $0 \leq i \leq \ell$. This follows from 4.6 for successor θ, and for limit θ, our current case, from the proof of 4.6. For that proof shows that for each fixed i there are most finitely many

$\gamma < \theta$ such that $\pi_{\gamma+1}(b_i^\gamma) \neq b_i^{\gamma+1}$, since this can occur only when the ordering $\leq_i^\gamma$ of that proof has $\mathcal{M}_\gamma$-cofinality equal to crit E_γ, but when that happens $\leq_i^{\gamma+1}$ also has $\mathcal{M}_{\gamma+1}$-cofinality crit $E_\gamma <$ crit $E_{\gamma+1}$. Thus we can find $\gamma < \theta$ such that for all i, and all η such that $\gamma < \eta < \theta$, $\pi_{\gamma\eta}(b_i^\gamma) = b_i^\eta$.

One can now easily check that $b_i^\theta = \pi_{\gamma\theta}(b_i^\gamma)$ for all i. This in turn implies that $\pi_{0\theta}(\bar{r})$ is the $n+1$st standard parameter of $(\mathcal{M}_\theta, \pi_{0\theta}(\bar{u}))$. The rest of (c) is easy. □

REMARK. Under the hypotheses of 4.7 (including that $\mathcal{M}$ is $n+1$-sound and $\rho_{n+1}^{\mathcal{M}} \leq \mathrm{crit}(\pi_{0\theta})$), we see that $\mathcal{M}_\theta$ is not $n+1$-sound. If $\mathrm{crit}(i_{0\theta}) \neq \tau^{\mathcal{M}_0}[\bar{\alpha}, \pi_{0\theta}(p_{n+1}(\mathcal{M}))]$ then $\forall \bar{\alpha} \in [\rho_{n+1}^{\mathcal{M}}]^{<\omega}$ and $\tau \in \mathrm{Sk}_{n+1}$.