§3. Squashed Mice

Let $\mathcal{M}$ be an active type III ppm. Let E be an extender over $\mathcal{M}$ with $\kappa = \mathrm{crit}\ E < \nu^{\mathcal{M}}$. Even if wellfounded, $\mathrm{Ult}(\mathcal{M}, E)$ may not be a ppm. The trouble is in the initial segment condition: if $i''_E \nu^{\mathcal{M}}$ is not cofinal in $i_E(\nu^{\mathcal{M}})$, then this condition will fail in $\mathrm{Ult}(\mathcal{M}, E)$. The problem seems to be that we are using too many functions in forming $\mathrm{Ult}(\mathcal{M}, E)$; we'd like to use only functions in $J^{\dot{E}^{\mathcal{M}}}_{\nu^{\mathcal{M}}}$ in order to get continuity of i_E at $\nu^{\mathcal{M}}$. Lemma 9.1 and the remarks following it give a fuller explanation. This leads to

Definition 3.0.1. ($\mathcal{M}$-squash) Let $\mathcal{M}$ be an active type III ppm. Let F be the extender coded by $\dot{F}^{\mathcal{M}}$ and $\nu = \nu^{\mathcal{M}}$. Then

$$\mathcal{M}^{\mathrm{sq}} = (J^{\dot{E}^{\mathcal{M}}}_{\nu}, \in, \dot{E}^{\mathcal{M}} \restriction \nu, F \restriction \nu)$$

The symbol $\mathcal{M}^{\mathrm{sq}}$ stands for "$\mathcal{M}$-squash". The term "squashed mouse" was invented by Dodd for use in a similar, but more complicated, context.

Recall that $\nu^{\mathcal{M}}$ is a cardinal of $\mathcal{M}$ in the type III case, so that $\mathcal{M}^{\mathrm{sq}}$ includes all sets which have hereditarily cardinality $< \nu^{\mathcal{M}}$ in $\mathcal{M}$. Our next lemma shows that $\mathcal{M}^{\mathrm{sq}}$ is amenable.

Lemma 3.1. *Let $\mathcal{M}$ be an active type* III *ppm. Then there are cofinally many $\gamma < \nu^{\mathcal{M}}$ such that $\dot{E}^{m}_{\gamma} = F \restriction \gamma$ where F is the extender coded by $\dot{F}^{\mathcal{M}}$.*

Proof. Let $\kappa = \mathrm{crit}\ F$, and let $\eta - 1$ be a generator of F. By the initial segment condition, there is a $\gamma < \mathrm{OR}^{\mathcal{M}}$ such that $\dot{E}^{\mathcal{M}}_{\gamma}$ exists and is the trivial completion of $F \restriction \eta$. (Alternative (b) of the initial segment condition cannot hold as η is a successor ordinal.) Now the natural map π from $\mathrm{Ult}(\mathcal{M}, F \restriction \eta)$ into $\mathrm{Ult}(\mathcal{M}, F)$ has critical point $\geq \eta$, and hence $\mathrm{crit}(\pi) \geq \gamma$ since $\gamma = (\eta^+)^{\mathrm{Ult}(\mathcal{M}, F\restriction\eta)}$. This implies that $F \restriction \gamma$ is the trivial completion of $F \restriction \eta$, which is $\dot{E}^{\mathcal{M}}_{\gamma}$.

To see this let G be the trivial completion of $F \restriction \eta$. We have

$$\begin{tikzcd} \mathcal{M} \arrow[r, "i_F"] \arrow[dr, "i_G"'] & \mathrm{Ult}(M, F) \\ & \mathrm{Ult}(\mathcal{M}, F \restriction \eta) \arrow[u, "\pi"'] \end{tikzcd}$$

and for $a \in [\gamma]^{<\omega}$, x appropriate,

$$\begin{aligned} (a, x) \in G &\Leftrightarrow a \in i_G(x) \\ &\Leftrightarrow \pi(a) \in \pi(i_G(x)) \\ &\Leftrightarrow a \in i_F(x) \\ &\Leftrightarrow (a, x) \in F\,. \end{aligned}$$

Since there are arbitrarily large $\eta < \nu^{\mathcal{M}}$ such that $\eta - 1$ is a generator of F, this completes the proof. $\square$

So $\mathcal{M}^{sq}$ is amenable. Moreover, the definition of $\nu^{\mathcal{M}}$ guarantees that the rest of $\mathcal{M}$ can be recovered from $\mathcal{M}^{sq}$ by taking an ultrapower.

If E is an extender over $\mathcal{M}$ with crit $E < \nu^{\mathcal{M}}$, we'll have

$$\begin{tikzcd} \mathcal{M} \arrow[r] & \mathrm{Ult}(\mathcal{M}, E) \\ \mathcal{M}^{sq} \arrow[u, "\mathrm{id}"] \arrow[r] & \mathrm{Ult}(\mathcal{M}^{sq}, E) \arrow[u] \end{tikzcd}$$

and $\mathrm{Ult}(\mathcal{M}^{sq}, E) = \mathcal{N}^{sq}$ for some $\mathcal{N} \subseteq \mathrm{Ult}(\mathcal{M}, E)$. But $\mathcal{N} \neq \mathrm{Ult}(\mathcal{M}, E)$ is possible, and this is what leads us to iterate on the squashed level.

As we shall iterate $\mathcal{M}^{sq}$ and not $\mathcal{M}$, the appropriate definability hierarchy is based on $\mathcal{M}^{sq}$, not $\mathcal{M}$ as in §2. Note every $\mathcal{M}$-definable subset of $\nu^{\mathcal{M}}$ is definable over $\mathcal{M}^{sq}$.

DEFINITION 3.1.1. $\mathcal{N}$ is an sppm iff $\mathcal{N} = \mathcal{M}^{sq}$ for some active type III ppm $\mathcal{M}$.

We now introduce a language appropriate for sppm.

DEFINITION 3.1.2. $\mathcal{L}^*$ is the language of set theory with additional 1- place predicate symbol $\dot{E}$, 2-place predicate symbol $\dot{F}$, and constant symbol $\dot{\mu}$.

We interpret $\mathcal{L}^*$ in an sppm

$$\mathcal{N} = (J_\nu^{\vec{E}}, \in, \vec{E}, F)$$

by setting $\dot{E}^{\mathcal{N}} = \vec{E}$, $\dot{F}^{\mathcal{N}} = F$, and $\dot{\mu}^{\mathcal{N}} = \text{crit } F$.

As sppm are amenable with respect to their predicates, we can work with the usual notions of Σ_0 and Σ_1.

DEFINITION 3.1.3. (a) A formula of $\mathcal{L}^*$ is Σ_0 iff it is built up from atomic formulae using $\wedge, \vee, \neg$, $\exists x \in y$, and $\forall x \in y$.

(b) The Σ_n and $\prod_n$ formulae of $\mathcal{L}^*$ are also as usual.

We want now to say "I am an sppm" with a simple formula.

DEFINITION 3.1.4. A *P-formula* is a formula of $\mathcal{L}^*$ of the form

$$\theta(\bar{v}) = \forall x \exists y (x \subset y \wedge \psi(y) \wedge \forall a \in x \, \exists b \in y \, \varphi(a, b, \bar{v}),$$

where ψ is Σ_1 without x free in it, and φ is Σ_0 without x or y free in it.

Thus a P formula can say a little more than that there are cofinally many y (under $\subseteq$) with a Σ_1 property. We aren't sure how necessary the little more is, but as the preservation lemma still goes through, there's no harm in it.

Lemma 3.2. *Let $\mathcal{M}$ and $\mathcal{N}$ be transitive $\mathcal{L}^*$ structures, and $\pi : \mathcal{M} \to \mathcal{N}$, and ψ be a P formula.*

(a) *If π is a Σ_1 embedding and $\mathcal{N} \models \psi[\pi(\bar{a})]$, then $\mathcal{M} \models \psi[\bar{a}]$.*

(b) *If π is a cofinal (i.e. $|\mathcal{N}| = \bigcup \operatorname{ran} \pi$) Σ_0 embedding and $\mathcal{M} \models \psi[\bar{a}]$, then $\mathcal{N} \models \psi[\pi(\bar{a})]$.*

One can't quite say "I am an sppm" with a P sentence, since the decoding of $\mathcal{M}$ from $\mathcal{M}^{sq}$ requires taking an ultrapower, and we can't capture the wellfoundedness of this ultrapower. We do get

Lemma 3.3. *There is a P sentence ψ of $\mathcal{L}^*$ such that*

(a) *If $\mathcal{N}$ is an sppm, then $\mathcal{N} \models \psi$.*

(b) *If $\mathcal{N}$ is transitive and $\mathcal{N} \models \psi$, then $\dot{F}^{\mathcal{N}}$ is a pre-extender over $\mathcal{N}$; moreover, if $\operatorname{Ult}(\mathcal{N}, \dot{F}^{\mathcal{N}})$ is wellfounded then $\mathcal{N}$ is an sppm or $\mathcal{N}$ is "of superstrong type", that is $i_F^{\mathcal{N}}(\text{crit } F) = \text{length } F = OR^{\mathcal{N}}$).*

Proof (Sketch). By Dodd-Jensen we have a P sentence θ_1 whose transitive models $\mathcal{N}$ are those of the form $\mathcal{N} = (J_\nu^{\dot{E}^{\mathcal{N}}}, \dots)$, ν a limit ordinal.

Let θ_2 be the Π_1 sentence of $\mathcal{L}^*$ asserting that $\dot{E}^{\mathcal{N}}$ is good at all $\alpha < \mathrm{OR}^{\mathcal{N}}$.

Let θ_3 be the Π_1 sentence: $\forall a \forall x (\dot{F}(a, x) \Rightarrow a \in [\mathrm{OR}]^{<\omega} \wedge x \subseteq [\dot{\mu}]^{\operatorname{card} a})$

Let θ_4 be the P sentence: There are cofinally many ordinals γ such that $\gamma \in \operatorname{dom} \dot{E}$ and $J_\gamma^{\dot{E}} \cap E_\gamma = \dot{F} \restriction \gamma \cap J_\gamma^{\dot{E}}$.

It may seem that "$\dot{\mu}^+$ exists" is Σ_2, but we can say with θ_5:

$$\exists \text{ ordinal } \alpha \text{ such that } \dot{\mu} < \alpha \text{ and } \{(\beta_0, \beta_1) \mid J_{\dot{\mu}}^{\dot{E}} \models \beta_1 = \beta_0^+\} \in \dot{F}_{\{\dot{\mu}, \alpha\}}.$$

We claim $\psi = \bigwedge_{i \leq 5} \theta_i$ is as desired. Clearly, if $\mathcal{N}$ is an sppm, then $\mathcal{N} \models \bigwedge_{i \leq 5} \theta_i$.

Now suppose $\mathcal{N}$ is a transitive $\mathcal{L}^*$ structure such that $\mathcal{N} \models \bigwedge_{i \leq 5} \theta_i$. As $\mathcal{N} \models \theta_4$, we see that $\dot{F}^{\mathcal{N}} = F$ is a pre-extender over $\mathcal{N}$. Suppose that $\operatorname{Ult}(\mathcal{N}, F)$ is wellfounded, and that $i_F^{\mathcal{N}}(\text{crit } F) > \mathrm{OR}^{\mathcal{N}}$. Let

$$\nu = \mathrm{OR}^{\mathcal{N}}$$

$$\alpha = (\nu^+)^{\operatorname{Ult}(\mathcal{N}, F)}$$

$$G = \text{the } (\dot{\mu}^{\mathcal{N}}, \alpha) \text{ extender derived from } i_F^{\mathcal{N}} : \mathcal{N} \to \operatorname{Ult}(\mathcal{N}, F)$$

$$\mathcal{M} = \left(J_\alpha^{i_F(\dot{E}^{\mathcal{N}})}, \in, i_F(\dot{E}^{\mathcal{N}}) \restriction \alpha, G\right).$$

Note that α exists since $i_F(\dot\mu^{\mathcal{N}}) > \nu$.

We claim $\mathcal{M}$ is an active type III ppm, and $\nu = \nu^{\mathcal{M}}$. For this, note $i_F(\dot E^{\mathcal{N}}) \upharpoonright \alpha$ is good at all $\beta < \alpha$, since $\dot E^{\mathcal{N}}$ is good at all $\beta < \dot\mu^{\mathcal{N}}$. So it is enough to check $i_F(\dot E^{\mathcal{N}}) \upharpoonright \alpha^\frown G$ is good at α.

Clearly $\mathcal{M}$ is strongly acceptable. G is a pre-extender over $\mathcal{M}$ as G is a pre-extender over $\mathcal{N}$ and

$$P(\dot\mu^{\mathcal{N}}) \cap \mathcal{N} = P(\dot\mu^{\mathcal{N}}) \cap \mathrm{Ult}(\mathcal{N}, F) = P(\dot\mu^{\mathcal{N}}) \cap \mathcal{M}.$$

The ordinal ν satisfies condition 3 of good at α since $P(\dot\mu^{\mathcal{N}}) \cap |\mathcal{M}| \subseteq J_\nu^{i_F(\dot E^{\mathcal{N}})}$ as $\mathcal{N} \models \theta_5$. Since G is derived from $i_F^{\mathcal{N}}$, $\mathrm{Ult}(\mathcal{N}, G) = \mathrm{Ult}(\mathcal{N}, F)$; on the other hand $\mathcal{N}$ and $\mathcal{M}$ agree up to ν (i.e. $J_\nu^{i_F(\dot E^{\mathcal{N}})} = J_\nu^{\dot E^{\mathcal{N}}}$) as $\mathcal{N} \models \theta_4$ so $\mathrm{Ult}(\mathcal{M}, G)$ agrees with $\mathrm{Ult}(\mathcal{N}, G)$ up to $i_F^{\mathcal{N}}(\nu) = i_G^{\mathcal{M}}(\nu)$, so that $\alpha = \nu^+$ in $\mathrm{Ult}(\mathcal{M}, G)$. This verifies 3(a). For 3(b), note $G \upharpoonright \nu = F$, and that if $\beta < \alpha$ then for some $a \subseteq \nu$ and $f : [\dot\mu^{\mathcal{N}}]^{\mathrm{card}\, a} \to \dot\mu^{\mathcal{N}}$ such that $f \in \mathcal{N}$ we have

$$[a, f]_F^{\mathcal{N}} = \beta,$$

so

$$[a, f]_G^{\mathcal{N}} = \beta,$$

so

$$[a, f]_G^{\mathcal{M}} = \beta.$$

This is enough to give 3(b). Finally, ν is the least ordinals satisfying clause 3 since if $\gamma < \nu$, then $G \upharpoonright \gamma = F \upharpoonright \gamma \in \mathcal{M}$ by the fact that $\mathcal{N} \models \theta_4$.

It is easy to see the coherence condition 4 is satisfied. The initial segment condition (only 5(a) is relevant) is satisfied as $\mathcal{N} \models \theta_4$ and $i_F^{\mathcal{N}}(\dot E^{\mathcal{N}}) \upharpoonright \nu = \dot E^{\mathcal{N}}$.

Thus $\mathcal{M}$ is an active type III ppm with $\nu = \nu^{\mathcal{M}}$. Clearly $\mathcal{N} = \mathcal{M}^{sq}$. □

Remark. It is annoying that we must include the possibility that $\mathcal{N} \models \psi$ be "of superstrong type", but our attempts to strengthen ψ so as to exclude this have not succeeded. Notice that if $\mathcal{N} \models \psi$ is of superstrong type, then a standard argument gives

$$(J_\nu^{\dot E^{\mathcal{N}}}, \in, \dot E^{\mathcal{N}}) \models \mathrm{ZFC} + \dot\mu^{\mathcal{N}} \text{ is a Shelah limit of Shelah cardinals}.$$

($\nu = \mathrm{OR}^{\mathcal{N}}$). So $\mathcal{N}$ is far above any mice our theory can handle anyway.

The rest of this section is an obvious parallel to §2. Because sppm are amenable, we could adopt a very literal version of the Dodd-Jensen approach here (in particular, we could stick to the usual Σ_n hierarchy); however, for the sake of internal consistency, we shall adopt the approach of §2.

Skolem terms and projecta.

DEFINITION 3.3.1. $\mathcal{L}^{**}$ is $\mathcal{L}^*$ together with binary relation symbols $\dot{T}_n$ for $1 \leq n < \omega$.

We define the quasi-Σ_n formulae for $n \geq 1$.

DEFINITION 3.3.2.

(a) The $q\Sigma_1$ formulae of $\mathcal{L}^{**}$ are precisely the Σ_1 formulae of $\mathcal{L}^*$.

(b) A formula $\theta(\bar{v})$ of $\mathcal{L}^{**}$ is $q\Sigma_{n+1}$, where $n \geq 1$, iff

$$\theta(\bar{v}) = \exists a \exists b(\dot{T}_n(a,b) \wedge \varphi(a,b,\bar{v}))$$

where φ is $q\Sigma_1$.

DEFINITION 3.3.3. For $\varphi(v_0 \cdots v_{k+1})$ an $\mathcal{L}^{**}$ formulae, $\tau_\varphi(v_0 \cdots v_k)$ is the basic Skolem term associated to φ. Having interpreted φ in an sppm $\mathcal{N}$, we set

$$\tau_\rho^{\mathcal{N}}[a_0 \cdots a_k] = \begin{cases} <^{\mathcal{N}} & \text{least } b \text{ such that } \mathcal{N} \models \varphi[\bar{a}, b] \\ 0 & \text{if no such } b \text{ exists}. \end{cases}$$

DEFINITION 3.3.4. SK_n (for $n \geq 1$) is the smallest class of terms containing all τ_φ for φ $q\Sigma_n$ and closed under composition.

DEFINITION 3.3.5. A $\mathcal{L}^{**}$ formula is *generalized* $q\Sigma_n$ iff it results from substituting terms in SK_n for free variables of a $q\Sigma_n$ formula. (The substitution must be such that no free variable of a term becomes bound in the result.)

DEFINITION 3.3.6. Let $\mathcal{M}$ be an sppm. Then for $n \geq 1$

(a) $\mathrm{Th}_n^{\mathcal{M}}(X) = \{(\varphi, \bar{a}) \mid \varphi \text{ is generalized } q\Sigma_n \text{ and } \bar{a} \in X^{<\omega} \text{ and } \mathcal{M} \models \varphi[\bar{a}]\}$.

(b) $\rho_n^{\mathcal{M}} =$ least $\alpha \leq \mathrm{OR}^{\mathcal{M}}$ such that for some $p \in |\mathcal{M}|$, $\mathrm{Th}_n^{\mathcal{M}}(\alpha \cup \{p\}) \notin |\mathcal{M}|$.

(c) $\dot{T}_n^{\mathcal{M}}(a,b)$ iff $a = \langle \alpha, q \rangle$ for some $\alpha < \rho_n^{\mathcal{M}}$ such that $b = \mathrm{Th}_n^{\mathcal{M}}(\alpha \cup \{q\})$.

We define the classes of relations $q\Sigma_n^{\mathcal{M}}$, etc., in the obvious way. It is easy to see that $q\Sigma_n^{\mathcal{M}}$ is closed under $\exists$, $\wedge$, and $\vee$, and that $(q\Sigma_n^{\mathcal{M}} \cup q\prod_n^{\mathcal{M}}) \subseteq q\Sigma_{n+1}^{\mathcal{M}}$, uniformly over all sppm. One can also show $\neg\dot{T}_n^{\mathcal{M}}$ is a $q\Sigma_{n+1}^m$ relation (uniformly) in parameter $\rho_n^{\mathcal{M}}$.

Hulls.

For $\mathcal{M}$ a sppm and $X \subseteq |\mathcal{M}|$ and $n \geq 1$ let

$$H_n^{\mathcal{M}}(X) = \text{the transitive collapse of } \{\tau^{\mathcal{M}}[\bar{a}] \mid \bar{a} \in X^{<\omega} \text{ and } \tau \in \mathrm{SK}_n\}$$
$$\mathcal{H}_n^{\mathcal{M}}(X) = (H_n^{\mathcal{M}}(X), \in, \pi''(\dot{E}^{\mathcal{M}}), \pi''(\dot{F}^{\mathcal{M}}))$$

where π is the collapse map.

Lemma 3.4. *Let $\mathcal{M}$ be an sppm such that $\forall\nu \in \mathrm{OR}^{\mathcal{M}}\,(J_\nu^{\dot{E}^{\mathcal{M}}} \models$ there are no Shelah cardinals). Then for any $X \subseteq |\mathcal{M}|$ and $n \geq 1$, $\mathcal{H}_n^{\mathcal{M}}(X)$ is an sppm.*

PROOF. By Lemmas 3.2 and 3.3. Strictly speaking, we haven't packed enough wellfoundedness of ultrapowers into being an sppm to be able just to quote 3.3(b), but the proof of 3.3(b) requires only the wellfoundedness we have.

Lemma 2.7 carries over verbatim.

Lemma 3.5. *Lemma 2.7 remains true if one replaces "ppm which is passive or active of types I or II" by "sppm" and "$\mathrm{r}\Sigma_n$" by "$q\Sigma_n$".*

Standard parameters, solid parameters, and Cores.

The definitions and results of §2 carry over verbatim. (The only "results" here are Lemmas 2.8, 2.9.) We shall say no more.

Premice.

DEFINITION 3.5.1. Let $\mathcal{M} = \mathcal{J}_\alpha^{\vec{E}}$ be a ppm. We say $\mathcal{M}$ is a premouse iff for all $\beta < \alpha$,

(1) $\mathcal{J}_\beta^{\vec{E}}$ is passive or active of types I or II $\Rightarrow$ $\mathcal{J}_\beta^{\vec{E}}$ is ω-sound, and

(2) $\mathcal{J}_\beta^{\vec{E}}$ is active type III $\Rightarrow$ $(\mathcal{J}_\beta^{\vec{E}})^{\mathrm{sq}}$ is ω-sound.

Notice that a premouse need not itself be ω-sound.

We shall eventually build an $\vec{E}$ such that every $\mathcal{J}_\alpha^{\vec{E}}$ is a premouse.