§2. Fine Structure

This and the next section explain some basic facts about the fine structure of definability over potential premice. Most of the notions and results here are due to Dodd and Jensen ([DJ4]). There is one important difference, however: because E_α measures only sets in $J_\alpha^{\vec{E}\restriction\alpha}$, the hydras of [DJ4] disappear; moreover, we never have to iterate in order to extend measures. This simplifies the theory considerably.

Let $\mathcal{M} = \mathcal{J}_\beta^{\vec{E}}$. If $\nu^{\mathcal{M}}$ is a successor ordinal, we let $\gamma^{\mathcal{M}}$ be the witness to 5(a) or 5(b) with respect to the trivial completion of $E_\beta \restriction \nu^{\mathcal{M}} - 1$. That is, let G be the trivial completion of $E_\beta \restriction \eta$, where $\eta \le \nu^{\mathcal{M}} - 1$ is the natural length of $E_\beta \restriction \nu^{\mathcal{M}}$.

Remark. If $\eta < \nu^{\mathcal{M}} - 1$, then $G = \dot{F}^{\mathcal{M}} \restriction \nu^{\mathcal{M}} - 1$. The proof uses the fact, which is not hard to prove, that if η is a limit of generators and is not itself a generator then η is not in S.

If 5 (a) of the definition of good at α holds, we set

$$\gamma^{\mathcal{M}} = \mathrm{lh}\, G = \text{the unique } \xi \in \mathrm{dom}\ \vec{E}^{\mathcal{M}} \text{ such that } G = \dot{E}_\xi^{\mathcal{M}}\,.$$

If 5(b) holds, then $\eta \in \mathrm{dom}\ \vec{E}^{\mathcal{M}}$ and G is on the sequence of $\mathrm{Ult}(\mathcal{M}, E_\eta^{\mathcal{M}})$. We set

$$\begin{aligned} (b, g) &= \text{the first pair } (a, f) \text{ in order of} \\ &\qquad \text{construction of } \mathcal{M} \text{ such that } G = [a, f]_{E_\eta}^{\mathcal{M}}, \text{ and} \\ \gamma^{\mathcal{M}} &= (\eta, b, g)\,. \end{aligned}$$

Definition 2.0.1. Let $\mathcal{M}$ be an active ppm, and $\kappa = \mathrm{crit}\ F$, where F is the new extender introduced by $\mathcal{M}$. Then

(a) $\mathcal{M}$ is type I iff $\nu^{\mathcal{M}} = (\kappa^+)^{\mathcal{M}}$

(b) $\mathcal{M}$ is type II iff $\nu^{\mathcal{M}}$ is a successor ordinal ($\nu^{\mathcal{M}} - 1 > (\kappa^+)^{\mathcal{M}}$ follows from the definition of $\nu^{\mathcal{M}}$),

(c) $\mathcal{M}$ is type III iff $\nu^{\mathcal{M}}$ is a limit ordinal and $\nu^{\mathcal{M}} > (\kappa^+)^{\mathcal{M}}$.

The definability hierarchy we study in this section is not appropriate for active type III ppm. The reason is explained at the beginning of §3, where we study a different hierarchy appropriate for such ppm. In this section we restrict our attention to ppm which are passive or active of types I or II.

Definition 2.0.2. $\mathcal{L}$ is the language of set theory with additional constant symbols $\dot{\nu}$, $\dot{\gamma}$, $\dot{\mu}$, additional 1-ary predicate symbol $\dot{E}$, and 3-ary predicate symbol $\dot{F}$.

If $\mathcal{M}$ is a ppm, then we interpret $\mathcal{L}$ in $\mathcal{M}$ as follows: Suppose that the structure $\mathcal{M} = (J_\alpha^{\vec{E}}, \in, \vec{E} \restriction \alpha, \tilde{E}_\alpha)$ is active. Then

$$\begin{aligned} \dot{E}^{\mathcal{M}} &= \vec{E} \restriction \alpha, & \dot{F}^{\mathcal{M}} &= \tilde{E}_\alpha, \\ \dot{\nu}^{\mathcal{M}} &= \nu^{\mathcal{M}}, & \dot{\mu}^{\mathcal{M}} &= \text{crit } E_\alpha, \end{aligned}$$
$$\dot{\gamma}^{\mathcal{M}} = \begin{cases} \gamma & \text{if } \mathcal{M} \text{ is type II} \\ 0 & \text{if } \mathcal{M} \text{ is type I} \end{cases}$$

Suppose $\mathcal{M} = (J_\alpha^{\vec{E}}, \in, \vec{E} \restriction \alpha)$ is passive. Then

$$\dot{E}^{\mathcal{M}} = \vec{E} \restriction \alpha, \ \dot{F}^{\mathcal{M}} = \emptyset$$
$$\dot{\mu}^{\mathcal{M}} = \dot{\nu}^{\mathcal{M}} = \dot{\gamma}^{\mathcal{M}} = 0.$$

An active ppm is not amenable with respect to its interpretation of $\dot{F}$. For this reason, the ultrapower of such a ppm via functions belonging to the ppm may not satisfy Łos' theorem for Σ_0 formulae of $\mathcal{L}$, and we must start our Levy-like hierarchy at a smaller class of formulae.

DEFINITION 2.0.3. $r\Sigma_0$ (or restricted Σ_0) is the smallest class of $\mathcal{L}$ formulae containing the atomic formulae of $\mathcal{L}$, all Σ_0 formulae of $\mathcal{L} - \{\dot{F}\}$, and closed under $\wedge$, $\vee$, $\neg$, and bounded quantification over finite sets. [That is, if $\theta(x, G, \bar{v})$ is $r\Sigma_0$, so are $\psi(G, \bar{v})$ and $\varphi(G, \bar{v})$, where $\psi(G, \bar{v}) =$ "G is finite $\wedge$ $\exists x \in G\, \theta$" and $\varphi(G, \bar{v}) =$ "G is finite $\wedge$ $\forall x \in G\, \theta$"]

DEFINITION 2.0.4. Let $\theta(\bar{v})$ be an $\mathcal{L}$ formula; then θ is $r\Sigma_1$ iff $\theta = \exists u \varphi(u, \bar{v})$, where φ is $r\Sigma_0$.

We could continue and define $r\Sigma_n$, $r\Pi_n$ for $n > 1$ by counting quantifier alternations. However, we shall reserve "$r\Sigma_n$" and "$r\Pi_n$" for $n > 1$, for different, more useful classes of formulae.

For $\mathcal{M}$ a ppm, $r\Sigma_n^{\mathcal{M}}$ is the class of relations on the universe of $\mathcal{M}$ definable over $\mathcal{M}$ by a $r\Sigma_n$ formula ($n = 0, 1$). If $\mathcal{M}$ is passive, then $r\Sigma_1^{\mathcal{M}} = \Sigma_1^{\mathcal{M}}$, the usual Levy class. The relativised classes $r\Sigma_n^{\mathcal{M}}(X)$, for $X \subseteq |\mathcal{M}|$, and $r\Sigma_n^{\mathcal{M}} = r\Sigma_n^{\mathcal{M}}(|\mathcal{M}|)$, are as usual.

Notice that $r\Sigma_1^{\mathcal{M}}$ is closed under $\exists$, $\wedge$, $\vee$, and bounded quantification over finite sets.

The following normal form theorem makes clearer what an $r\Sigma_1$ formulae can assert of an active ppm.

Lemma 2.1. *If $\varphi(\bar{v})$ is any $r\Sigma_1$ formula then there is a Σ_1 formula $\psi(a, b, \delta, \bar{v})$ of $\mathcal{L} - \{\dot{F}\}$ such that for all active ppm $\mathcal{M}$*

$$(*) \qquad \mathcal{M} \models \forall \bar{v}\, (\varphi(\bar{v}) \Leftrightarrow \exists a, b, \delta[\dot{F}(a, b, \delta) \wedge \psi(a, b, \delta, \bar{v})]).$$

Remark. That is, an $r\Sigma_1$ formula asserts there is a "small chunk" of the new extender having a Σ_1 in $(\mathcal{L} - \{\dot{F}\})$ property.

PROOF. We show first that if $\theta(\bar{v})$ is a $r\Sigma_0$ formula and $\varphi = \theta$ or $\varphi = \neg\theta$ then there is ψ as in (*). This is by induction on θ. E.g. for $\varphi = \neg\dot{F}(x, y, z)$, we have

$$\begin{aligned}\mathcal{M} \models \forall x, y, z\big(\neg\dot{F}(x, y, z) \Leftrightarrow \exists a, b, \delta\big[\dot{F}(a, b, \delta)\wedge \\ \wedge\,(z \text{ is not an ordinal } \vee y \text{ is not a function } \vee \\ \vee \operatorname{dom} y \neq \operatorname{dom} b \vee (y = b \wedge z = \delta \wedge a \neq x))\big]\big).\end{aligned}$$

E. g. for the "$\forall x \in G$" step, let $\psi(a, b, \delta, x, G, \bar{v})$ be Σ_1 in $\mathcal{L}$ - $\{\dot{F}\}$; then we can rewrite

$$G \text{ finite } \wedge \forall x \in G\, \exists a, b, \delta[\dot{F}(a, b, \delta) \wedge \psi(a, b, \delta, x, G, \bar{v})]$$

as

$$(**) \quad \exists a, b, \delta\big[\dot{F}(a, b, \delta) \wedge \exists k < \omega\, \exists \text{ sequences } \bar{x}, \bar{a}, \bar{b}, \bar{\delta} \text{ of length } k \\ (G = \{x_1, \ldots, x_k\} \wedge \psi^*(\bar{x}, \bar{a}, \bar{\delta}, k, b))\big]$$

where ψ^* is the formula

$$\forall i \leq k\big(b_i \text{ is a function } \wedge \operatorname{dom} b_i = \operatorname{dom} b \wedge \operatorname{ran} b_i \subseteq \operatorname{ran} b \\ \wedge\, \delta_i < \delta \wedge a_i = a \cap ([\delta_i]^{<\omega} \times \operatorname{ran} b_i) \wedge \psi(a_i, b_i, \delta_i, x_i, G, \bar{v})\big).$$

Then the formula (**) is equivalent to

$$\exists a, b, \delta \exists k < \omega\, \exists \text{ sequences } \bar{x}, \bar{a}, \bar{b}, \bar{\delta} \text{ of length } k \\ \big[\dot{F}(a, b, \delta) \wedge G = \{x_1, \ldots, x_k\} \wedge \psi^*(\bar{x}, \bar{a}, \bar{\delta}, k, b)\big]$$

Thus every $r\Sigma_0$ formula φ can be put in the form given by (*). This easily implies the same is true of $r\Sigma_1$ φ. □

As a corollary, $r\Sigma_1$ satisfaction is $r\Sigma_1$, uniformly over all active ppm:

Corollary 2.2. *There is an $r\Sigma_1$ formula $\theta(v_0, v_1)$ such that whenever $\mathcal{M}$ is an active ppm*

$$\mathcal{M} \models \theta[i, \langle a_0 \cdots a_k\rangle] \quad \textit{iff } i \textit{ is the Gödel number of an } r\Sigma_1 \textit{ formula } \varphi \\ \textit{and } \mathcal{M} \models \varphi[a_0 \cdots a_k].$$

Of course, $r\Sigma_1 = \Sigma_1$ satisfaction over passive ppm is also uniformly $r\Sigma_1$.

Henceforth we will use the same letter for the Gödel number of a formula as we use for the formula, so that the displayed line in the corollary would start "$\mathcal{M} \models \theta[\varphi, \langle a_0 \cdots a_k\rangle]$."

Lemmas 2.1 and 2.2 hold also for type III active ppm, setting $\dot{\gamma}^{\mathcal{M}} = 0$ for $\mathcal{M}$ type III, but once again we have no use for this fact.

Remark. Let $\mathcal{M} = \mathcal{J}_\alpha^{\vec{E}}$ be active. We can construct an amenable structure $\mathcal{M}^*$ as follows. Define E_α^* to be the set of quadruples (γ, ξ, a, x) such that

$$(\alpha > \gamma > \nu^{\mathcal{M}}) \wedge (\dot{\mu}^{\mathcal{M}} < \xi < \dot{\mu}^{+\mathcal{M}}) \wedge \\ \wedge \left(E_\alpha \cap ([\nu^{\mathcal{M}}]^{<\omega} \times J_\xi^{\vec{E}}) \in J_\gamma^{\vec{E}}\right) \wedge \left((a, x) \in (E_\alpha \cap ([\gamma]^{<\omega} \times J_\xi^{\vec{E}}))\right).$$

Our remark on p. 10 shows that $\mathcal{M}^*$ is amenable; moreover one can easily see that $r\Sigma_1^{\mathcal{M}}$ is the usual Σ_1 over $\mathcal{M}^*$. This fact (which we didn't notice until we had written much of these notes) doesn't seem to simplify the definability analysis to follow. It might be used to give the analysis a more conventional look.

For any ppm $\mathcal{M}$, let $<^{\mathcal{M}}$ be the usual order of construction (so if $\mathcal{M} = \mathcal{J}_\alpha^{\vec{E}}$ and $\mathcal{N} = \mathcal{J}_\beta^{\vec{E}}$ where $\alpha < \beta$, then $<^{\mathcal{N}}$ end-extends $<^{\mathcal{M}}$).

Lemma 2.3. *There are Σ_1 formulae of $\mathcal{L} - \{\dot{F}\}$ defining uniformly over all ppm $\mathcal{J}_\alpha^{\vec{E}}$ the functions*

$$\beta \mapsto \mathcal{J}_\beta^{\vec{E}}, \quad \beta \mapsto <^{\mathcal{J}_\beta^{\vec{E}}}.$$

PROOF. See Dodd-Jensen [DJ1]. □

The formulae of the lemma are in $\mathcal{L} - \{\dot{F}\}$ hence $r\Sigma_1$. To apply [DJ1], notice every ppm is amenable with respect to its interpretation of $\dot{E}$ (and in fact $\dot{E}^{\mathcal{M}} \in \mathcal{M}$ if $\mathcal{M} = \mathcal{J}_{\alpha+1}^{\vec{E}}$).

Skolem terms and projecta.

Following Magidor and Silver, we shall introduce Skolem terms into our language as a convenience. The existence of the amenable structure $\mathcal{M}^*$, which we described following 2.2, shows that $r\Sigma_1^{\mathcal{M}}$ relations admit $r\Sigma_1$ uniformizations, and this means that we could avoid the Skolem terms if we cared to do so. There seems to be no great advantage to either approach. We shall also define the classes $r\Sigma_n^{\mathcal{M}}$ for $n > 1$, and introduce a predicate $\dot{T}_n$ related to $r\Sigma_n$ satisfaction. Like Magidor and Silver, we shall work directly with the classes $r\Sigma_n^{\mathcal{M}}$, rather than with Σ_1 definability over master code structures. However, $\dot{T}_n^{\mathcal{M}}$ is closely related to the nth master code of M, and its use in constructing $r\Sigma_{n+1}^{\mathcal{M}}$ relations makes this difference more apparent than real.

DEFINITION 2.3.1. $\mathcal{L}^+$ is $\mathcal{L}$ together with additional 2-ary predicate symbols $\dot{T}_n$ for $1 \leq n < \omega$.

The interpretation $\dot{T}_n^{\mathcal{M}}$, for $\mathcal{M}$ a ppm, will be defined shortly.

DEFINITION 2.3.2. Let $\theta(\bar{v})$ be a formula of $\mathcal{L}^+$.

(a) θ is $r\Sigma_1$ iff θ is a formula of $\mathcal{L}$ which is $r\Sigma_1$ in our former sense.

(b) θ is $r\Sigma_{n+1}$ (where $n \geq 1$) iff there is a $r\Sigma_1$ formula $\psi(a, b, \bar{v})$ of $\mathcal{L}$ such that

$$\theta = \exists a \exists b (\dot{T}_n(a, b) \wedge \psi(a, b, \bar{v}))\,.$$

DEFINITION 2.3.3. Let $\varphi = \varphi(v_0, \ldots, v_k, v_{k+1})$ be a formula of $\mathcal{L}^+$. Then $\tau_\varphi(v_0, \ldots v_k)$ is the *basic Skolem term* associated to φ.

Given that we have interpreted φ in a ppm $\mathcal{M}$ (which we have not as yet done in general), we interpret τ_φ as follows:

$$\tau_\varphi^{\mathcal{M}}(a_0 \cdots a_k) = \begin{cases} <^{\mathcal{M}} \text{ least } b \text{ such that } \mathcal{M} \models \varphi[\bar{a}, b] & \text{if such } b \text{ exists} \\ 0 & \text{otherwise} \end{cases}$$

DEFINITION 2.3.4. For $n \geq 1$, Sk_n (the class of level n Skolem terms) is the smallest class which contains τ_φ for each $r\Sigma_n$ formula φ and is closed under composition.

DEFINITION 2.3.5. A formula φ of $\mathcal{L}^+$ is *generalized* $r\Sigma_n$ for $n \geq 1$ iff φ results from an $r\Sigma_n$ formula ψ by substituting terms in Sk_n for free variables in ψ.

(The substitution of τ into ψ must be such that no variable free in τ becomes bound in the resulting φ.)

We can now define the predicate $\dot{T}_n^{\mathcal{M}}$ for $\mathcal{M}$ a ppm; simultaneously, we define the nth projectum $\rho_n^{\mathcal{M}}$ of $\mathcal{M}$.

DEFINITION 2.3.6. Let $\mathcal{M}$ be a ppm and $n \geq 1$. Then

(a) $\mathrm{Th}_n^{\mathcal{M}}(X) = \{\langle \varphi, \bar{a} \rangle \mid \bar{a} \in X^{<\omega}$ and φ is a generalized $r\Sigma_n$ formula and $\mathcal{M} \models \varphi[\bar{a}]\}$,

(b) $\rho_n^{\mathcal{M}}$ is the least ordinal $\rho \subseteq |\mathcal{M}|$ such that $\mathrm{Th}_n^{\mathcal{M}}(\rho \cup \{q\}) \notin |\mathcal{M}|$ for some $q \in |\mathcal{M}|$,

(c) The predicate $\dot{T}_n^{\mathcal{M}}(a, b)$ holds if and only if $a = \langle \alpha, q \rangle$ for some $\alpha < \rho_n^{\mathcal{M}}$ and $q \in \mathcal{M}$, and $b = \mathrm{Th}_n^{\mathcal{M}}(\alpha \cup \{q\})$.

Of course, $\dot{T}_n$ is essentially 3-ary. It's present form is a relic of an earlier version of these notes, where we set $\dot{T}_n^{\mathcal{M}}(a, b) \Leftrightarrow b = \mathrm{Th}_n^{\mathcal{M}}(a)$. That earlier version led to a problem (showing $r\Sigma_n$ ultrapowers give rise to $r\Sigma_{n+1}$ elementary embeddings).

Remark. $\rho_n^{\mathcal{M}} = \mathrm{OR}^{\mathcal{M}}$ is possible.

The definition above is by induction on n, as (a) for n depends upon (c) for $1 \leq m < n$.

We define the classes $r\Sigma_n^{\mathcal{M}}$, $r\Pi_n^{\mathcal{M}}$, $r\Delta_n^{\mathcal{M}}$, for $\mathcal{M}$ a ppm, in the obvious way. The relativised and boldface versions of these classes are also defined in the obvious way.

Notice that if $\rho_n^{\mathcal{M}} \leq \omega$, then in fact $\rho_n^{\mathcal{M}} = 0$, and $r\Sigma_{n+1}^{\mathcal{M}}$ trivializes. We are not interested in $r\Sigma_{n+1}^{\mathcal{M}}$ when $\rho_n^{\mathcal{M}} = 0$ (although we are definitely interested in $r\Sigma_n^{\mathcal{M}}$ in this case). We shall tacitly assume henceforth that, in any discussion of a class of the form $r\Sigma_{n+1}^{\mathcal{M}}$, $\rho_n^{\mathcal{M}} > 0$.

It is easy to see that $r\Sigma_n^{\mathcal{M}}$ is closed under $\exists$, $\wedge$, and $\vee$, and that $(r\Sigma_n^{\mathcal{M}} \cup r\Pi_n^{\mathcal{M}}) \subseteq r\Sigma_{n+1}^{\mathcal{M}}$, for any ppm $\mathcal{M}$. Moreover, the closure and inclusion are uniform over all ppm (there is a recursive translation procedure acting on formulae of the appropriate type). It is clear that $\neg\dot{T}_n^{\mathcal{M}}$ is $r\Sigma_{n+1}^{\mathcal{M}}$ in the parameter $\rho_n^{\mathcal{M}}$, uniformly over all $\mathcal{M}$.

It follows that the class of sets definable by $r\Sigma_{n+1}$ formulae would be unchanged if we modified definition 2.3.3 by allowing allowed any formula of the form $\exists a, b(\dot{T}_n(a,b) \wedge \psi(a,b,\bar{v}))$ where ψ is a boolean combination of $r\Sigma_n$ formulae. A similar argument shows that we could also have restricted ψ to be Σ_1 in $\mathcal{L} - \{\dot{F}, \dot{E}\}$.

Hulls.

DEFINITION 2.3.7. Let $\mathcal{M}$ be a ppm, $n \geq 1$, and $X \subseteq |\mathcal{M}|$. Then

$$S_n^{\mathcal{M}}(X) = \{\tau^{\mathcal{M}}(\bar{a}) \mid \tau \in \mathrm{Sk}_n \wedge \bar{a} \in X^{<\omega}\},$$
$$H_n^{\mathcal{M}}(X) = \pi'' S_n^{\mathcal{M}}(X) \quad \text{where } \pi \text{ is the transitive collapse,}$$
$$\mathcal{H}_n^{\mathcal{M}}(X) = (H_n^{\mathcal{M}}(X), \in, \pi''(\dot{E}^{\mathcal{M}}), \pi''(\dot{F}^{\mathcal{M}})).$$

(The last predicate occurs only if $\mathcal{M}$ is active).

We shall show $\mathcal{H}_n^{\mathcal{M}}(X)$ is a ppm. To this end, recall the Q formulae of [DJ4]. One virtue of these formulae is that they go down under Σ_1 embeddings and up under cofinal Σ_0 embeddings. We now define the appropriate analog in our situation.

DEFINITION 2.3.8. Let $\mathcal{M}$ be a ppm, and $\pi : \mathcal{M} \to \mathcal{P}$ be an $r\Sigma_0$ embedding, with $\mathcal{P}$ transitive. We say π is *cofinal* iff

(a) $\forall\, y \in |\mathcal{P}|\, \exists x\ (y \subseteq \pi(x))$, and

(b) $\pi''(\dot{\mu}^+)^{\mathcal{M}}$ is cofinal in $\pi((\dot{\mu}^+)^{\mathcal{M}})$.

Recall here $\dot{\mu}^{\mathcal{M}} = 0$ if $\mathcal{M}$ is passive, so that (b) is trivially true then. If $\mathcal{M}$ is active, $\dot{\mu}^{\mathcal{M}} = \mathrm{crit}\ \dot{F}^{\mathcal{M}}$.

DEFINITION 2.3.9. An rQ formula is one of the form:

$$\forall x \forall \theta < \dot{\mu}^+\ \exists y \exists \nu\ (x \subseteq y \wedge (\theta \leq \nu < \dot{\mu}^+) \wedge \varphi(y, \nu, \bar{u}))$$

where φ is $r\Sigma_1$ and does not have x or θ free.

Interpreted in a ppm $\mathcal{M}$, an rQ formula asserts that, in the product order on $(\dot{\mu}^+)^{\mathcal{M}} \times |\mathcal{M}|$ determined by the inclusion order on the factors, there are cofinally many pairs (ν, y) with an $r\Sigma_1^{\mathcal{M}}$ property. (If $\mathcal{M}$ is passive, this reduces to asserting that, under inclusion, there are cofinally many $y \in |\mathcal{M}|$ with an $r\Sigma_1^{\mathcal{M}}$ property.) So we have clearly

Lemma 2.4. *Let $\varphi(\bar{v})$ be an rQ formula, and $\pi : \mathcal{M} \to \mathcal{P}$. Then*

(a) *If π is an $r\Sigma_1$ embedding and $\mathcal{P} \models \varphi[\pi(\bar{a})]$, then $\mathcal{M} \models \varphi[\bar{a}]$.*
(b) *If π is a cofinal $r\Sigma_0$ embedding and $\mathcal{M} \models \varphi[\bar{a}]$, then $\mathcal{P} \models \varphi[\pi(\bar{a})]$.*

The preservation properties given in 2.4 are interesting because one can say with an rQ sentence: "I am a (passive/active) ppm".

Lemma 2.5. *There are rQ sentences φ_1, φ_2, φ_3 such that if $\mathcal{M}$ is a transitive $\mathcal{L}$-structure, then*

(a) *$\mathcal{M} \models \varphi_1$ holds if and only if $\mathcal{M}$ is passive and $\langle \dot{\mu}^{\mathcal{M}}, \dot{\nu}^{\mathcal{M}}, \dot{\gamma}^{\mathcal{M}} \rangle = \langle \mu^{\mathcal{M}}, \nu^{\mathcal{M}}, \gamma^{\mathcal{M}} \rangle$;*
(b) *$\mathcal{M} \models \varphi_2$ holds if and only if $\mathcal{M}$ is active type* I *and $\langle \dot{\mu}^{\mathcal{M}}, \dot{\nu}^{\mathcal{M}}, \dot{\gamma}^{\mathcal{M}} \rangle = \langle \mu^{\mathcal{M}}, \nu^{\mathcal{M}}, \gamma^{\mathcal{M}} \rangle$;*
(c) *$\mathcal{M} \models \varphi_3$ holds if and only if $\mathcal{M}$ is active type* II *and $\langle \dot{\mu}^{\mathcal{M}}, \dot{\nu}^{\mathcal{M}}, \dot{\gamma}^{\mathcal{M}} \rangle = \langle \mu^{\mathcal{M}}, \nu^{\mathcal{M}}, \gamma^{\mathcal{M}} \rangle$.*

PROOF. We construct φ_3, the other constructions being slightly simpler.

We shall use the fact that every $r\Pi_1$ formula can be put in rQ form; we leave it to the reader to check this.

By Dodd-Jensen [DJ4] there is an rQ sentence θ_1 of $\mathcal{L} - \{\dot{F}\}$ whose transitive models $\mathcal{M}$ are precisely those of the form $(J_\alpha^{\dot{E}^{\mathcal{M}}}, \in, \dot{E}^{\mathcal{M}}, \dot{F}^{\mathcal{M}})$. We add to θ_1 the $r\Pi_1$ sentence of $\mathcal{L} - \{\dot{F}\}$ stating that $\mathcal{M}$ is strongly acceptable and that $P(\dot{\mu})^{\mathcal{M}} \subseteq (J_{\dot{\nu}})^{\mathcal{M}}$.

Now we define a rQ sentence θ_2 asserting that $\dot{F}^{\mathcal{M}}$ codes a pre-extender over $\mathcal{M}$. The pre-extender coded is $\bigcup\{a \mid \exists b, \delta\, \dot{F}^{\mathcal{M}}(a, b, \delta)\}$. The formula θ_2 is the conjunction of the formulas (i–vi) below:

(i) "there are cofinally many (θ, γ) in the product order on $\dot{\mu}^+ \times \mathrm{OR}$ such that $\exists a\, \exists b(\dot{F}(a, b, \gamma) \wedge b$ is a function from $\dot{\mu}$ onto $P(\dot{\mu}) \cap J_\theta^{\dot{E}})$."
(ii) $\forall a, b, \gamma, a', b', \gamma'$ (if $\dot{F}(a, b, \gamma) \wedge \gamma' \leq \gamma \wedge b'$ is a function with dom $b' = \dot{\mu}$ and ran $b' \subseteq$ ran $b \wedge a' = a \cap ([\gamma']^{<\omega} \times \operatorname{ran} b')$, then $\dot{F}(a', b', \gamma')$).
(iii) $\dot{F}(a, b, \gamma) \Rightarrow \gamma \in \mathrm{OR}$ and $b : \dot{\mu} \to P(\dot{\mu})$ and $a \subseteq [\gamma]^{<\omega} \times \operatorname{ran} b$ and, letting $E_c = \{x \mid (c, x) \in a\}$, the E_c's are compatible $\dot{\mu}$-complete measures on $[\dot{\mu}]^{\operatorname{card} c}$ "as far as sets in ran b are concerned".
(iv) $\dot{F}(a, b, \gamma) \wedge \dot{F}(a', b, \gamma) \Rightarrow a = a'$.
(v) (Normality) $(\forall f : [\dot{\mu}]^n \to \dot{\mu})(\forall b : \dot{\mu} \to (P([\dot{\mu}]^n) \cup P([\dot{\mu}]^{n+1})))$ [if b is f-closed then $\forall a, \delta(\dot{F}(a, b, \delta) \Rightarrow a = \langle E_c \mid c \in [\delta]^{<\omega} \rangle$ is normal with

respect to f)] (where "b is f-closed" stands for the formula: $\forall A \in \operatorname{ran} b \cap P([\dot\mu]^n) \forall i < n \{\langle \alpha_1 \ldots \alpha_i, \beta, \alpha_{i+1} \ldots \alpha_n\rangle \mid \langle \alpha_1 \ldots \alpha_n\rangle \in A \wedge \beta = f(\alpha_1 \ldots \alpha_n)\} \in \operatorname{ran} b$).

So far, condition (i) is rQ while (ii)-(v) are actually $r\Pi_1$, and we have asserted enough to ensure that $\mathrm{Ult}(\mathcal{M}, F)$ makes sense whenever $\mathcal{M} \models$ (i)-(v), where $F = \bigcup\{a \mid \dot F^{\mathcal{M}}(a, b, \delta)$ for some $b, \delta\}$. Normality guarantees $\mathrm{OR}^{\mathcal{M}} \subseteq \omega fp(\mathrm{Ult})$, but we must have $\mathrm{OR}^{\mathcal{M}} \in \mathrm{wfp}(\mathrm{Ult})$ for pre-extenderhood. From condition 3 of goodness at α ($\alpha = \mathrm{OR}^{\mathcal{M}}$), we know that we want to assert that $[\{\dot\nu\}^{\mathcal{M}}, f]_F^{\mathcal{M}} = \alpha$ where $f(\beta) = (\beta^+)^{\mathcal{M}}$ for $\beta < \dot\mu^{\mathcal{M}}$. The next clauses in θ_2 do this.

(vi) $\forall$ ordinals $\delta > \dot\nu\ \forall \gamma > \delta\ \forall a, b$ (if $\dot F(a, b, \gamma)$ and $\{(\alpha, \beta) \mid J_{\dot\mu}^{\dot E} \models \operatorname{card} \beta \leq \alpha\} = x$ is in $\operatorname{ran} b$, then $(\{\dot\nu, \delta\}, x) \in a$).

We have to say finally that there is no function "between" $f(\beta) = \beta^+$ on the $\dot\nu$th coordinate and the projection functions on arbitrary coordinates.

(vii) For cofinally many pairs (θ, γ) in the product order on $\dot\mu^+ \times \mathrm{OR}$ there are a, b and δ such that

$$\dot F(a, b, \delta) \wedge \delta > \gamma \wedge \forall n < \omega \left(P([\dot\mu]^n) \cap J_\theta^{\dot E} \subseteq \operatorname{ran} b\right)$$

and for all functions $f \in J_\theta^{\dot E}$ such that $f : [\dot\mu]^n \to \dot\mu$, and for all $c \in [\gamma]^{<\omega}$ such that $c = \{\eta_1 \cdots \eta_n\}$ for some ordinals $\eta_1 < \cdots < \eta_n$ with $\eta_i = \dot\nu$, and

$$(c, \{(\alpha_1 \ldots \alpha_n) \mid f(\alpha_1 \ldots \alpha_n) < (\alpha_i^+)^{J_{\dot\mu}^{\dot E}}\}) \in a$$

there is an ordinal ξ such that $\gamma \leq \xi < \delta$ and

$$(c \cup \{\xi\}, \{(\alpha_1 \cdots \alpha_{n+1}) \mid f(\alpha_1 \cdots \alpha_n) < \alpha_{n+1}\}) \in a\,.$$

The formula in (vii) is rQ. To see that if $\mathcal{M}$ satisfies (i–vii) then $f(\beta) = (\beta^+)^{\mathcal{M}}$, on the $\dot\nu^{\mathcal{M}}$ coordinate, represents $\mathrm{OR}^{\mathcal{M}}$ in Ult, notice that as $\dot\mu^{\mathcal{M}}$ is a cardinal of $\mathcal{M}$, strong acceptability implies $((\alpha_i^+)^{J_{\dot\mu}^{\dot E}})^{\mathcal{M}} = (\alpha_i^+)^{\mathcal{M}}$ for $\alpha_i < \dot\mu^{\mathcal{M}}$. We leave to the reader the not entirely trivial fact that any active ppm satisfies (vii).

Let θ_2 be the conjunction of (i)-(vii). If $\mathcal{M}$ satisfies $\theta_1 \wedge \theta_2$, then $\mathcal{M}$ satisfies conditions 1, 2, and part of 3 of good at α, for $\alpha = \mathrm{OR}^{\mathcal{M}}$. We capture the rest of condition 3 with θ_3:

θ_3: There are cofinally many $\gamma \in \mathrm{OR}$ such that $\exists a, b, \delta(\dot F(a, b, \delta) \wedge \delta > \gamma \wedge \exists f : [\dot\mu]^n \to \dot\mu\ \exists c \in [\dot\nu]^n$ such that $\dot\nu - 1 \in c$ and

$$\left(c \cup \{\gamma\}, \{(\alpha_1 \ldots \alpha_n, \beta) \mid f(\alpha_1 \ldots \alpha_n) = \beta \wedge J_{\dot\mu}^{\dot E} \models \operatorname{card}(\beta) \leq \alpha_n\}\right) \in a\,.$$

Moreover, $\dot\nu - 1$ is a generator of $\dot F$; that is $\forall a,\ b,\ \delta\ \forall f : [\dot\mu]^n \to \dot\mu$

$$\forall c \subseteq \dot\nu - 1 \left(c \cup \{\dot\nu - 1\},\ \{(\alpha_1 \ldots \alpha_n, \beta) \mid f(\alpha_1 \ldots \alpha_n) = \beta\}\right) \notin a\,.$$

The formula θ_3 is the conjunction of an $r Q$ sentence and an $r\Pi_1$ sentence, so θ_3 in rQ. One can check that if $\mathcal{M} \models \theta_3$, $\dot{\nu}^{\mathcal{M}} - 1$ is the largest generator of $\dot{F}^{\mathcal{M}}$. Notice here that if $\gamma \geq \dot{\nu}$ satisfies the displayed clause of θ_3, then there are no generators between γ and $\dot{\nu}$.

Recall that we are working with a type II ppm $\mathcal{M}$, so that $\dot{\nu}^{\mathcal{M}} - 1$ exists.

We can capture coherence, which is condition 4 of good at α, with an $r\Pi_1$ sentence θ_4: θ_4 just says $\forall a, b, \delta\ (\dot{F}(a, b, \delta) \Rightarrow$ "a is coherent as far as sets in b go"). We omit further detail.

Condition 5 is a disjunction of two possibilities, (a) and (b), and we accordingly set $\theta_5 = \psi_1 \vee \psi_2$. The formula ψ_1, asserting that clause 5a holds, is "$\dot{\gamma} \geq \dot{\nu} - 1$ and $\dot{\gamma} \in \operatorname{dom} \dot{E}$ and $\forall a, b(\dot{F}(a, b, \dot{\nu} - 1) \Rightarrow a \subseteq \dot{E}_{\dot{\gamma}})$ and $\forall \xi < \dot{\gamma}$ (ξ a generator of $\dot{E}_{\dot{\gamma}} \Rightarrow \xi < \dot{\nu} - 1$)." The formula ψ_1 is $r\Pi_1$ (its third conjunct is the only one not Σ_0 in $\mathcal{L} - \{\dot{F}\}$).

The formula ψ_2, asserting that clause 5b holds, says that $\dot{\gamma} = (\eta, b, g)$, where if we set $G = [b, g]_{\dot{E}_\eta}$ then η is the natural length of G and is in $\operatorname{dom}(\dot{E})$, the conjunction of the following three formulas holds:

$$\forall a, b(\dot{F}(a, b, \dot{\nu} - 1) \Rightarrow a \subseteq G)$$

$$g(\bar{u}) \text{ is on } \dot{E} \text{ for } (\dot{E}_\eta)_b \text{ a.e. } \bar{u}$$

$$\forall \xi < \operatorname{lh} G(\xi \text{ a generator of } G \Rightarrow \xi < \dot{\nu} - 1)$$

and finally $G \neq [a, f]_{\dot{E}_\eta}$ whenever (a, f) is constructed before (b, g). We leave it to the reader to see that the formula ψ_2 is also $r\Pi_1$.

The formula $\theta_5 = \psi_1 \vee \psi_2$ captures (5) for the "last" proper initial segment of $\dot{F}^{\mathcal{M}}$. Together with the Π_0 in $\mathcal{L} - \{\dot{F}\}$ assertion that $\dot{E}^{\mathcal{M}}$ is good at all $\beta < \alpha$, θ_5 captures (5).

Let φ be the Π_1 assertion that $\dot{E}^{\mathcal{M}}$ is good at all $\beta < \mathrm{OR}^{\mathcal{M}}$. Then $\varphi \wedge \bigwedge_{i \leq 5} \theta_i$ is the desired rQ sentence. This completes the proof of 2.5. □

Corollary 2.6. *Let $\mathcal{M}$ be a ppm which is passive or active of types* I *or* II. *Then*

(a) *if $\pi : \mathcal{H} \to \mathcal{M}$ is an $r\Sigma_1$ embedding, then $\mathcal{H}$ is a ppm of the same type as $\mathcal{M}$ and $\pi(\mu^{\mathcal{H}}) = \mu^{\mathcal{M}}$, $\pi(\nu^{\mathcal{H}}) = \nu^{\mathcal{M}}$, and $\pi(\gamma^{\mathcal{H}}) = \gamma^{\mathcal{M}}$,*

(b) *if $\pi : \mathcal{M} \to \mathcal{P}$ is either a cofinal $r\Sigma_0$ embedding or a "Σ_1 over $r\Sigma_1$" embedding (see the proof of 2.7 for the definition) then $\mathcal{P}$ is a ppm of the same type as $\mathcal{M}$, and $\pi(\mu^{\mathcal{M}}) = \mu^{\mathcal{P}}$, $\pi(\nu^{\mathcal{M}}) = \nu^{\mathcal{P}}$, and $\pi(\gamma^{\mathcal{M}}) = \gamma^{\mathcal{P}}$.*

The natural embedding $\pi : \mathcal{H}_n^{\mathcal{M}}(X) \to \mathcal{M}$ is clearly $r\Sigma_1$, so it follows that $\mathcal{H}_n^{\mathcal{M}}(X)$ is a ppm of the same type as $\mathcal{M}$. The next lemma shows that in certain circumstances π in fact preserves generalized $r\Sigma_n$ formulae.

Lemma 2.7. *Let $\mathcal{M}$ be a ppm which is passive or active type* I *or* II. *Let $\mathcal{H} = \mathcal{H}_n^{\mathcal{M}}(X)$, where $X \subseteq |\mathcal{M}|$ and $n \geq 1$. Suppose that if $n \geq 2$, then*

$$\rho_{n-1}^{\mathcal{M}} < \mathrm{OR}^{\mathcal{M}} \Rightarrow \exists q \in X\,(\mathrm{Th}_{n-1}^{\mathcal{M}}(\rho_{n-1}^{\mathcal{M}} \cup \{q\}) \notin |\mathcal{M}|)$$

and if $n \geq 3$, then

$$\rho_{n-2}^{\mathcal{M}} < OR^{\mathcal{M}} \Rightarrow \rho_{n-2}^{\mathcal{M}} \in X \wedge \exists q \in X\,(\mathrm{Th}_{n-2}^{\mathcal{M}}(\rho_{n-2}^{\mathcal{M}} \cup \{q\}) \notin |\mathcal{M}|)\,.$$

Let $\pi : \mathcal{H} \to \mathcal{M}$ be the inverse of the collapse. Then

(a) *$\mathcal{H} \models \varphi[\bar{a}]$ iff $\mathcal{M} \models \varphi[\pi(\bar{a})]$ for φ generalized $r\Sigma_n$ and $\bar{a} \in H$*

(b) *for $1 \leq i \leq n-2$*

$$\rho_i^{\mathcal{M}} = \begin{cases} \mathrm{OR}^{\mathcal{M}} & \text{if } \rho_i^{\mathcal{H}} = OR^{\mathcal{H}} \\ \pi(\rho_i^{\mathcal{H}}) < OR^{\mathcal{M}} & \text{if } \rho_i^{\mathcal{H}} < \mathrm{OR}^{\mathcal{H}} \end{cases}$$

(c) $$\rho_{n-1}^{\mathcal{H}} = \begin{cases} \text{the least } \alpha \text{ such that } \pi(\alpha) \geq \rho_{n-1}^{\mathcal{M}} \\ OR^{\mathcal{H}} \text{ if no such } \alpha \text{ exists.} \end{cases}$$

PROOF. For $i \geq 0$ and $k \geq 1$, we say a formula φ is Σ_k over (generalized) $r\Sigma_i$ iff

$$\varphi = \exists v_1 \forall v_2 \exists v_3 \cdots Q_k v_k \psi$$

where ψ is a Boolean combination of (generalized) $r\Sigma_i$ formulae and $Q_k = \exists$ or $\forall$ as appropriate. (Here generalized $r\Sigma_0 = r\Sigma_0$.)

We show by induction on i that for $0 \leq i \leq n-1$

(i) If φ is Σ_{n-i} over generalized $r\Sigma_i$, then for all $\bar{a} \in H$

$$\mathcal{H} \models \varphi[\bar{a}] \Leftrightarrow \mathcal{M} \models \varphi[\pi(\bar{a})]\,,$$

(ii) for $1 \leq i \leq n-1$ and $a, b \in H$

$$\mathrm{Th}_i^{\mathcal{H}}(a) = b \text{ iff } \mathrm{Th}_i^{\mathcal{M}}(\pi(a)) = \pi(b)\,,$$

(iii) for $1 \leq i \leq n-2$, (b) of the statement of the lemma holds, and for $i = n-1$, (c) holds

(iv) if φ is generalized $r\Sigma_{i+1}$, then for all $\bar{a} \in H$

$$\mathcal{H} \models \varphi[\bar{a}] \Leftrightarrow \mathcal{M} \models \varphi[\pi(\bar{a})]\,.$$

Proof of (i). If θ is Σ_{n-i} over generalized $r\Sigma_i$, then the translation procedure mentioned earlier gives us an $r\Sigma_n$ formula θ^* which is equivalent to θ over all ppm. As $\mathrm{ran}\,\pi$ is closed under $\tau_{\theta^*}^{\mathcal{M}}$, we see that if $\exists x(\mathcal{M} \models \theta[x, \pi(b)])$, then $\exists x \in \operatorname{ran}\pi(\mathcal{M} \models \theta[x, \pi(b)])$. But now π is elementary with respect to all generalized $r\Sigma_i$ formulae (trivially if $i = 0$, or by the induction hypothesis (iv) if $i > 0$). So the usual induction on the length of the quantifier prefix in φ gives (i).

Proof of (ii). First we observe that for any $i \geq 1$ there is a Π_1 over $r\Sigma_i$ formula $\sigma(v_0, v_1)$ such that for any ppm $\mathcal{P}$

$$\mathcal{P} \models \sigma[a, b] \Leftrightarrow \mathrm{Th}_i^{\mathcal{P}}(a) = b\,.$$

To see this, notice first that there is a recursive function associating to each term $\tau \in \mathrm{Sk}_i$ a Σ_1 over $r\Sigma_i$ formula θ_τ such that $\tau^{\mathcal{P}}[\bar{a}] = b$ iff $\mathcal{P} \models \theta_\tau[\bar{a}, b]$, for all ppm $\mathcal{P}$. For basic τ, say $\tau = \tau_\varphi$, let $\theta_\tau(\bar{u}, v)$ be the formula

$$\big(\varphi(\bar{u}, v) \wedge \forall w < v\ \neg\varphi(\bar{u}, w)\big) \vee \big(v = 0 \wedge \forall w\ \neg\varphi(\bar{u}, w)\big)\,.$$

In this case θ_τ is a Boolean combination of $r\Sigma_i$ formulae. The extension of $\tau \mapsto \theta\tau$ to all of Sk_i is obvious. Notice second that $r\Sigma_i$ satisfaction is uniformly $r\Sigma_i$ over all ppm.

It then follows that generalized $r\Sigma_i$ satisfaction is uniformly Σ_1 over $r\Sigma_i$, as well as uniformly Π_1 over $r\Sigma_i$, over all ppm. This gives us the desired formula σ.

Clause (ii) follows easily from (i) and the existence of σ.

proof of (iii). We first prove clause (b) for $i \leq n-3$. Consider for example the first equivalence. The statement "$\rho_i^{\mathcal{M}} = \mathrm{OR}^{\mathcal{M}}$" can be expressed

$$\mathcal{M} \models \forall\alpha \in \mathrm{OR}\ \forall q \exists b\, \sigma(\alpha \cup \{q\}, b)$$

where σ is the formula asserting that $b = \mathrm{Th}_i^{\mathcal{P}}(a)$ from part (ii). This sentence is Π_3 over $r\Sigma_i$, so true in $\mathcal{M}$ iff true in $\mathcal{H}$ as $i \leq n-3$ and we have induction hypothesis (i) at i. A similar calculation gives the second equivalence.

Clause (b) for $i = n-2$ comes from a similar calculation. If $\rho_i^{\mathcal{M}} = \mathrm{OR}^{\mathcal{M}}$ then as we have just seen this is expressible by a Π_3 over $r\Sigma_i$ sentence which, since true in $\mathcal{M}$, must go down to $\mathcal{H}$ by induction hypothesis (i). If $\rho_i^{\mathcal{M}} < \mathrm{OR}^{\mathcal{M}}$, then by hypothesis $\rho_i^{\mathcal{M}}$ and a suitable parameter p are in $\operatorname{ran}(\pi)$. We get $\mathcal{M} \models \forall b \neg\sigma(\rho_i^{\mathcal{M}} \cup \{p\}, b)$, which is Π_2 over $r\Sigma_i$ and thus goes down to $\mathcal{H}$, showing $\rho_i^{\mathcal{H}} \leq \pi^{-1}(\rho_i^{\mathcal{M}})$. The second implication comes from a similar calculation.

Finally, in the case $i = n-1$ we must prove (c).

Let $\pi(\alpha) < \rho_{n-1}^{\mathcal{M}}$; we claim $\alpha < \rho_{n-1}^{\mathcal{H}}$. For let $q \in |\mathcal{H}|$. Then

$$\begin{aligned}\mathrm{Th}_{n-1}^{\mathcal{M}}(\pi(\alpha) \cup \{\pi(q)\}) = \text{unique } c \text{ such that } &\exists a, b(\dot{T}_{n-1}^{\mathcal{M}}(a, b) \wedge \\ &a = \pi(\alpha) \cup \{\pi(q)\} \wedge b = c)\end{aligned}$$

so we can find $b \in |\mathcal{H}|$ such that

$$\mathrm{Th}_{n-1}^{\mathcal{M}}(\pi(\alpha) \cup \{\pi(q)\}) = \pi(b)\,.$$

But then $\mathrm{Th}^{\mathcal{H}}_{n-1}(\alpha \cup \{q\}) = b$ by (ii), so $\mathrm{Th}^{\mathcal{H}}_{n-1}(\alpha \cup \{q\}) \in |\mathcal{H}|$, and as q was arbitrary, $\alpha < \rho^{\mathcal{H}}_{i-1}$.

On the other hand, if $\pi(\alpha) \geq \rho^{\mathcal{M}}_{i-1}$, then by hypothesis we have a $p \in \operatorname{ran} \pi$ such that

$$\mathrm{Th}^{\mathcal{M}}_{n-1}(\rho^{\mathcal{M}}_{n-1} \cup \{p\}) \notin |\mathcal{M}| .$$

Let $\pi(q) = p$. Then $\mathrm{Th}^{\mathcal{H}}_{n-1}(\alpha \cup \{q\}) \notin |\mathcal{H}|$, so $\alpha \geq \rho^{\mathcal{H}}_{n-1}$.

Finally, we prove (iv) at i. Notice first that

$$\dot{T}^{\mathcal{H}}_i(a, b) \quad \text{iff} \quad \dot{T}^{\mathcal{M}}_i(\pi(a), \pi(b)) .$$

For suppose $\dot{T}^{\mathcal{H}}_i(a, b)$. Then $a = \langle \alpha, q \rangle$ where $\alpha < \rho^{\mathcal{H}}_i$, and $b = \mathrm{Th}^{\mathcal{H}}_i(\alpha \cup \{q\})$. By (ii), $\pi(b) = \mathrm{Th}^{\mathcal{M}}_i(\pi(\alpha) \cup \{\pi(q)\})$, and by (iii) $\pi(\alpha) < \rho^{\mathcal{M}}_i$. Thus $\dot{T}^{\mathcal{M}}_i(\pi(a), \pi(b))$. The converse is equally easy.

It follows at once that π is $r\Sigma_{n+1}$ elementary. Suppose for example that $\mathcal{M} \models \exists a, b(\dot{T}_i(a, b) \wedge \psi(a, b, \pi(\bar{c})))$. Then applying the proper term in Sk_n to $\pi(\bar{c})$ we get $a, b \in$ ran π such that $\dot{T}^{\mathcal{M}}_i(a, b) \wedge \psi(a, b, \pi(\bar{c}))$, and we're done.

As the graph of any basic $\tau \in \mathrm{Sk}_{i+1}$ is definable by a boolean combination of $r\Sigma_{i+1}$ formulae, uniformly over all ppm, we see now that π is generalized $r\Sigma_{i+1}$ elementary.

This completes the proof of lemma 2.7.

Standard parameters and cores.

DEFINITION 2.7.1. A *parameter* of $\mathcal{M}$ is a sequence $\langle \alpha_0, \ldots, \alpha_k \rangle$ of ordinals of $\mathcal{M}$ such that $\alpha_0 > \alpha_1 > \cdots > \alpha_k$.

DEFINITION 2.7.2. $<_{\text{lex}}$ is the lexicographic wellordering of all parameters (i.e. of all descending sequences of ordinals).

DEFINITION 2.7.3. Given a ppm $\mathcal{M}$ with $\rho^{\mathcal{M}}_k < \mathrm{OR}^{\mathcal{M}}$ and given $q \in |\mathcal{M}|$, the *kth standard parameter of* $(\mathcal{M}, q)$ is the $<_{\text{lex}}$ least parameter p of $\mathcal{M}$ such that $\mathrm{Th}^{\mathcal{M}}_k(\rho^{\mathcal{M}}_k \cup \{p, q\}) \notin |\mathcal{M}|$.

We now define two useful properties a parameter might possess, solidity and universality. We shall eventually show that the appropriate standard parameters associated to the levels of the model we shall construct are solid and universal.

Solid parameters.

DEFINITION 2.7.4. Let $r = \langle \alpha_0 \cdots \alpha_\ell \rangle$ be a descending sequence of ordinals, $\mathcal{M}$ a ppm which is passive or active of types I or II, and $q \in |\mathcal{M}|$, and $1 \leq k < \omega$. We say r is k *solid over* $(\mathcal{M}, q)$ iff for all $i \leq \ell$

$$\mathrm{Th}^{\mathcal{M}}_k(\alpha_i \cup \{\langle \alpha_0 \cdots \alpha_{i-1} \rangle, q\}) \in |\mathcal{M}| .$$

We are interested in the case that r is the kth standard parameter of $(\mathcal{M}, q)$. Notice that in this case $\alpha_\ell \geq \rho_k^{\mathcal{M}}$, and for any finite $s \subseteq \alpha_i$, $\mathrm{Th}_k^{\mathcal{M}}(\rho_k^{\mathcal{M}} \cup s \cup \{\langle\alpha_0 \cdots \alpha_{i-1}\rangle, q\}) \in |\mathcal{M}|$ simply by the $<_{\mathrm{lex}}$ minimality of r. Solidity is the uniform version of this closure property of $\mathcal{M}$: $\langle\alpha_0 \ldots \alpha_\ell\rangle$ is k-solid over $(\mathcal{M}, q)$ iff $\pi \in \mathcal{M}$, where $\pi(s, i) = \mathrm{Th}_k^{\mathcal{M}}(\rho_k^{\mathcal{M}} \cup s \cup \{\langle\alpha_0 \ldots \alpha_{i-1}\rangle, q\})$, for all $i \leq \ell$ and finite $s \subseteq \alpha_i$.

Solidity is useful because it is easier to show solidity is preserved by the appropriate embeddings than to show standardness is.

Universal parameters.

DEFINITION 2.7.5. Let $\mathcal{M}$ be a ppm, $q \in |\mathcal{M}|$, r a parameter of $\mathcal{M}$, and $1 \leq k < \omega$. We say that r is *k-universal over* $(\mathcal{M}, q)$ iff whenever $A \in |\mathcal{M}|$ and $A \subseteq \rho_k^{\mathcal{M}}$, there is some term $\tau \in \mathrm{Sk}_k$ and $\bar{\alpha} \in \rho_k^{<\omega}$ such that

$$A = \tau^{\mathcal{M}}[\bar{\alpha}, r, q] \cap \rho_k^{\mathcal{M}} .$$

Again, we are interested in the case r is the kth standard parameter of $(\mathcal{M}, q)$. The k-universality of r will be used to show r remains the standard parameter in a certain hull of $\mathcal{M}$; the argument is given in the next lemma.

Lemma 2.8. *Let $\pi : \mathcal{H} \to \mathcal{M}$ be generalized $r\Sigma_k$ elementary , where $\mathcal{M}$ is a ppm and $1 \leq k < \omega$. Suppose $\rho_k^{\mathcal{M}} \subseteq \mathrm{OR}^{\mathcal{H}}$ and $\pi \restriction \rho_k^{\mathcal{M}} =$ identity. Suppose also that $\pi(r)$ is the kth standard parameter $(\mathcal{M}, \pi(q))$, and $\pi(r)$ is k-solid and k-universal over $(\mathcal{M}, \pi(q))$. Then $\rho_k^{\mathcal{H}} = \rho_k^{\mathcal{M}}$, r is the kth standard parameter of $(\mathcal{H}, q)$, and r is k-universal over $(\mathcal{H}, q)$.*

PROOF. For $\alpha \leq \rho_k^{\mathcal{M}}$ we have $\mathrm{Th}_k^{\mathcal{H}}(\alpha \cup \{s\}) = \mathrm{Th}_k^{\mathcal{M}}(\alpha \cup \{\pi(s)\})$, moreover the theory in question can be regarded as a subset of α. If $\alpha < \rho_k^{\mathcal{M}}$, then as $\rho_k^{\mathcal{M}}$ is a cardinal of $\mathcal{M}$ and $\mathcal{M}$ is strongly acceptable, $\mathrm{Th}_k^{\mathcal{M}}(\alpha \cup \{\pi(s)\})$ belongs to $\mathcal{H}$. This shows that $\rho_k^{\mathcal{M}} \leq \rho_k^{\mathcal{H}}$. But $\mathrm{Th}_k^{\mathcal{H}}(\rho_k^{\mathcal{M}} \cup \{\langle r, q\rangle\}) \notin |\mathcal{H}|$, as otherwise, letting $A \subseteq \rho_k^{\mathcal{M}}$ code it, we have $A = \pi(A) \cap \rho_k^{\mathcal{M}} \in |\mathcal{M}|$, and so $\mathrm{Th}_k^{\mathcal{M}}(\rho_k^{\mathcal{M}} \cup \{\pi(r), \pi(q)\}) \in |\mathcal{M}|$, a contradiction. Thus $\rho_k^{\mathcal{M}} = \rho_k^{\mathcal{H}}$.

We have $\mathrm{Th}_k^{\mathcal{H}}(\rho_k^{\mathcal{H}} \cup \{r, q\}) \notin |\mathcal{H}|$, so to see that r is the kth standard parameter of $(\mathcal{H}, q)$, suppose $s <_{\mathrm{lex}} r$. So $\pi(s) <_{\mathrm{lex}} \pi(r)$, and we have $A \subseteq \rho_k^{\mathcal{M}}$, $A \in |\mathcal{M}|$, such that A codes $\mathrm{Th}_k^{\mathcal{M}}(\rho_k^{\mathcal{M}} \cup \{\pi(s), \pi(q)\})$. The k universality of $\pi(r)$ over $(\mathcal{M}, \pi(q))$ easily implies $A \in \mathcal{H}$, and so $\mathrm{Th}_k^{\mathcal{H}}(\rho_k^{\mathcal{H}} \cup \{s, q\}) \in |\mathcal{H}|$, as desired.

It is routine to check that r is k-universal over $(\mathcal{H}, q)$. □

We now define by induction on $k \geq 0$

$$\mathfrak{C}_k(\mathcal{M}) = \text{the } k\text{th core of } \mathcal{M} ,$$
$$\rho_k(\mathcal{M}) = \text{the } k\text{th core projectum of } \mathcal{M} ,$$
$$p_k(\mathcal{M}) = \text{the } k\text{th core parameter of } M .$$

We shall assume that certain parameters we encounter in the course of the definition are solid and universal; otherwise we stop the induction. This assumption may not be necessary for a sensible definition, but it is true of the ppm we are interested in, as we shall show later.

DEFINITION 2.8.1. Let $\mathcal{M}$ be a ppm which is passive or active of types I or II. We define $\mathfrak{C}_k(\mathcal{M})$, $\rho_k(\mathcal{M})$, and $p_k(\mathcal{M})$ by induction on k.

$k = 0$: Let $\rho_0(\mathcal{M}) = \mathrm{OR}^{\mathcal{M}}$, $\mathfrak{C}_0(\mathcal{M}) = \mathcal{M}$, and $p_0(\mathcal{M}) = \varnothing$.

$k = 1$: Let $r =$ first standard parameter of $(\mathcal{M}, \varnothing)$. Suppose r is 1-universal over $(\mathcal{M}, \varnothing)$; otherwise stop the induction. Let

$$\pi : \mathcal{H}_1^{\mathcal{M}}(\rho_1^{\mathcal{M}} \cup \{r\}) \to \mathcal{M}$$

be the inverse of the collapse. If $\pi^{-1}(r)$ is not 1-solid over $(\mathcal{H}_1^{\mathcal{M}}(\rho_1^{\mathcal{M}} \cup \{r\}), \varnothing)$ then stop the induction, and otherwise set

$$\begin{aligned} \rho_1(\mathcal{M}) &= \rho_1^{\mathcal{M}} \\ p_1(\mathcal{M}) &= \langle r, \varnothing \rangle \\ \mathfrak{C}_1(\mathcal{M}) &= \mathcal{H}_1^{\mathcal{M}}(\rho_1^{\mathcal{M}} \cup \{r\}). \end{aligned}$$

Notice that $p_1(\mathcal{M}) = \pi(\langle s, q \rangle)$, where s is the first standard parameter of $(\mathfrak{C}_1(\mathcal{M}), q)$, and s is 1-solid and 1-universal over $(\mathfrak{C}_1(\mathcal{M}), q)$. (This follows from Lemma 2.8.)

$k > 1$: Suppose we are given

$$p_{k-1}(\mathcal{M}) = \langle s, q \rangle$$

where s is the k-1st standard parameter of $(\mathfrak{C}_{k-1}(\mathcal{M}), q)$ and s is $k-1$ solid and $k-1$ universal over $(\mathfrak{C}_{k-1}(\mathcal{M}), q)$. Let

$$s = \langle \alpha_0 \cdots \alpha_\ell \rangle$$

and

$$b_i = \mathrm{Th}_{k-1}^{\mathfrak{C}_{k-1}(\mathcal{M})}(\alpha_i \cup \{\alpha_0, \ldots, \alpha_{i-1}, q\})$$

for $0 \le i \le \ell$, so that $b_i \in |\mathfrak{C}_{k-1}(\mathcal{M})|$ by solidity. Set

$$u_{k-1}(\mathcal{M}) = u = \begin{cases} \langle s, q, b_0 \cdots b_\ell \rangle & \text{if } \rho_{k-2}^{\mathfrak{C}_{k-1}(\mathcal{M})} = \mathrm{OR}^{\mathfrak{C}_{k-1}(\mathcal{M})} \\ \langle s, q, b_0 \cdots b_\ell, \rho_{k-2}^{\mathfrak{C}_{k-1}(\mathcal{M})} \rangle & \text{otherwise}. \end{cases}$$

Let r be the kth standard parameter of $(\mathfrak{C}_{k-1}(\mathcal{M}), u)$. If r is not k-solid and k-universal over $(\mathfrak{C}_{k-1}(\mathcal{M}), u)$, then stop the induction. If it is, consider

$$\pi : \mathcal{H}_k^{\mathfrak{C}_{k-1}(\mathcal{M})}\left(\rho_k^{\mathfrak{C}_{k-1}(m)} \cup \{r, u\}\right) \to \mathfrak{C}_{k-1}(\mathcal{M}),$$

the inverse of the collapse. Suppose that $\pi^{-1}(r)$ is k-solid over

$$\left(\mathcal{H}_k^{\mathfrak{C}_{k-1}(\mathcal{M})}\left(\rho_k^{\mathfrak{C}_{k-1}(\mathcal{M})} \cup \{r,u\}\right), \pi^{-1}(u)\right) ;$$

if not then we stop the induction. Set then

$$\rho_k(\mathcal{M}) = \rho_k^{\mathfrak{C}_{k-1}(\mathcal{M})},$$
$$p_k(\mathcal{M}) = \langle r,u\rangle ,$$

and

$$\mathfrak{C}_k(\mathcal{M}) = \mathcal{H}_k^{\mathfrak{C}_{k-1}(\mathcal{M})}(\rho_k^{\mathfrak{C}_{k-1}(\mathcal{M})} \cup \{r,u\}).$$

DEFINITION 2.8.2. A ppm $\mathcal{M}$ ***is k-solid*** iff $\mathfrak{C}_k(\mathcal{M})$ is defined.

DEFINITION 2.8.3. $\mathcal{M}$ is ***k-sound*** iff $\mathcal{M}$ is k-solid and $\mathfrak{C}_i(\mathcal{M}) = \mathcal{M}$ for all $i \leq k$. $\mathcal{M}$ is a *core ppm* or: completely sound, or ω-sound, iff $\mathcal{M}$ is k sound for all $k < \omega$ such that $\rho_{k-1}^{\mathcal{M}} \neq 0$.

All levels of the model we eventually construct will be ω sound (for active type III levels this will be given a meaning in the next section). Nevertheless, one must consider ppm which are not ω-sound, as iteration of a core ppm can produce them.

Notice that if $\mathcal{M}$ is k-sound, then $\rho_i(\mathcal{M}) = \rho_i^{\mathcal{M}}$ for $i \leq k+1$. If $\mathcal{M}$ is not $i-1$ sound, then $\rho_i(\mathcal{M}) \neq \rho_i^{\mathcal{M}}$ is possible.

We record in a definition some properties of the natural embedding π mapping $\mathfrak{C}_{k+1}(\mathcal{M})$ into $\mathfrak{C}_k(\mathcal{M})$.

DEFINITION 2.8.4. Let $\pi : \mathcal{M} \to \mathcal{N}$, and $k \leq \omega$. We call π a ***k-embedding*** iff

(a) $\mathcal{M}$ and $\mathcal{N}$ are k-sound, (b) π is $r\Sigma_{k+1}$ elementary, (c) $\pi(p_i(\mathcal{M})) = p_i(\mathcal{N})$ for all $i \leq k$, (d) $\pi(\rho_i(\mathcal{M})) = \rho_i(\mathcal{N})$ for all $i \leq k-1$, and $\rho_k(\mathcal{N}) = \sup \pi''\rho_k(\mathcal{M})$.

(We adopt the convention that $\pi(\mathrm{OR}^{\mathcal{M}}) = \mathrm{OR}^{\mathcal{N}}$ in the previous definition.)

Lemma 2.9. *Let $\mathcal{M}$ be a $k+1$ solid ppm, and $\pi : \mathfrak{C}_{k+1}(\mathcal{M}) \to \mathfrak{C}_k(\mathcal{M})$ be the inverse of the collapse. Then π is a k-embedding ; moreover $\mathfrak{C}_{k+1}(\mathcal{M})$ is $k+1$ sound and $\pi(\rho_{k+1}(\mathfrak{C}_{k+1(\mathcal{M})})) = \rho_{k+1}(\mathfrak{C}_k(\mathcal{M})$.*

Remark. It is easy to see that if $\pi : \mathcal{M} \to \mathcal{N}$ is a k-embedding, then $\pi(u_k(\mathcal{M})) = u_k(\mathcal{N})$.

Appendix to §2

We close this section by relating the $r\Sigma_n$ hierarchy to the more traditional hierarchy involving master codes and iterated Σ_1 definability.

First, the use of generalized $r\Sigma_{n+1}$ formulae rather than just pure $r\Sigma_{n+1}$ formulae in defining $\mathrm{Th}_{n+1}^{\mathcal{M}}(X)$ does not affect the value of $\rho_{n+1}^{\mathcal{M}}$, at least if $\mathcal{M}$ is n sound.

Lemma 2.10. *Let $\mathcal{M}$ be a ppm and $n \geq 0$. Let $q \in |\mathcal{M}|$, and suppose in the case that $n \geq 1$ that $\mathcal{M} = \mathcal{H}_n^{\mathcal{M}}(\rho_n^{\mathcal{M}} \cup \{q\})$. Suppose that*

$$\mathrm{Th}_{n+1}^{\mathcal{M}}(\alpha \cup \{q\}) \cap \{\langle\varphi, \bar{a}\rangle \mid \varphi \textit{ is pure } r\Sigma_{n+1}\}$$

is a member of $|\mathcal{M}|$. Then in fact $\mathrm{Th}_{n+1}^{\mathcal{M}}(\alpha \cup \{q\}) \in |\mathcal{M}|$.

Proof. We give the proof in full only for $n = 0$ and $\mathcal{M}$ passive and $\mathrm{OR}^{\mathcal{M}} = \omega\lambda$, for λ a limit. So let us make those assumptions. Let

$$\mathcal{M} = (J_\lambda^{\vec{E}}, \in, \vec{E}) \qquad (\lambda \quad \text{limit})$$

and

$$\mathcal{M}_\beta = (J_\beta^{\vec{E}}, \in, \vec{E} \restriction \beta)$$

for $\beta < \lambda$. For $\tau \in \mathrm{Sk}_1$, $\beta < \lambda$, and $\bar{u} \in |\mathcal{M}_\beta|^{<\omega}$, we say that $\tau(\bar{u})$ changes value at β iff

$$\tau^{\mathcal{M}_\beta}[\bar{u}] \neq \tau^{\mathcal{M}_{\beta+1}}[\bar{u}].$$

Notice that if $\tau_\varphi \in \mathrm{Sk}_1$ is a basic term, then $\tau_\varphi(\bar{u})$ changes value finitely often, since the new value precedes the old in the order of construction (unless the old is 0). It follows that for any $\tau \in \mathrm{Sk}_1$, $\tau(\bar{u})$ changes value finitely often. Notice also that there is a recursive map $(\tau, n) \mapsto \theta_{\tau,n}$ associating to each $\tau \in \mathrm{Sk}_1$ and $n < \omega$ an $r\Sigma_1$ formula $\theta_{\tau,n}$ such that $\mathcal{M} \models \theta_{\tau,n}[\bar{u}, a]$ if and only if $\tau(\bar{u})$ changes value at least n times and a is the nth value of $\tau(\bar{u})$. Now, letting

$$P = \mathrm{Th}_1^{\mathcal{M}}(\alpha \cup \{q\}) \cap \{\langle\varphi, \bar{a}\rangle \mid \varphi \text{ is pure } r\Sigma_{n+1}\}$$

we can compute $\mathrm{Th}_1^{\mathcal{M}}(\alpha \cup \{q\})$ from P inside $\mathcal{M}$ as follows: Given a potential member $\langle\varphi, \bar{a}\rangle$ of $\mathrm{Th}_1^{\mathcal{M}}(\alpha \cup \{q\})$, which we write as $\psi(\tau_1(\bar{u}) \cdots \tau_k(\bar{u}))$ where ψ is pure $r\Sigma_1$ and $\tau_1 \cdots \tau_k \in \mathrm{Sk}_1$ and $\bar{u} \in (\alpha \cup \{q\})^{<\omega}$, find first numbers $n_1 \cdots n_k < \omega$ such that

$$\bigwedge_{i \leq k} \exists a\, \theta_{\tau_{i,n_i}}(\bar{u}, a) \in P$$

and

$$\bigvee_{i \leq k} \exists a\, \theta_{\tau_{i,n_i+1}}(\bar{u}, a) \notin P\,.$$

Then $\langle\varphi, \bar{a}\rangle \in \mathrm{Th}_1^{\mathcal{M}}(\alpha \cup \{q\}$ iff

$$\exists a_1 \cdots \exists a_k \left(\bigwedge_{i \leq k} \theta_{\tau_i, n_i}(\bar{u}, a_i) \wedge \psi(a_1 \cdots a_k) \right)$$

is a member of P.

If $M = (J_{\alpha+1}^{\vec{E}}, \in, \vec{E})$, then we can use the $S_{\omega\alpha+n}^{\vec{E}}$, for $n < \omega$, as we used the M_β's of the previous argument.

If M is active, then we can use the fact that $r\Sigma_1^m = \Sigma_1$ over M^*, where M^* is the amenable structure associated to M (cf. the remark following Corollary 2.2.) We can then ramify Σ_1 over M^* as above to carry out the proof.

This finishes the case $n = 0$. Now let $n \geq 1$. If $\tau \in \mathrm{Sk}_n$ and $\bar{\beta} \in (\rho_n^{\mathcal{M}})^{<\omega}$, then we call the triple $(\tau, \bar{\beta}, q)$ a *name* of $\tau^{\mathcal{M}}[\bar{\beta}, q]$. We are assuming every member of $|\mathcal{M}|$ has a name. If θ is $r\Sigma_{n+1}$ and $a_1 \cdots a_k$ are names and $\beta < \rho_n^{\mathcal{M}}$, then we let "$\mathrm{Th}_n^{\mathcal{M}}(\beta \cup \{q\})$ witnesses that $\theta(a_1 \cdots a_k)$" have the obvious meaning. Namely, if

$$\theta = \exists a, b(\dot{T}_n(a, b) \wedge \psi(a, b, v_1 \cdots v_k))$$

and

$$a_i = (\tau_i, \bar{\beta}_i, q),$$

then $\mathrm{Th}_n^{\mathcal{M}}(\beta \cup \{q\})$ witnesses $\theta(a_1 \cdots a_k)$ iff there is a $\gamma < \beta$ and there are names $(\sigma_1, \bar{\eta}_1, q)$ and $(\sigma_2, \bar{\eta}_2, q)$ such that, first, the following sentences are in $\mathrm{Th}_n^{\mathcal{M}}(\beta \cup \{q\})$:

(a) $\sigma_1(\bar{\eta}_1, q) = \langle \gamma, x \rangle$ for some x such that $\sigma_2(\bar{\eta}_2, q)$ is a complete generalized $r\Sigma_n$ theory of parameters in $\gamma \cup \{x\}$.

(b) $\psi(\sigma_1(\bar{\eta}_1, q), \sigma_2(\bar{\eta}_2, q), \tau_1(\bar{\beta}_1, q) \cdots \tau_k(\bar{\beta}_k, q))$.

(implicit here is that $\bar{\beta}_i, \bar{\eta}_i \in \beta^{<\omega}$), and second, if we let $\sigma_1^{*\mathcal{M}}[\bar{\eta}_1, q] =$ second coordinate of $\sigma_1^{\mathcal{M}}[\bar{\eta}_1, q]$, and we let $f(\varphi, \bar{\delta}) =$ a canonical name for $(\varphi, \bar{\delta}^\frown \sigma_1^{*\mathcal{M}}[\bar{\eta}_1, q])$ for each generalized $r\Sigma_n$ formula φ and $\bar{\delta} \in \gamma^{<\omega}$, then

$$\text{“}f(\varphi, \bar{\delta}) \in \sigma_2(\bar{\eta}_2, q)\text{”} \in \mathrm{Th}_n^{\mathcal{M}}(\beta \cup \{q\}) \text{iff “}\varphi(\bar{\delta}, \sigma^*(\bar{\eta}_1, q))\text{”} \in \mathrm{Th}_n^{\mathcal{M}}(\beta \cup \{q\})$$

Remark. We have taken some liberties above, as $\mathrm{Th}_n^{\mathcal{M}}(\beta \cup \{q\})$ is not literally speaking a set of sentences.

Now for $\tau \in \mathrm{Sk}_{n+1}$ and $a_1 \cdots a_k$, b names and $\beta < \rho_n^{\mathcal{M}}$ let

$$\tau^\beta(a_1 \cdots a_k) = b \quad \text{iff} \quad \mathrm{Th}_n^{\mathcal{M}}(\beta \cup \{q\}) \text{ witnesses } \tau(a_1 \cdots a_k) = b$$

where the right hand side is interpreted in the spirit above. We can use the τ^β's to carry out the argument given in the case $n = 0$. □

The calculations just indicated also give

Lemma 2.11. *Let $\mathcal{M}$ be a ppm, $q \in |\mathcal{M}|$, and $\mathcal{M} = \mathcal{H}_n^{\mathcal{M}}(\rho_n^{\mathcal{M}} \cup \{q\})$ where $n \geq 1$. Let*

$$A = \mathrm{Th}_n^{\mathcal{M}}(\rho_n^{\mathcal{M}} \cup \{q\}) \cap \{(\varphi, \bar{c}) \mid \varphi \quad \textit{is pure } r\Sigma_n\},$$

coded in a natural way as a subset of $\rho_n^{\mathcal{M}}$. Then, letting

$$\mathcal{N} = (J_{\rho_n^{\mathcal{M}}}^{\dot{E}^{\mathcal{M}}}, \in, \dot{E}^{\mathcal{M}} \restriction \rho_n^{\mathcal{M}}, A)$$

$\mathcal{N}$ is amenable, and for all $B \subseteq \rho_n^{\mathcal{M}}$, B is $r\Sigma_{n+1}^{\mathcal{M}}$ iff B is Σ_1 over $\mathcal{N}$.

PROOF. From $A \cap \alpha$ we can compute $\mathrm{Th}_n^{\mathcal{M}}(\alpha \cup \{q\})$ in a simple way, as in the preceding lemma. Thus we may as well assume $A = \mathrm{Th}_n^{\mathcal{M}}(\rho_n^{\mathcal{M}} \cup \{q\})$. Now suppose

$$\eta \in B \Leftrightarrow \mathcal{M} \models \varphi[\eta, x]$$

and let $x = \sigma[\bar{\beta}, q]$, $\sigma \in \mathrm{Sk}_n$, $\bar{\beta} \in (\rho_n^{\mathcal{M}})^{<\omega}$, where φ is $r\Sigma_{n+1}$. Then

$$\eta \in B \Leftrightarrow \exists \beta < \rho_n^{\mathcal{M}}(\mathrm{Th}_n^{\mathcal{M}}(\beta \cup \{q\}) \quad \text{witnesses} \quad \varphi(\eta^*, \sigma(\bar{\beta}, q)))$$

where η^* is a canonical name for η. This shows B is Σ_1 over $\mathcal{N}$. The converse is easy. □