§9. A general iterability theorem

In this section we give a full proof of the iterability facts we have used. The proof results from an amalgamation of §12 of [FSIT], §4 of [IT], and §2 of this paper. Given a premouse $\mathcal{M}$ of the construction $\mathbb{C}$ of [FSIT], and an iteration tree $\mathcal{T}$ on $\mathcal{M}$, §12 of [FSIT] shows how to use the background extenders of $\mathbb{C}$ to "enlarge $\mathcal{T}$" to an iteration tree $\mathcal{U}$ on V. The good behavior of $\mathcal{U}$ guarantees that of $\mathcal{T}$. That $\mathcal{U}$ is well-behaved is shown in §4 of [IT], by realizing in V the models $\mathcal{M}_\alpha^{\bar{\mathcal{U}}}$ occurring on countable elementary submodels $\bar{\mathcal{U}}$ of $\mathcal{U}$. However, since the construction $\mathbb{C}$ of the present paper does not involve background extenders over V, we cannot in the current situation enlarge $\mathcal{T}$ to a tree on V. Instead, we shall run the enlargement process of [FSIT] and the realization process of [IT] simultaneously, making do with the partial background extenders of $\mathbb{C}$ as in §2.

We have also re-organized and streamlined the construction of §4 of [IT]. Moreover, in order to cover all our applications, we shall consider more than just iteration trees on premice.

Definition 9.1. *A creature is a structure which is either a premouse, a psuedo-premouse, or a bicephalus.*

Let $\mathbb{C}$ be the construction of §1, that is,

$$\mathbb{C} = \langle \mathcal{N}_\xi \mid \xi < \Omega \wedge \mathcal{N}_\xi \text{ is defined} \rangle .$$

Definition 9.2. *$\mathcal{M}$ is a creature of $\mathbb{C}$ just in case for some j, ξ*

(a) *$\mathcal{M} = \mathfrak{C}_j(\mathcal{N}_\xi)$, or*

(b) *$\mathcal{M} = (\mathfrak{C}_\omega(\mathcal{N}_\xi), F)$, $\mathcal{M}$ is a psuedo-premouse, and letting $\kappa = crit(F)$, $\forall \mathcal{A} \subseteq P(\kappa)^{\mathcal{M}} (|\mathcal{A}| \leq \omega \Rightarrow F$ has a certificate on $\mathcal{A})$, or*

(c) *$\mathcal{M} = (\mathfrak{C}_\omega(\mathcal{N}_\xi), F_0, F_1)$, $\mathcal{M}$ is a bicephalus, and letting $i \in \{0,1\}$ and $\kappa_i = crit(F_i)$, $\forall \mathcal{A} \subseteq P(\kappa_i)^{\mathcal{M}}$ $(|\mathcal{A}| \leq \omega \Rightarrow F_i$ has a certificate on $\mathcal{A})$.*

We say $\mathcal{M}$ *is $\mathbb{C}$-exotic* just in case condition (a) above fails to hold.

If $\mathcal{M}$ is a creature of $\mathbb{C}$ which is not a premouse, then $\mathcal{M}$ must be $\mathbb{C}$-exotic, but we do not know whether the converse is true. If $\mathcal{M} = \mathfrak{C}_j(\mathcal{N}_\xi)$, then we call (j, ξ) an index (in $\mathbb{C}$) for $\mathcal{M}$; a non-exotic $\mathcal{M}$ can have more than one such index, but all its indices have the same second coordinate. If $\mathcal{M}$ is $\mathbb{C}$-exotic, it must be of the form $(\mathfrak{C}_\omega(\mathcal{N}_\xi), F)$ or $(\mathfrak{C}_\omega(\mathcal{N}_\xi), F_0, F_1)$, and then we say $(0, \xi)$ is an index (in $\mathbb{C}$) for $\mathcal{M}$. A $\mathbb{C}$-exotic creature of $\mathbb{C}$ has exactly one index in $\mathbb{C}$. By ind($\mathcal{M}$) we mean the common second coordinate of all indices of $\mathcal{M}$.

Recall that a coarse premouse is a structure $\mathcal{M} = (M, \in, \delta)$ such that M is transitive, power admissible, satisfies choice, infinity, and the full separation schema, satisfies the full collection schema for domains contained in V_δ, and such that $\omega\delta = \delta$ and ${}^\omega M \subseteq M$.

Definition 9.3. *If $\mathcal{M}$ is a coarse premouse, then $\mathbb{C}^{\mathcal{M}} = \langle \mathcal{N}_\xi^{\mathcal{M}} \mid \xi < \delta^{\mathcal{M}}$ and $\mathcal{N}_\xi^{\mathcal{M}}$ exists$\rangle$ is the construction of §1 as done inside $\mathcal{M}$, up to stage $\delta^{\mathcal{M}}$.*

Thus $\mathbb{C} = \mathbb{C}^{\mathcal{M}}$ for all coarse premice $\mathcal{M}$ such that $\delta^{\mathcal{M}} = \Omega$ and $V_\Omega^{\mathcal{M}} = V_\Omega$. Notice that for any coarse premouse $\mathcal{M}$, $\mathcal{N}_\xi^{\mathcal{M}} \in V_{\delta^{\mathcal{M}}}^{\mathcal{M}}$ whenever $\mathcal{N}_\xi^{\mathcal{M}}$ exists. Further , there are in $V_{\delta^{\mathcal{M}}}$ certificates for all extenders put into models of $\mathbb{C}^{\mathcal{M}}$.

By convention, all creatures are 0-sound. The notion of a weak 0-embedding extends in an obvious way to creatures which are not premice. If $\pi : \mathcal{M} \to \mathcal{N}$ and $\mathcal{M}$ and $\mathcal{N}$ are psuedo-premice, then π is a weak 0-embedding just in case π is $r\Sigma_0$ elementary, and for some cofinal $X \subseteq \mathrm{OR}^{\mathcal{M}}$, π is $r\Sigma_1$ elementary on parameters from X. If $\pi : \mathcal{M} \to \mathcal{N}$ where $\mathcal{M} = (\mathcal{M}', F_0, F_1)$ and $\mathcal{N} = (\mathcal{N}', G_0, G_1)$ are bicephali, then π is a weak 0-embedding just in case it is a weak 0-embedding from $(\mathcal{M}', F_0)$ to $(\mathcal{N}', G_0)$ and a weak 0-embedding from $(\mathcal{M}', F_1)$ to $(\mathcal{N}', G_1)$. For $k > 0$, we shall consider k-soundness and weak k-embeddings only as applied to premice.

Definition 9.4. *Let $\mathcal{M}$ be a creature and let $k \le \omega$; then $(\mathcal{R}, Q, \pi)$ is a k-realization of $\mathcal{M}$ just in case $\mathcal{R}$ is a coarse premouse and*

(a) *Q is a creature of $\mathbb{C}^{\mathcal{R}}$ of the same type as $\mathcal{M}$, and if $k > 0$ then $\mathcal{M}$ is premouse and $Q = \mathfrak{C}_k(\mathcal{N}_\xi)^{\mathcal{R}}$ for some ξ,*

(b) *π is a weak k-embedding from $\mathcal{M}$ into Q, and*

(c) *π, $\mathcal{M} \in \mathcal{R}$.*

In the situation of 9.4, if $\mathcal{M}$ is a premouse then the ordinal ξ as in (a) is determined uniquely by Q and k.

If $\mathcal{M}$ is a creature and $\omega\beta = \mathrm{OR}^{\mathcal{M}}$, then we set $\mathcal{J}_\beta^{\mathcal{M}} = \mathcal{M}$. If $\omega\beta < \mathrm{OR}^{\mathcal{M}}$, then we let $\mathcal{J}_\beta^{\mathcal{M}}$ be the unique premouse Q such that Q is an initial segment of $\mathcal{M}$ and $\omega\beta = \mathrm{OR}^Q$.

Definition 9.5. *Two creatures $\mathcal{M}$ and $\mathcal{N}$ agree below γ just in case for all $\beta < \gamma$, $\mathcal{J}_\beta^{\mathcal{M}} = \mathcal{J}_\beta^{\mathcal{N}}$.*

We wish to consider iteration trees whose base is a family of creatures.

Definition 9.6. *A phalanx of creatures is a pair*

$$(\langle(\mathcal{M}_\beta, k_\beta) \mid \beta \le \alpha\rangle, \langle(\nu_\beta, \lambda_\beta) \mid \beta < \alpha\rangle)$$

such that for all $\beta \le \alpha$

(1) *$\mathcal{M}_\beta$ is a creature, $k_\beta \le \omega$, and if $k_\beta \ne 0$ then $\mathcal{M}_\beta$ is a k_β-sound premouse;*

(2) *if $\beta < \gamma < \alpha$, then $\nu_\beta < \nu_\gamma$;*

(3) *if $\beta < \gamma \le \alpha$, then λ_β is the least $\eta \ge \nu_\beta$ such that $\mathcal{M}_\gamma \models \eta$ is a cardinal, and moreover, $\rho_{k_\gamma}(\mathcal{M}_\gamma) > \lambda_\beta$;*

(4) *$\lambda_\beta \le OR^{\mathcal{M}_\beta}$; and*

(5) *if* $\beta < \gamma \leq \alpha$, *then* $\mathcal{M}_\beta$ *agrees with* $\mathcal{M}_\gamma$ *below* λ_β.

If $\mathcal{B}$ is a phalanx of creatures, say $\mathcal{B} = (\langle(\mathcal{M}_\beta, k_\beta) \mid \beta \leq \alpha\rangle, \langle(\nu_\beta, \lambda_\beta) \mid \beta < \alpha\rangle)$, then we set $\mathcal{M}^{\mathcal{B}}_\beta = \mathcal{M}_\beta$, $\deg^{\mathcal{B}}(\beta) = k_\beta$, $\nu(\beta, \mathcal{B}) = \nu_\beta$ and $\lambda(\beta, \mathcal{B}) = \lambda_\beta$. We also set $lh(\mathcal{B}) = \alpha + 1$. Notice that, because of (3), the $\lambda(\beta, \mathcal{B})$'s are determined by the $\nu(\beta, \mathcal{B})$'s and the $\mathcal{M}^{\mathcal{B}}_\beta$'s.

If $\mathcal{B}$ is a simple phalanx in the sense of §6, then $\mathcal{B}$ becomes a phalanx in the sense of 9.6 if we set $k_\beta = \omega$ for all $\beta < lh(\mathcal{B})$, and $\nu(\beta, \mathcal{B}) = \lambda(\beta, \mathcal{B})$ for all $\beta + 1 < lh(\mathcal{B})$. The notion of an iteration tree on a simple phalanx, as defined in §6, extends in an obvious way to phalanxes as defined in 9.6.

Definition 9.7. *Let $\mathcal{T}$ be an iteration tree of length $\theta + 1$ on a phalanx $\mathcal{B}$ of length $\alpha + 1$. Then:*

(i) (a) *for* $\beta \leq \alpha$, $\deg^{\mathcal{T}}(\beta) = \deg^{\mathcal{B}}(\beta)$, *and for* $\alpha < \beta \leq \theta$, $\deg^{\mathcal{T}}(\beta)$ *is the unique* $k \leq \omega$ *such that* $\mathcal{M}^{\mathcal{T}}_\beta = Ult_k((\mathcal{M}^{\mathcal{T}}_\beta)^*, E^{\mathcal{T}}_\beta)$ *if* β *is a successor, and the eventual value of* $\deg^{\mathcal{T}}(\gamma)$ *for* $\gamma T \beta$ *sufficiently large if* β *is a limit*;

(b) *for* $\beta < \alpha$, $\nu(\beta, \mathcal{T}) = \nu(\beta, \mathcal{B})$ *and* $\lambda(\beta, \mathcal{T}) = \lambda(\beta, \mathcal{B})$, *while for* $\alpha \leq \beta < \theta$, $\nu(\beta, \mathcal{T}) = \nu(E^{\mathcal{T}}_\beta)$ *and* $\lambda(\beta, \mathcal{T}) = lh(E^{\mathcal{T}}_\beta)$.

(ii) $\Phi(\mathcal{T})$ *is the unique phalanx $\mathcal{D}$ such that $lh(\mathcal{D}$ such that $lh(\mathcal{D}) = \theta + 1$ and*

(a) $\mathcal{M}^{\mathcal{D}}_\beta = \mathcal{M}^{\mathcal{T}}_\beta$ *and* $\deg^{\mathcal{D}}(\beta) = \deg^{\mathcal{T}}(\beta)$ *for all* $\beta \leq \theta$,

(b) $\nu(\beta, \mathcal{D}) = \nu(\beta, \mathcal{T})$ *and* $\lambda(\beta, \mathcal{D}) = \lambda(\beta, \mathcal{T})$ *for all* $\beta < \theta$.

A realization of a phalanx $\mathcal{B}$ will be a family of realizations of the creatures occurring in $\mathcal{B}$. We shall demand that these realizations agree with one another in a certain way. In order to explain this agreement condition, we now recall the terminology associated to "resurrection" in §12 of [FSIT].

Let $\mathcal{M}$ be a creature $\omega\alpha = \mathrm{OR}^{\mathcal{M}}$, and $t \leq \omega$. Suppose $t = 0$ if $\mathcal{M}$ is not a premouse. Let $\omega\lambda \leq \mathrm{OR}^{\mathcal{M}}$. Set

$$\langle\beta_0, k_0\rangle = \langle\lambda, 0\rangle,$$

and

$$\begin{aligned}\langle\beta_{i+1}, k_{i+1}\rangle = {} & \text{lexicographically least pair } \langle\beta, k\rangle. \text{ such that} \\ & \langle\lambda, 0\rangle \leq \langle\beta_i, k_i\rangle \leq_{lex} \langle\beta, k\rangle \leq_{\text{lex}} \langle\alpha, t\rangle \\ & \text{and } \rho_k(\mathcal{J}^{\mathcal{M}}_\beta) < \rho_{k_i}(\mathcal{J}^{\mathcal{M}}_{\beta_i}),\end{aligned}$$

where $\langle\beta_{i+1}, k_{i+1}\rangle$ is undefined if no such pair exists. Let i be largest such that $\langle\beta_i, k_i\rangle$ is defined; then we call $\langle\langle\beta_0, k_0\rangle, \ldots, \langle\beta_i, k_i\rangle\rangle$ the (t, λ) *dropdown sequence of* $\mathcal{M}$. It is clear that if $\langle\langle\beta_e, k_e\rangle \mid e \leq i\rangle$ is the (t, λ) dropdown sequence of $\mathcal{M}$, then $\langle\beta_e, k_e\rangle <_{\text{lex}} \langle\beta_{e+1}, k_{e+1}\rangle$ for all $e < i$, and $0 < k_e < \omega$ for all $e \leq i$ such that $e > 0$. Also, letting $\omega\alpha = \mathrm{OR}^{\mathcal{M}}$,

$$\{\rho_k(\mathcal{J}^{\mathcal{M}}_\beta) \mid \lambda \leq \beta \wedge (\beta, k) \leq_{\text{lex}} (\alpha, t) \wedge \rho_k(\mathcal{J}^{\mathcal{M}}_\beta) \leq \lambda\} = \{\rho_{k_e}(\mathcal{J}^{\mathcal{M}}_{\beta_e}) \mid e \leq i\}.$$

The following lemma is proved in §12 of [FSIT]. We gave its proof in a typical special case in Lemma 2.6 of these notes.

Lemma 9.8. *Let $\mathcal{M}$ be a creature of $\mathbb{C}$ with index (t,ξ), let $\langle\langle\beta_e,k_e\rangle \mid e\le i\rangle$ be the (t,λ) dropdown sequence of $\mathcal{M}$. Then there is a unique $\gamma\le\xi$ such that $\mathcal{J}^{\mathcal{M}}_{\beta_i}$ is a creature of $\mathbb{C}$ with index (k_i,γ).*

Now let $\mathcal{M}$ be a creature of $\mathbb{C}$ with index (t,ξ), and let $\omega\lambda\le \mathrm{OR}^{\mathcal{M}}$. We define the $(\mathcal{M},t,\xi)$ *resurrection sequence for* λ as follows. Let $\langle\beta,k\rangle$ be the last term in the (t,λ) dropdown sequence of $\mathcal{M}$. If $\langle\beta,k\rangle=\langle\lambda,0\rangle$, then the $(\mathcal{M},t,\xi)$ resurrection sequence for λ is empty. If $\langle\beta,k\rangle\ne\langle\lambda,0\rangle$ (so that $k>0$), then let $\gamma\le\xi$ be unique so that $\mathcal{J}^{\mathcal{M}}_{\beta}=\mathfrak{C}_k(\mathcal{N}_\gamma)$, as given by 9.8. Let

$$\pi:\mathfrak{C}_k(\mathcal{N}_\gamma)\to\mathfrak{C}_{k-1}(\mathcal{N}_\gamma)$$

be the canonical embedding. Then the $(\mathcal{M},t,\xi)$ resurrection sequence for λ is $\langle\beta,k,\gamma,\pi\rangle^\frown s$, where s is the $(\mathfrak{C}_{k-1}(\mathcal{N}_\gamma),k-1,\gamma)$ resurrection sequence for $\pi(\lambda)$. Here, as usual, if $\lambda=\mathrm{OR}\cap\mathfrak{C}_k(\mathcal{N}_\gamma)$, then $\pi(\lambda)=\mathrm{OR}\cap\mathfrak{C}_{k-1}(\mathcal{N}_\gamma)$ by convention. Notice $(\gamma,k-1)<_{\mathrm{lex}}(\xi,t)$, so this is indeed a legitimate inductive definition.

Now suppose $\mathcal{M}$ is a creature of $\mathbb{C}$ with index (t,ξ), $\omega\lambda\le\mathrm{OR}^{\mathcal{M}}$, $\langle\langle\beta_e,k_e\rangle\mid e\le i\rangle$ is the (t,λ) dropdown sequence of $\mathcal{M}$, and $\langle\langle\delta_e,\ell_e,\gamma_e,\pi_e\rangle\mid e\le s\rangle$ is the $(\mathcal{M},t,\xi)$ resurrection sequence for λ. As explained in §12 of [FSIT], we can find stages

$$i\le e_1<e_2<\cdots<e_{i-1}=s$$

such that for $1\le j\le i-1$,

$$\langle\delta_{e_j},\ell_{e_j}\rangle=\pi_{e_j-1}\circ\pi_{e_j-2}\circ\cdots\circ\pi_0(\langle\beta_{i-j},k_{i-j}\rangle)\,.$$

We set $e_0=0$, and interpret "$\pi_{e_0-1}\circ\cdots\circ\pi_0$" as standing for the identity embedding; this makes the equation just displayed true for $j=0$ as well. Set

$$\sigma_{i-j}=\pi_{e_j}\circ\pi_{e_j-1}\circ\cdots\circ\pi_0$$

so that

$$\sigma_{i-j}:\mathcal{J}^{\mathcal{M}}_{\beta_{i-j}}\to\mathfrak{C}_{\ell_{e_j}-1}(\mathcal{N}_{\gamma_{e_j}})$$

is an $\ell_{e_j}-1$ embedding, for $0\le j\le i-1$. In order to simplify the indexing a bit, we set $\tau_{i-j}=\gamma_{e_j}$ for $0\le j\le i-1$. Notice that $k_{i-j}=\ell_{e_j}$. Thus, setting $p=i-j$, we have that for $1\le p\le i$,

$$\sigma_p:\mathcal{J}^{\mathcal{M}}_{\beta_p}\to\mathfrak{C}_{k_p-1}(\mathcal{N}_{\tau_p})$$

is a k_p-1 embedding. Let us set $\mathrm{Res}_p=\mathfrak{C}_{k_p-1}(\mathcal{N}_{\tau_p})$.

Definition 9.9. *In the situation described above, we call (σ_p, Res_p) the pth partial resurrection of λ from stage (t,ξ).*

The partial resurrections of λ from stage (t,ξ) agree with one another in the following way. For $1 \leq p \leq i$, let

$$\kappa_p = \rho_{k_p}(\mathcal{J}^{\mathcal{M}}_{\beta_p}).$$

Then one can check without too much difficulty that $\kappa_1 > \kappa_2 > \cdots > \kappa_i$, and

$$p < q \Rightarrow \sigma_p \restriction \kappa_{q-1} = \sigma_q \restriction \kappa_{q-1},$$

and

$$p < q \Rightarrow \mathrm{Res}_p \text{ and } \mathrm{Res}_q \text{ agree below } \sup(\sigma_q'' \kappa_{q-1}).$$

Definition 9.10. *In the situation described above, we call (σ, Res) the complete resurrection of λ from $(\mathcal{M},t,\xi)$ if and only if*

(a) the $(\mathcal{M},t,\xi)$ resurrection sequence for λ is empty, and $(\sigma, Res) = (identity, \mathcal{M})$, or

(b) the $(\mathcal{M},t,\xi)$ resurrection sequence for λ is nonempty, and $(\sigma, Res) = (\sigma_1, Res_1)$.

Notice that in either case of 9.10, Res is a creature of $\mathbb{C}$ with index (k,γ), for some $(\gamma,k) \leq_{\text{lex}} (\xi,t)$. If Res is $\mathbb{C}$-exotic, then 9.10 (a) must hold.

Of course, the notions associated to resurrection can be interpreted in any coarse premouse $\mathcal{R}$, using $\mathbb{C}^{\mathcal{R}}$, and not just in V. We shall do this in the following.

Let (σ, Res) be the complete resurrection of λ from $(\mathcal{M},t,\xi)$. Suppose $\mathcal{J}^{\mathcal{M}}_{\lambda}$ is active, which is a case of particular interest. If $(t,\xi) = (0,\lambda)$, then $\mathcal{M} = \mathcal{J}^{\mathcal{M}}_{\lambda} = \mathrm{Res}$, and σ is the identity. Otherwise, $\langle\lambda, 1\rangle \leq_{\text{lex}} \langle\xi, t\rangle$, so $\langle\beta_1, k_1\rangle = \langle\lambda, 1\rangle$. It follows that $\mathrm{Res} = \mathcal{N}_\gamma$ for some $\gamma \leq \xi$, and $\sigma : \mathcal{J}^{\mathcal{M}}_{\lambda} \to \mathcal{N}_\gamma$ is a 0-embedding.

Definition 9.11. *Let $\mathcal{B}$ be a phalanx of length $\alpha + 1$. Then a realization of $\mathcal{B}$ is a sequence $\langle(\mathcal{R}_\beta, Q_\beta, \pi_\beta) \mid \beta \leq \alpha\rangle$ such that*

(1) for all $\beta \leq \alpha$, $(\mathcal{R}_\beta, Q_\beta, \pi_\beta)$ is a $\deg^{\mathcal{B}}(\beta)$-realization of $\mathcal{M}^{\mathcal{B}}_\beta$, and

(2) if $\beta < \gamma \leq \alpha$, and τ is the unique ordinal ξ such that $(\deg^{\mathcal{B}}(\beta), \xi)$ is an index of Q_β in $\mathbb{C}^{\mathcal{R}_\beta}$, and $\lambda_\beta = \lambda(\beta, \mathcal{B})$, and $(\sigma^\beta, Res^\beta)$ is the complete resurrection of $\pi_\beta(\lambda_\beta)$ from $(Q_\beta, \deg^{\mathcal{B}}(\beta), \tau)$, and $\nu_\beta = \nu(\beta, \mathcal{B})$, then

(a) $V^{\mathcal{R}_\beta}_\mu = V^{\mathcal{R}_\gamma}_\mu$, and $V^{\mathcal{R}_\beta}_{\mu+1} \subseteq V^{\mathcal{R}_\gamma}_{\mu+1}$, for $\mu = \sigma^\beta \circ \pi_\beta(\nu_\beta)$,

(b) Res^β agrees with Q_γ below $\sigma^\beta \circ \pi_\beta(\lambda_\beta)$,

(c) $(\sigma^\beta \circ \pi_\beta) \restriction \lambda_\beta = \pi_\gamma \restriction \lambda_\beta$, and

(d) $(\sigma^\beta \circ \pi_\beta)(\lambda_\beta) \leq \pi_\gamma(\lambda_\beta)$.

If $\mathcal{E} = \langle(\mathcal{R}_\beta, Q_\beta, \pi_\beta) \mid \beta \leq \alpha\rangle$ is a realization of $\mathcal{B}$, then we write $\mathcal{R}^{\mathcal{E}}_\beta$ for $\mathcal{R}_\beta$, etc.

Definition 9.12. *Let $\mathcal{B}$ be a phalanx of length $\alpha + 1$, $\mathcal{E}$ a realization of $\mathcal{B}$, and $\mathcal{T}$ a putative iteration tree on $\mathcal{B}$. Let $\alpha + 1 \leq \gamma < lh\ \mathcal{T}$, and let $\beta \leq \alpha + 1$ and $\beta T \gamma$. We call a pair $(\mathcal{P}, \sigma)$ an $\mathcal{E}$-realization of $\mathcal{M}^{\mathcal{T}}_\gamma$ if and only if*

(1) $(\mathcal{R}^{\mathcal{E}}_{\beta}, \mathcal{P}, \sigma)$ *is a* $\deg^{\mathcal{T}}(\gamma)$ *realization of* $\mathcal{M}^{\mathcal{T}}_{\gamma}$,

(2) *if* Q_β *has index* $(\deg^{\mathcal{B}}(\beta), \xi)$ *and* $\mathcal{P}$ *has index* $(\deg^{\mathcal{T}}(\gamma), \theta)$ *in the construction of* $\mathcal{R}^{\mathcal{E}}_{\beta}$, *then* $\theta \leq \xi$, *and*

$$D^{\mathcal{T}} \cap [\beta, \gamma]_T \neq \phi \Leftrightarrow \theta < \xi\,,$$

(3) *if* $D^{\mathcal{T}} \cap [\beta, \gamma]_T = \phi$ *and* $\deg^{\mathcal{T}}(\gamma) = \deg^{\mathcal{B}}(\beta)$, *then* $\mathcal{P} = Q_\beta$ *and* $\pi^{\mathcal{E}}_{\beta} = \sigma \circ i^{\mathcal{T}}_{\beta,\gamma}$.

Definition 9.13. *Let* $\mathcal{T}$ *be an iteration tree on* $\mathcal{B}$, *and* b *a maximal branch of* $\mathcal{T}$ *such that* $D^{\mathcal{T}} \cap b$ *is finite. Then an* $\mathcal{E}$*-realization of* b *is just an* $\mathcal{E}$*-realization of* $\mathcal{M}^{\mathcal{S}}_{\gamma}$, *where* $\gamma = \sup b$ *and* $\mathcal{S}$ *is the putative iteration tree of length* $\gamma + 1$ *such that* $\mathcal{S} \restriction \gamma = \mathcal{T} \restriction \gamma$ *and* $b = \{\eta \mid \eta S \gamma\}$. *We say* b *is* $\mathcal{E}$*-realizable iff there is an* $\mathcal{E}$*-realization of* b.

We can now state the main result of this section. Recall that a cutoff point of a coarse premouse $(M, \in, \delta)$ is an ordinal $\theta \in M$ such that $(V^M_\theta, \in, \delta)$ is a coarse premouse. We say that $\mathcal{M}$ has α cutoff points if the order type of the set of cutoff points of $\mathcal{M}$ is at least α.

Theorem 9.14. *Let* $\mathcal{B}$ *be a hereditarily countable phalanx, and let* $\mathcal{E}$ *be a realization of* $\mathcal{B}$ *such that* $\forall \alpha < lh(\mathcal{B})$ $(\mathcal{R}^{\mathcal{E}}_{\alpha}$ *has* $\delta^{\mathcal{R}^{\mathcal{E}}_{\alpha}}$ *cutoff points). Let* $\mathcal{T}$ *be a countable putative normal iteration tree on* $\mathcal{B}$. *Then either*

(1) $\mathcal{T}$ *has a maximal,* $\mathcal{E}$*-realizable branch, or*

(2) $\mathcal{T}$ *has a last model* $\mathcal{M}^{\mathcal{T}}_{\gamma}$, *and this model is* $\mathcal{E}$*-realizable.*

Proof. Fix $\mathcal{E}_0$, a realization of $\mathcal{B}_0$ as in the hypotheses, and $\mathcal{T}$ a putative normal iteration tree of countable length θ on $\mathcal{B}_0$. We shall consider no iteration trees but $\mathcal{T}$ in the proof to follow, and so we set $\mathcal{M}_\beta = \mathcal{M}^{\mathcal{T}}_{\beta}$, $E_\beta = E^{\mathcal{T}}_{\beta}$, $\nu_\beta = \nu(\beta, \mathcal{T})$, $\lambda_\beta = \lambda(\beta, \mathcal{T})$, and $\deg(\beta) = \deg^{\mathcal{T}}(\beta)$. Let $n^* : \theta \longrightarrow \omega$ be one-one, and set

$$n(\alpha) = \inf\{n^*(\beta) \mid \alpha = \beta \quad \text{or} \quad \alpha T \beta\}\,.$$

Clearly $\alpha T \beta \Rightarrow n(\alpha) \leq n(\beta)$, and for λ a limit $< \theta$, $n(\lambda)$ is the eventual value of $n(\beta)$ for all sufficiently large $\beta T \lambda$. Notice that if $n(\alpha) = n(\beta)$, then $\alpha T \beta$ or $\beta T \alpha$ or $\alpha = \beta$. Also, for b a branch of T,

$$b \text{ is maximal } \Leftrightarrow \sup\{n(\alpha) \mid \alpha \in b\} = \omega\,.$$

For $\alpha, \beta < \theta$, we say

$$\begin{aligned}\alpha \quad \text{survives at} \quad \beta \qquad &\Leftrightarrow [\alpha = \beta \vee (\alpha T \beta \wedge n(\alpha) = n(\beta) \wedge \\ &\forall \gamma(\alpha < \gamma < \beta \wedge \gamma \notin (\alpha, \beta)_T) \Rightarrow n(\alpha) < n(\gamma)))]\,.\end{aligned}$$

It is easy to see that if α survives at β and β survives at γ, then α survives at γ. Also, if α survives at γ and $\alpha T \beta T \gamma$, then α survives at β and β survives at γ. One can also easily see that if λ is a limit, then all sufficiently large $\beta T\lambda$ survive at λ, and that for b a branch of T,

$$b \text{ is maximal } \Leftrightarrow \forall\alpha \in b\, \exists\beta \in b\ (\alpha < \beta \wedge \alpha \text{ doesn't survive at } \beta).$$

Let $lh\ (\mathcal{B}_0) = \alpha_0 + 1$.

For each $\beta \leq \alpha_0$, letting $k = \deg^{\mathcal{B}}(\beta)$, choose a cofinal $Y_\beta \subseteq \rho_k(\mathcal{M}_\beta^{\mathcal{B}_0})$ such that $\pi_\beta^{\mathcal{E}_0}$ is $r\Sigma_{k+1}$ elementary on parameters from Y_β. Next, for $\alpha_0 < \beta < \theta$, we define Y_β by induction. If T-pred$(\beta) = \gamma$ and $\beta \notin D^{\mathcal{T}}$ and $\deg(\beta) = \deg(\gamma)$, then set $Y_\beta = i_{\gamma\beta}{}''Y_\gamma$. If T-pred$(\beta) = \gamma$ but $\beta \in D^{\mathcal{T}}$ or $\deg(\beta) < \deg(\gamma)$, then set $Y_\beta = i_\beta^*{}''(\mathcal{M}_\beta^*)$. Finally, if β is a limit ordinal $< \theta$, let $Y_\beta =$ common value of $i''_{\gamma\beta}Y_\gamma$ for all sufficiently large $\gamma T\beta$.

The idea here is that in a copying construction beginning from $\mathcal{E}_0$, Y_β is the subset of $\mathcal{M}_\beta$ on which we expect $r\Sigma_{k+1}$ elementarity of the copy map, for $k = \deg(\beta)$. We call a k realization $(\mathcal{R}, Q, \pi)$ of $\mathcal{M}_\beta$ a $(k, \boldsymbol{Y})$ *realization* just in case π is $r\Sigma_{k+1}$ elementary on Y_β. A realization Σ of $\Phi(\mathcal{T} \restriction \alpha+1)$ is a $\boldsymbol{Y}$ *realization* just in case $\forall\beta \leq \alpha((\mathcal{R}_\beta^{\mathcal{E}}, Q_\beta^{\mathcal{E}}, \pi_\beta^{\mathcal{E}})$ is a $(\deg(\beta), \boldsymbol{Y})$ realization of $\mathcal{M}_\beta)$. All realizations we consider in the proof to follow will be $\boldsymbol{Y}$-realizations.

Let $\alpha < \theta$ and let $(\mathcal{R}, Q, \pi)$ be a $\deg(\alpha)$ realization of $\mathcal{M}_\alpha$. We shall define a tree $U = U(\alpha, \mathcal{R}, Q, \pi)$. Roughly speaking, U tries to build a maximal branch b of T such that $\alpha \in b$, together with a realizing map σ for $\mathcal{M}_b^{\mathcal{T}}$ which extends π. More precisely, we put a triple $(\langle\beta_0, \ldots, \beta_n\rangle, \langle\varphi_0, \ldots, \varphi_n\rangle, \langle Q_0, \ldots, Q_n\rangle)$ into U just in case

(1) $\beta_0 = \alpha$, $\varphi_0 = \pi$, and $Q_0 = Q$,

and for all $i < n$,

(2) $\beta_i T \beta_{i+1}$ and β_i does not survive at β_{i+1},

(3) $\operatorname{ind}^{\mathcal{R}}(Q_{i+1}) \leq \operatorname{ind}^{\mathcal{R}}(Q_i)$, and $D^{\mathcal{T}} \cap (\beta_i, \beta_{i+1}]_T \neq \phi$ iff $\operatorname{ind}^{\mathcal{R}}(Q_{i+1}) < \operatorname{ind}^{\mathcal{R}}(Q_i)$; moreover, if $D^{\mathcal{T}} \cap (\beta_i, \beta_{i+1}]_T = \phi$ and $\deg^{\mathcal{T}}(\beta_i) = \deg^{\mathcal{T}}(\beta_{i+1})$, then $Q_i = Q_{i+1}$, and

(4) $(\mathcal{R}, Q_{i+1}, \varphi_{i+1})$ is a $(\deg(\beta_{i+1}), \boldsymbol{Y})$ realization; moreover, if $D^{\mathcal{T}} \cap (\beta_i, \beta_{i+1}]_T \neq \phi$ and $\deg(\beta_i) = \deg(\beta_{i+1})$, then $\varphi_i = \varphi_{i+1} \circ i^{\mathcal{T}}_{\beta_i, \beta_{i+1}}$.

Suppose that $(\langle\beta_i \mid i \in \omega\rangle, \langle\varphi_i \mid i \in \omega\rangle, \langle Q_i \mid i \in \omega\rangle)$ is an infinite branch of $U(\alpha, \mathcal{R}, Q, \pi)$. Set $b = \{\eta \mid \exists i(\eta T \beta_i)\}$; then (1) and (2) guarantee that b is a maximal branch of $\mathcal{T}$ such that $\alpha \in b$. By condition (3), $D^{\mathcal{T}} \cap b$ is finite, and Q_i is eventually constant as $i \to \omega$, say with value Q_∞. Condition (3) also guarantees $Q_\infty = Q_0 = Q$ in the case $D^{\mathcal{T}} \cap (b - (\alpha+1)) = \phi$ and $\deg(\alpha) = \deg$ (b) (i.e., $\deg(\eta) = \deg(\alpha)$ for all $\eta \in b - (\alpha+1)$). Finally, let $y \in \mathcal{M}_b^{\mathcal{T}}$, and let k be large enough that $D^{\mathcal{T}} \cap (b - \beta_k) = \phi$, $\deg(\beta_k) = \deg$ (b), and $y = i^{\mathcal{T}}_{\beta_k b}(x)$ for some $x \in \mathcal{M}_{\beta_k}$. We then set $\sigma(y) = \varphi_k(x)$. By condition (4), σ is a well-defined weak deg (b) embedding from $\mathcal{M}_b$ into Q_∞. Moreover, if $D^{\mathcal{T}} \cap (b - (\alpha+1)) = \phi$ and $\deg(\alpha) = \deg(b)$, then $\pi = \sigma \circ i^{\mathcal{T}}_{\alpha b}$.

So if for some $\alpha \leq \alpha_0$, $U(\alpha, \mathcal{R}_\alpha^{\mathcal{E}_0}, Q_\alpha^{\mathcal{E}_0}, \pi_\alpha^{\mathcal{E}_0})$ has an infinite branch, then conclusion (1) of 9.14 holds. We therefore assume henceforth that for all $\alpha \leq \alpha_0$, $U(\alpha, \mathcal{R}_\alpha^{\mathcal{E}_0}, Q_\alpha^{\mathcal{E}_0}, \pi_\alpha^{\mathcal{E}_0})$ is wellfounded. Notice that $U(\alpha, \mathcal{R}_\alpha^{\mathcal{E}_0}, Q_\alpha^{\mathcal{E}_0}, \pi_\alpha^{\mathcal{E}_0})$ belongs to $\mathcal{R}_\alpha^{\mathcal{E}_0}$, and has size $< \delta^{\mathcal{R}_\alpha^{\mathcal{E}_0}}$ in $\mathcal{R}_\alpha^{\mathcal{E}_0}$.

Notice that if $\alpha < \theta$, then there are only finitely many $\gamma < \theta$ such that $\alpha \leq \gamma$ and T-pred$(\gamma) \leq \alpha$ and T-pred(γ) survives at γ. [If not, then we can fix $k < n(\alpha)$ such that $k = n(\gamma) = n(T\text{-pred}(\gamma))$ for infinitely many γ such that T-pred$(\gamma) < \alpha < \gamma$. Fix two distinct such γ's, say γ_0 and γ_1. Then γ_0 and γ_1 are T-incomparable, yet $n(\gamma_0) = n(\gamma_1)$. This contradicts the definition of n.] For $\alpha \leq \beta < \theta$, we define

$$c(\alpha, \beta) = |\{\gamma \mid \beta \leq \gamma < \theta \wedge \text{T-pred}(\gamma) \leq \alpha \ \wedge \text{T-pred}(\gamma) \text{ survives at } \gamma\}|.$$

Definition 9.15. *Let $\gamma < \theta = lh\ \mathcal{T}$, and let $\mathcal{E}$ be a realization of $\Phi(\mathcal{T} \restriction \gamma)$. We say $\mathcal{E}$ has enough room iff $\forall\, \alpha < \gamma$*

(a) *$\mathcal{U}(\alpha, \mathcal{R}^{\mathcal{E}}_\alpha, Q^{\mathcal{E}}_\alpha, \pi^{\mathcal{E}}_\alpha)$ is wellfounded, and*

(b) *$\mathcal{R}^{\mathcal{E}}_\alpha$ has $\omega \cdot rank(U(\alpha, \mathcal{R}^{\mathcal{E}}_\alpha, Q^{\mathcal{E}}_\alpha, \pi^{\mathcal{E}}_\alpha)) + c(\alpha, \gamma)$ cutoff points.*

Definition 9.16. *Let $\alpha < \gamma \leq \theta$; then α is a break point at γ iff whenever β is a successor ordinal such that $\alpha < \beta \leq \gamma$ and T-pred$(\beta) \leq \alpha$, then T-pred(β) does not survive at β.*

We can now prove our main lemma, which concerns the extendibility of realizations of the phalanxes determined by initial segments of $\mathcal{T}$.

Lemma 9.17. *Let $\alpha_0 \leq \alpha < \eta < \theta$, and let $\mathcal{E}$ be a realization of $\Phi(\mathcal{T} \restriction \alpha + 1)$ such that $\mathcal{E}$ has enough room. Then:*

(1) *Suppose α is a break point at η. Then there is a realization $\mathcal{F}$ of $\Phi(\mathcal{T} \restriction \eta + 1)$ such that $\mathcal{F} \restriction \alpha + 1 = \mathcal{E}$, $\mathcal{F}$ has enough room, and $\mathcal{R}^{\mathcal{F}}_\eta \in \mathcal{R}^{\mathcal{E}}_\alpha$.*

(2) *Suppose that for some $\delta \leq \alpha$, δ survives at η, and let δ_0 be the largest such ordinal δ. Then there is a realization $\mathcal{F}$ of $\Phi(\mathcal{T} \restriction \eta + 1)$ such that $\mathcal{F} \restriction \delta_0 = \mathcal{E} \restriction \delta_0$, $\mathcal{F}$ has enough room, and*

(a) *$\mathcal{R}^{\mathcal{F}}_\eta = \mathcal{R}^{\mathcal{E}}_{\delta_0}$ and $ind^{\mathcal{R}^{\mathcal{F}}_\eta}(Q^{\mathcal{F}}_\eta) \leq ind^{\mathcal{R}^{\mathcal{E}}_\alpha}(Q^{\mathcal{E}}_\alpha)$,*

(b) *$D^{\mathcal{T}} \cap (\alpha, \eta]_T \neq \phi \Rightarrow ind^{\mathcal{R}^{\mathcal{F}}_\eta}(Q^{\mathcal{F}}_\eta) < ind^{\mathcal{R}^{\mathcal{E}}_\alpha}(Q^{\mathcal{E}}_\alpha)$, and*

(c) *if $D^{\mathcal{T}} \cap (\alpha, \eta]_T = \phi$ and $deg^{\mathcal{T}}(\alpha) = deg^{\mathcal{T}}(\eta)$, then $Q^{\mathcal{E}}_\alpha = Q^{\mathcal{F}}_\eta$ and $\pi^{\mathcal{E}}_\alpha = \pi^{\mathcal{F}}_\eta \circ i^{\mathcal{T}}_{\alpha\eta}$.*

Proof. By induction on η. First, supposing 9.17 known for $\eta \leq \gamma$, we prove it for $\gamma + 1$. So let $\alpha_0 \leq \alpha < \gamma + 1$, and let $\mathcal{E}$ realize $\Phi(\mathcal{T} \restriction \alpha + 1)$ and have enough room. Let $\beta = T$-pred$(\gamma + 1)$.

We shall ultimately consider two cases in the construction of the desired $\mathcal{F}$ realizing $\Phi(\mathcal{T} \restriction \gamma + 2)$: the case that for some $\delta \leq \alpha$, δ survives at $\gamma + 1$, and the case that α is a break point at $\gamma + 1$ and β does not survive at $\gamma + 1$. Ostensibly there is a third case, the case that α is a break point at $\gamma + 1$ and β survives at $\gamma + 1$, but this case reduces easily to case one. For in this third case, $\alpha < \beta < \gamma + 1$. Since α is a break point at β, induction hypothesis 9.17 (1) gives us a $\mathcal{G}$ realizing $\Phi(\mathcal{T} \restriction \beta + 1)$, having enough room, and such that $\mathcal{E} = \mathcal{G} \restriction \alpha + 1$ and $\mathcal{R}^{\mathcal{G}}_\beta \in \mathcal{R}^{\mathcal{E}}_\alpha$. Now case one gives us an $\mathcal{F}$ realizing $\Phi(\mathcal{T} \restriction \gamma + 2)$, having enough room, and such that $\mathcal{F} \restriction \beta = \mathcal{G} \restriction \beta$ and $\mathcal{R}^{\mathcal{F}}_{\gamma+1} = \mathcal{R}^{\mathcal{G}}_\beta$. Clearly, $\mathcal{F}$ is as required in 9.17 (1) with $\eta = \gamma + 1$.

The desired $\mathcal{F}$ will come from a realization $\mathcal{H}$ of $\Phi(\mathcal{T} \upharpoonright \gamma+1)$ which we now define. The definition depends on which of the two cases we are in.

Case 1. For some $\delta \leq \alpha$, δ survives at $\gamma+1$.

Let δ_0 be the largest such δ. Since $\beta = T\text{-pred}(\gamma+1)$, $\delta_0 = \beta$ or ($\delta_0 T \beta$ and δ_0 survives at β). Let $\mathcal{G} = \mathcal{E}$ if $\delta_0 = \beta$, and otherwise let $\mathcal{G}$ be the realization of $\Phi(\mathcal{T} \upharpoonright \beta+1)$ given by our induction hypothesis 9.17 (2), with $\eta = \beta$. Since β survives at $\gamma+1$, either $\beta = \gamma$ or β is a break point at γ. [If $T\text{-pred}(\xi) \leq \beta < \xi \leq \gamma$ and $T\text{-pred}(\xi)$ survives at ξ, then $n(\beta) > n(\xi) = n(T\text{-pred}(\xi))$, so β doesn't survive at $\gamma+1$.] Let $\mathcal{H} = \mathcal{G}$ if $\beta = \gamma$, and otherwise let $\mathcal{H}$ be a realization of $\Phi(\mathcal{T} \upharpoonright \gamma+1)$ such that $\mathcal{H} \upharpoonright \beta+1 = \mathcal{G}$ as given by our induction hypothesis 9.17 (1), with $\eta = \gamma$.

Notice that in any case, $\mathcal{R}^{\mathcal{H}}_{\beta} = \mathcal{R}^{\mathcal{G}}_{\beta} = \mathcal{R}^{\mathcal{E}}_{\alpha}$. Also, if $D^{\mathcal{T}} \cap (\alpha, \beta]_T = \phi$ and $\deg^{\mathcal{T}}(\alpha) = \deg^{\mathcal{T}}(\beta)$, then $Q^{\mathcal{H}}_{\beta} = Q^{\mathcal{E}}_{\alpha}$ and $\pi^{\mathcal{E}}_{\alpha} = \pi^{\mathcal{H}}_{\beta} \circ i^{\mathcal{T}}_{\alpha\beta}$. Finally, $\mathrm{ind}^{\mathcal{R}^{\mathcal{H}}_{\beta}}(Q^{\mathcal{H}}_{\beta}) \leq \mathrm{ind}^{\mathcal{R}^{\mathcal{E}}_{\alpha}}(Q^{\mathcal{E}}_{\alpha})$, and if $D^{\mathcal{T}} \cap (\alpha, \beta]_T \neq \phi$, then $\mathrm{ind}^{\mathcal{R}^{\mathcal{H}}_{\beta}}(Q^{\mathcal{H}}_{\beta}) < \mathrm{ind}^{\mathcal{R}^{\mathcal{H}}_{\alpha}}(Q^{\mathcal{H}}_{\alpha})$.

Case 2. α is a break point at $\gamma+1$, and β does not survive at $\gamma+1$.

In this case, α is a break point at γ. If $\alpha = \gamma$ we let $\mathcal{H} = \mathcal{E}$. If $\alpha < \gamma$, we let $\mathcal{H}$ be the realization of $\Phi(\mathcal{T} \upharpoonright \gamma+1)$ given by induction hypothesis 9.17 (1). In either case, we have $\mathcal{E} = \mathcal{H} \upharpoonright \alpha+1$ and $\mathcal{R}^{\mathcal{H}}_{\gamma} \subseteq \mathcal{R}^{\mathcal{E}}_{\alpha}$.

Now, using $\mathcal{H}$, we produce the desired $\mathcal{F}$ realizing $\Phi(\mathcal{T} \upharpoonright \gamma+2)$. We shall have to consider the case split above again later, but for now we can run the two cases simultaneously. In order to clean up our notation a bit, we set $(Q_\eta, \mathcal{R}_\eta, \pi_\eta) = (Q^{\mathcal{H}}_\eta, \mathcal{R}^{\mathcal{H}}_\eta, \pi^{\mathcal{H}}_\eta)$ for all $\eta \leq \gamma$.

Let $j = \deg(\gamma)$, and let Q_γ have index (j, ξ) in $\mathbb{C}^{\mathcal{R}_\gamma}$. Let ($\sigma^\gamma$, Res^γ) be the complete resurrection of $\pi_\gamma(\lambda_\gamma)$ from (Q_γ, j, ξ), as computed in $\mathcal{R}_\gamma$, of course. Since $\gamma \geq \alpha_0$, $\lambda_\gamma = lh\ E^{\mathcal{T}}_\gamma$. If $\lambda_\gamma = \mathrm{OR}^{\mathcal{M}_\gamma}$, then as usual we set $\pi_\gamma(\lambda_\gamma) = \mathrm{OR}^{Q_\gamma}$.

Claim 1. If $\eta < \gamma$, then $\sigma^\gamma \upharpoonright \pi_\gamma(\lambda_\eta) = \text{identity}$.

Proof. Since $\Phi(\mathcal{T} \upharpoonright \gamma+1)$ is a phalanx, definition 9.6 guarantees that λ_η is a cardinal of $\mathcal{M}_\gamma$ and $\rho_j(\mathcal{M}_\gamma) > \lambda_\eta$. Since π_γ is a weak j-embedding, $\pi_\gamma(\lambda_\eta)$ is a cardinal of Q_γ and $\rho_j(Q_\gamma) > \pi_\gamma(\lambda_\eta)$. Also, $\pi_\gamma(\lambda_\eta) < \pi_\gamma(\lambda_\gamma)$. It follows that all projecta associated to the $(j, \pi_\gamma(\lambda_\gamma))$ dropdown sequence of Q_γ are $\geq \pi_\gamma(\lambda_\eta)$. □

Set

$$F = \sigma^\gamma \circ \pi_\gamma(E^{\mathcal{T}}_\gamma) = \text{ last extender of } \mathrm{Res}^\gamma\ ,$$

where if Res^γ is a bicephalus we choose the extender interpreting the same predicate symbol that E_γ interprets in $\mathcal{M}_\gamma$. We wish to consider $\mathrm{Ult}(Q^*_{\gamma+1}, F)$, where $Q^*_{\gamma+1}$ is the creature of $\mathbb{C}^{\mathcal{R}_\beta}$ we shall now define. Let $n = \deg(\beta)$, and $\langle(\eta_0, k_0), \ldots, (\eta_e, k_e)\rangle$ = the (n, λ_β) dropdown sequence of $\mathcal{M}_\beta$, and set

$$\kappa_i = \rho_{k_i}(\mathcal{J}^{\mathcal{M}_\beta}_{\eta_i})$$

for $0 \le i \le e$. The following claim relates these to the $(n, \pi_\beta(\lambda_\beta))$ dropdown sequence of Q_β. The claim is slightly complicated by the fact that π_β is only a *weak* n-embedding.

Claim 2. The $(n, \pi_\beta(\lambda_\beta))$ dropdown sequence of Q_β is

(a) $\langle(\pi_\beta(\eta_0), k_0), \ldots, (\pi_\beta(\eta_e), k_e)\rangle$ if $\kappa_e < \rho_n(\mathcal{M}_\beta)$,

(b) $\langle(\pi_\beta(\eta_0), k_0), \ldots, (\pi_\beta(\eta_e), k_e)\rangle^\frown u$, where $u = \phi$ or $u = (\eta, n)$ for $\omega\eta = \mathrm{OR}^{Q_\beta}$ if $\kappa_e = \rho_n(\mathcal{M}_\beta)$ but $(\omega\eta_e, k_e) \ne (\mathrm{OR}^{\mathcal{M}_\beta}, n)$, and

(c) $\langle(\pi_\beta(\eta_0), k_0), \ldots, (\pi_\beta(\eta_{e-1}), k_{e-1})\rangle^\frown u$, where $u = \phi$ or $u = (\pi_\beta(\eta_e), k_e) = (\omega\eta, n)$, for $\omega\eta = \mathrm{OR}^{Q_\beta}$ if $(\omega\eta_e, k_e) = (\mathrm{OR}^{\mathcal{M}_\beta}, n)$.

Remark. Note that $\kappa_e = \rho_n(\mathcal{M}_\beta)$ in case (c). If $e = 0$, then $n = 0 = k_0$ and $\eta_0 = \lambda_\beta = \omega\lambda_\beta = \mathrm{OR}^{\mathcal{M}_\beta}$. The $(n, \pi_\beta(\lambda_\beta))$ dropdown sequence for Q_β is then $\langle(\mathrm{OR}^{Q_\beta}, 0)\rangle$, which falls under case (c).

Remark. The $u = \phi$ case in (c) would not be necessary if π_β were a full n-embedding.

The claim follows quite easily from the fact that π_β is a weak (n, Y_β)-embedding. For (a), notice that $\pi_\beta'' \rho_n(\mathcal{M}_\beta) \le \rho_n(Q_\beta)$. Recall that π_β preserves cardinals, so that if for example $\omega\eta_e < \mathrm{OR}^{\mathcal{M}_\beta}$ then $\mathcal{M}_\beta \models \forall\gamma \ge \eta_e(\rho_\omega(\mathcal{J}_\gamma^{\dot{E}}) \ge \rho_{k_e}(\mathcal{J}_{\eta_e}^{\dot{E}}))$, and thus $Q_\beta \models \forall\gamma \ge \pi_\beta(\eta_e)(\rho_\omega(\mathcal{J}_\gamma^{\dot{E}}) \ge \pi_\beta(\kappa_e))$.

Let $\mu_0 = \mathrm{crit}(E_\gamma^{\mathcal{T}})$, and let

$$i = \begin{cases} e+1 & \text{if} \quad \mu_0 < \kappa_e\,, \\ \text{least } j \text{ s.t. } \kappa_j \le \mu_0\,, & \text{if } \kappa_e \le \mu_0\,. \end{cases}$$

Notice that since $\kappa_0 = \lambda_\beta > \mu_0$, $i > 0$.

Because $\mathcal{T}$ is maximal,

$$\mathcal{M}_{\gamma+1}^* = \begin{cases} \mathcal{J}_{\eta_i}^{\mathcal{M}_\beta} & \text{if} \quad i \le e\,, \\ \mathcal{M}_\beta & \text{if} \quad i = e+1\,, \end{cases}$$

and

$$\deg(\gamma+1) = \begin{cases} k_i - 1 & \text{if} \quad i \le e\,, \\ n & \text{if} \quad i = e+1\,. \end{cases}$$

Let $(\sigma_i^\beta, \mathrm{Res}_i^\beta)$ be the ith partial resurrection of λ_β from (Q_β, n, τ), where Q_β has index (n, τ) in $\mathbb{C}^{\mathcal{R}_\beta}$, if this resurrection is defined. (The resurrection is undefined if $i = e+1$, and defined if $i < e$ by claim 2. If $i = e$, $(\sigma_i^\beta, \mathrm{Res}_i^\beta)$ is undefined just in case $(\omega\eta_e, k_e) = (\mathrm{OR}^{\mathcal{M}_\beta}, n)$ and the conclusion of (c) of claim 2 holds with $u = \phi$.)

Now let

$$Q^*_{\gamma+1} = \begin{cases} \mathrm{Res}^\beta_i & \text{if } \mathrm{Res}^\beta_i \text{ is defined}, \\ Q_\beta & \text{otherwise} \end{cases}$$

$$\sigma = \begin{cases} \sigma^\beta_i & \text{if } \mathrm{Res}^\beta_i \text{ is defined}, \\ \text{identity} & \text{otherwise}. \end{cases}$$

Thus, in any case, $\sigma \circ (\pi_\beta \restriction \mathcal{M}^*_{\gamma+1})$ is a weak $\deg(\gamma+1)$ embedding from $\mathcal{M}^*_{\gamma+1}$ into $Q^*_{\gamma+1}$. Moreover, $\sigma \circ \pi_\beta$ is $r\Sigma_{\deg(\gamma+1)+1}$ elementary on Z, where $Z = Y_\beta$ if $\gamma+1 \notin D^{\mathcal{T}}$ and $\deg(\gamma+1) = \deg(\beta)$, and $Z =$ universe of $\mathcal{M}^*_{\gamma+1}$ otherwise.

Set $k = \deg(\gamma+1)$.

Claim 2.5. $\sigma \circ \pi_\beta$ is a weak k-embedding which is $r\Sigma_{k+1}$ elementary on ${i^*_{\gamma+1}}^{-1}(Y_{\gamma+1})$.

Proof. Assume first that Res^β_i is defined, so that $i \le e$, $\deg(\gamma+1) = k_i - 1$, and $\sigma = \sigma^\beta_i$ is a full $k_i - 1$ embedding. Looking at claim 2, we see that in all cases the domain of σ is $\mathcal{J}^{Q_\beta}_{\pi_\beta(\eta_i)}$, since we cannot have the situation in (c) with $i = e$ and $u = \phi$. But $\mathcal{M}^*_{\gamma+1} = \mathcal{J}^{\mathcal{M}_\beta}_{\eta_i}$, and $\pi_\beta \restriction \mathcal{M}^*_{\gamma+1}$ is a weak $k_i - 1$ embedding. In fact, if $\omega\eta_i < \mathrm{OR}^{\mathcal{M}_\beta}$, then $\pi_\beta \restriction \mathcal{M}^*_{\gamma+1}$ is fully elementary, and if $\omega\eta_i = \mathrm{OR}^{\mathcal{M}_\beta}$, then $k_i \le n$, so $\pi_\beta \restriction \mathcal{M}^*_{\gamma+1}$ is a weak k_i embedding. It follows that $\sigma \circ (\pi_\beta \restriction \mathcal{M}^*_{\gamma+1})$ is a weak $k_i - 1$ embedding from $\mathcal{M}^*_{\gamma+1}$ into $Q^*_{\gamma+1}$. Assume next that Res^β_i is undefined. Then either $i = e+1$, or we have the situation in (c) of claim 2 with $u = \phi$. In either case, $\deg(\gamma+1) \le n$. Also $\mathcal{M}^*_{\gamma+1} = \mathcal{M}_\beta$, $Q^*_{\gamma+1} = Q_\beta$, and σ=identity. Since π_β is a weak n-embedding, $\sigma \circ \pi_\beta$ is a weak $\deg(\gamma+1)$ embedding from $\mathcal{M}^*_{\gamma+1}$ into $Q^*_{\gamma+1}$.

Let $(\sigma^\beta, \mathrm{Res}^\beta)$ be the complete resurrection of $\pi_\beta(\lambda_\beta)$ from (Q_β, n, τ). Let ψ be the complete resurrection embedding for $\sigma(\pi_\beta(\lambda_\beta))$ from the appropriate tuple. (This tuple is $(Q^*_{\gamma+1}, n, \tau)$ if Res^β_i is undefined, and $(Q^*_{\gamma+1}, k_i - 1, \eta)$ where $\mathrm{Res}^\beta_i = \mathfrak{C}_{k_i-1}(\mathcal{N}_\eta)^{\mathcal{R}_\beta}$ otherwise.) Then

$$\psi : \mathcal{J}^{Q^*_{\gamma+1}}_{\sigma\circ\pi_\beta(\lambda_\beta)} \to \mathrm{Res}^\beta$$

and

$$\sigma^\beta = \psi \circ (\sigma \restriction \mathcal{J}^{Q_\beta}_{\pi_\beta(\lambda_\beta)}).$$

Claim 3. $\psi \restriction (\sup(\sigma \circ \pi_\beta''\kappa_{i-1})) = \text{identity}$.

Proof. Suppose first that Res^β_i exists, so that $i \le e$ and $\sigma = \sigma^\beta_i$. From claim 2 and the fact that π_β is a weak n-embedding we see that $\pi_\beta(\kappa_{i-1})$ is the projectum associated to the $(i-1)^s t$ element of the $(n, \pi_\beta(\lambda_\beta))$ drop-down sequence of Q_β. As we remarked earlier, ψ is therefore the identity on $\sup(\sigma^{\beta}_i{}''\pi_\beta(\kappa_{i-1}))$, and this implies the claim.

Suppose next that Res^β_i is undefined, so that either $i = e+1$ or $i = e$ and (c) of claim 2 holds with $u = \phi$. In either case the projectum associated to the last term of the $(n, \pi_\beta(\lambda_\beta))$ dropdown sequence of Q_β is at least $\sup(\pi_\beta''\kappa_{i-1})$. Thus $\sigma^\beta \restriction \sup(\pi_\beta''\kappa_{i-1}) =$ identity. But $\psi = \sigma^\beta$ and $\sigma =$ identity, so this implies the claim. □

Now let

$$\mu_1 = \begin{cases} (\mu_0^+)^{\mathcal{M}^*_{\gamma+1}} & \text{if } \mathcal{M}^*_{\gamma+1} \models \mu_0^+ \text{ exists}, \\ \mathrm{OR}^{\mathcal{M}^*_{\gamma+1}} & \text{otherwise}. \end{cases}$$

Claim 4. $\mu_1 \leq \lambda_\beta$, and if $\mu_1 = \mathrm{OR}^{\mathcal{M}^*_{\gamma+1}}$ then $\mathcal{M}^*_{\gamma+1} = \mathcal{J}^{\mathcal{M}_\beta}_{\lambda_\beta}$ and μ_0 is the largest cardinal of $\mathcal{M}^*_{\gamma+1}$.

Proof. If $\beta = \gamma$, then $(\mu_0^+)^{\mathcal{J}^{\mathcal{M}_\gamma}_{\lambda_\gamma}}$ exists (is $< \lambda_\gamma$) since E_γ has index λ_γ on the $\mathcal{M}_\gamma$ sequence. Also, $\mathcal{M}^*_{\gamma+1}$ is the shortest initial segment of $\mathcal{M}_\gamma$ over which a subset of μ_0 not in $\mathcal{J}^{\mathcal{M}_\gamma}_{\lambda_\gamma}$ is definable. Thus $\mu_1 = (\mu_0^+)^{\mathcal{M}^*_{\gamma+1}} = (\mu_0^+)^{\mathcal{J}^{\mathcal{M}_\gamma}_{\lambda_\gamma}} < \lambda_\gamma$, and $\lambda_\gamma \leq \mathrm{OR}^{\mathcal{M}^*_{\gamma+1}}$, which yields the claim.

Now let $\beta < \gamma$. We have $\mu_0 < \nu_\beta \leq \lambda_\beta$, and λ_β is a cardinal of $\mathcal{M}_\gamma$. Also $P(\mu_0) \cap \mathcal{M}_\gamma = P(\mu_0) \cap \mathcal{J}^{\mathcal{M}_\gamma}_{\lambda_\beta} = P(\mu_0) \cap \mathcal{J}^{\mathcal{M}_\beta}_{\lambda_\beta} = P(\mu_0) \cap \mathcal{M}^*_{\gamma+1}$. It follows that $\mu_1 \leq \lambda_\beta$. If $\mu_1 = \mathrm{OR}^{\mathcal{M}^*_{\gamma+1}}$, then as $\lambda_\beta \leq \mathrm{OR}^{\mathcal{M}^*_{\gamma+1}}$, $\mathcal{M}^*_{\gamma+1} = \mathcal{J}^{\mathcal{M}_\beta}_{\lambda_\beta}$ and μ_0 is the largest cardinal of $\mathcal{M}^*_{\gamma+1}$. □

From the proof above we see that if $\beta < \gamma$, then $\mu_1 = (\mu_0^+)^{\mathcal{M}_\gamma}$. Also, claim 4 implies $\mu_1 \leq \kappa_{i-1}$. If $\kappa_{i-1} = \lambda_\beta$ this is obvious. Otherwise κ_{i-1} is a cardinal of $\mathcal{J}^{\mathcal{M}_\beta}_{\lambda_\beta}$, since it is a projectum of some $\mathcal{J}^{\mathcal{M}_\beta}_\eta$ with $\eta \geq \lambda_\beta$. Since $\mu_0 < \kappa_{i-1}$ by the choice of i, $\mu_1 \leq \kappa_{i-1}$.

The next claim shows that Res^γ and $Q^*_{\gamma+1}$ have the agreement required for an application of the shift lemma.

Claim 5. (a) Res^γ agrees with $Q^*_{\gamma+1}$ below $\sup(\sigma \circ \pi_\beta''\mu_1)$,
(b) $\sigma^\gamma \circ \pi_\gamma \restriction \mu_1 = \sigma \circ \pi_\beta \restriction \mu_1$.

Proof.

Subclaim A. $Q^*_{\gamma+1}$ and Res^β agree below $\sup(\sigma \circ \pi_\beta''\mu_1)$, and $\sigma \circ \pi_\beta \restriction \mu_1 = \psi \circ \sigma \circ \pi_\beta \restriction \mu_1$.

Proof. This follows at once from claim 3 and the fact that $\mu_1 \leq \kappa_{i-1}$.

Subclaim A yields claim 5 at once in the case $\beta = \gamma$, so let us assume $\beta < \gamma$.

Subclaim B. If $\beta < \gamma$, then Res^β and Q_γ agree below $\sup(\sigma \circ \pi_\beta''\mu_1)$, and $\psi \circ \sigma \circ \pi_\beta \restriction \mu_1 = \pi_\gamma \restriction \mu_1$.

Proof. Recall that $\psi \circ \sigma \circ \pi_\beta = \sigma^\beta \circ \pi_\beta$. This subclaim therefore follows at once from the fact that $\mathcal{H}$ is a realization of $\Phi(\mathcal{T} \restriction \gamma + 1)$; see clause 2 of 9.11. Notice here that $\mu_1 \leq \lambda_\beta$ by claim 4.

Subclaim C. If $\beta < \gamma$, then Q_γ and Res^γ agree below $\sup(\sigma \circ \pi_\beta''\mu_1)$, and $\pi_\gamma \restriction \mu_1 = \sigma^\gamma \circ \pi_\gamma \restriction \mu_1$.

Proof. $\mu_1 \le \lambda_\beta$, and $\sigma \circ \pi_\beta \restriction \mu_1 = \pi_\gamma \restriction \mu_1$, so $\sup(\sigma \circ \pi_\beta{}''\mu_1) \le \pi_\gamma(\lambda_\beta)$. By claim 1, Q_γ and Res^γ agree below $\pi_\gamma(\lambda_\beta)$, and σ^γ is the identity there.

Together, A, B, and C yield claim 5. □

Let us define

$$\kappa = \sigma^\gamma \circ \pi_\gamma(\mu_0) = \sigma \circ \pi_\beta(\mu_0) = \mathrm{crit}\ F\,.$$

Thus $(\kappa^+)^{Q^*_{\gamma+1}} = \sigma \circ \pi_\beta(\mu_1)$, with the usual understanding if $\mu_1 = \mathrm{OR}^{\mathcal{M}^*_{\gamma+1}}$.

Claim 6. Res^γ agrees with $Q^*_{\gamma+1}$ below $(\kappa^+)^{Q^*_{\gamma+1}} \le (\kappa^+)^{\mathrm{Res}^\gamma}$.

Proof. We prove this slight strengthening of claim 5(a) in the same way that we proved 5(a). First, Res^β and $Q^*_{\gamma+1}$ agree below $(\kappa^+)^{Q^*_{\gamma+1}}$, and $(\kappa^+)^{Q^*_{\gamma+1}} \le (\kappa^+)^{\mathrm{Res}^\beta}$. This is because $\mu_0 < \kappa_{i-1}$, so $\sigma \circ \pi_\beta(\mu_0) = \kappa < \mathrm{crit}\ \psi$, so $(\kappa^+)^{Q^*_{\gamma+1}} \le \mathrm{crit}\ \psi$. This finishes the proof of claim 6 if $\beta = \gamma$, so suppose $\beta < \gamma$. Since $\mu_1 \le \lambda_\beta$, and $(\kappa^+)^{\mathrm{Res}^\beta} = \sigma^\beta \circ \pi_\beta(\mu_1)$, and $\mathcal{H}$ is a realization, we have Res^β agrees with Q_γ below $(\kappa^+)^{\mathrm{Res}^\beta}$ and $(\kappa^+)^{\mathrm{Res}^\beta} \le (\kappa^+)^{Q_\gamma}$. But Q_γ agrees with Res^γ below $\sigma^\gamma \circ \pi_\gamma(\lambda_\beta)$, and $(\kappa^+)^{Q_\gamma} \le \sigma^\gamma \circ \pi_\gamma(\lambda_\beta)$, which completes the proof. □

Claim 7. $V_\kappa^{\mathcal{R}_\beta} = V_\kappa^{\mathcal{R}_\gamma}$.

Proof. $\mu_0 < \nu_\beta$ because $\mathcal{T}$ is an iteration tree, so $\kappa = \sigma \circ \pi_\beta(\mu_0) = \sigma^\beta \circ \pi_\beta(\mu_0) < \sigma^\beta \circ \pi_\beta(\nu_\beta)$. The claim now follows from the fact than $\mathcal{H}$ is a realization; cf. 9.11 (2) (a). □

Now Res^γ is a creature of $\mathbb{C}^{\mathcal{R}_\gamma}$ with an index of the form $(0, \eta)$ in $\mathbb{C}^{\mathcal{R}_\gamma}$. Therefore $\mathcal{R}_\gamma$ has background certificates for the countable fragments of F. Let

$$(N, G) = \quad \text{some } (\sigma^\gamma \circ \pi_\gamma(\nu_\gamma),\ \mathrm{ran}(\sigma^\gamma \circ \pi_\gamma)) - \text{certificate for } F, \text{ as computed in } \mathcal{R}_\gamma\,.$$

Since $\mathrm{Ult}(N, G)$ is closed under ω-sequences, $\sigma^\gamma \circ \pi_\gamma \restriction \nu_\gamma \in \mathrm{Ult}(N, G)$. Let us fix $b \in [lh\ G]^{<\omega}$ and a function $\bar{u} \mapsto \pi(\bar{u})$ mapping $[\kappa]^{|b|}$ into $V_\kappa^{\mathcal{R}_\gamma}$ so that

$$\sigma^\gamma \circ \pi_\gamma \restriction (\lambda_\gamma + 1) = [b, \lambda\bar{u} \cdot \pi(\bar{u})]^N_G\,.$$

Suppose for a moment that case 1 of 9.17 applies, that is, that $\delta_0 \le \alpha$ and δ_0 survives at $\gamma + 1$. It follows that $c(\eta, \gamma + 2) < c(\eta, \gamma + 1)$ for all η such that $\beta \le \eta \le \gamma$. Therefore, for such η, $\mathcal{R}_\eta$ has $\omega \cdot \mathrm{rank}(\mathcal{U}(\eta, \mathcal{R}_\eta, Q_\eta, \pi_\eta)) + c(\eta, \gamma + 2) + 1$ cutoff points, because $\mathcal{H}$ has enough room. Let ξ_η be the last of these cutoff points, and set

$$\begin{aligned} \mathcal{R}^*_\eta \quad = \quad & \text{transitive collapse of} \\ & \mathrm{Hull}^{V_{\xi_\eta}}(V_{\sigma^\eta \circ \pi_\eta(\nu_\eta)} \cup \{\delta^{\mathcal{R}_\eta}, Q_\eta, \pi_\eta\} \cup \sigma^\eta \circ \pi_\eta(\lambda_\eta))\,, \\ & \text{as computed in } \mathcal{R}_\eta\,, \end{aligned}$$

and

$$(Q^*_\eta, \pi^*_\eta) = \text{image of } (Q_\eta, \pi_\eta) \text{ under collapse}.$$

Notice that $\mathcal{R}^*_\eta$ is coded by an element of $V^{\mathcal{R}_\eta}_{\sigma^\eta \circ \pi_\eta(\nu_\eta)+1}$, which is a subset of $\mathcal{R}_\gamma$ because $\mathcal{H}$ is a realization. (Note here $\sigma^\eta \circ \pi_\eta(\lambda_\eta)$ has cardinality $\sigma_\eta \circ \pi_\eta(\nu_\eta)$ in $\mathcal{R}_\eta$.) So $(\mathcal{R}^*_\eta, Q^*_\eta, \pi^*_\eta) \in \mathcal{R}_\gamma$ for all η such that $\beta \leq \eta \leq \gamma$. Set

$$\mathcal{H}^* = \langle (\mathcal{R}^*_\eta, Q^*_\eta, \pi^*_\eta) \mid \beta \leq \eta \leq \gamma \rangle .$$

Then $\mathcal{H}^* \in \mathcal{R}_\gamma$ since $\mathcal{R}_\gamma$ is closed under ω sequences. Clearly, $\mathcal{H}^*$ is coded by a member of $V^{\mathcal{R}_\gamma}_{\sigma^\gamma \circ \pi_\gamma(\nu_\gamma)+1}$. It is easy to check that $(\mathcal{H} \restriction \beta)^\frown \mathcal{H}^*$ is a realization of $\Phi(\mathcal{T} \restriction \gamma+1)$. It may not have enough room as a realization of $\Phi(\mathcal{T} \restriction \gamma+1)$, of course, because we have dropped an ordinal on coordinates η such that $\beta \leq \eta \leq \gamma$. Since G is $\sigma^\gamma \circ \pi_\gamma(\nu_\gamma)+1$ strong in $\mathcal{R}_\gamma$, $\mathcal{H}^* \in \mathrm{Ult}(N, G)$. We may suppose our finite support b was chosen so that for some function $\bar{u} \mapsto \mathcal{H}^*(\bar{u})$ mapping $[\kappa]^{|b|}$ into V_κ,

$$\mathcal{H}^* = [b, \lambda\bar{u} \cdot \mathcal{H}^*(\bar{u})]^N_G .$$

If there is no $\delta \leq \alpha$ such that δ survives at $\gamma+1$, then $\mathcal{H}^*$ is undefined.

Let $k = \deg(\gamma+1)$, and $Q'_{\gamma+1} = \mathrm{Ult}_k(Q^*_{\gamma+1}, F)$. The ultrapower makes sense by claim 6, and it is wellfounded because F has background certificates in $\mathcal{R}_\gamma$, and $\mathcal{R}_\gamma$ is ω-closed. Let $\tau : \mathcal{M}_{\gamma+1} \to Q'_{\gamma+1}$ be given by the shift lemma, that is,

$$\tau\left([a,f]^{\mathcal{M}^*_{\gamma+1}}_{E_\gamma}\right) = [\sigma^\gamma \circ \pi_\gamma(a), \sigma \circ \pi_\beta(f)]^{Q^*_{\gamma+1}}_F .$$

(Here, if $k > 0$, then $\sigma \circ \pi_\beta(f_{r,q}) = f_{r,\sigma\circ\pi_\beta(q)}$ for all terms $r \in Sk_k$ and $q \in \mathcal{M}^*_{\gamma+1}$. For simplicity, we shall use the $k=0$ ultrapower notation.) By the shift lemma, $Q'_{\gamma+1}$ agrees with Q_γ below $\sigma^\gamma \circ \pi_\gamma(\lambda_\gamma)$, and $\tau \restriction \lambda_\gamma = \sigma^\gamma \circ \pi_\gamma \restriction \lambda_\gamma$. Also, τ is a weak k-embedding which is $r\,\Sigma_{\sigma+1}$ elementary on $Y_{\gamma+1}$. We now use the countable completeness of G to reflect τ below κ.

Let $\{x_n \mid n < \omega\}$ be an enumeration of the universe of $\mathcal{M}_{\gamma+1}$, and let $x_n = [\bar{a}_n, \bar{f}_n]^{\mathcal{M}^*_{\gamma+1}}_{E_\gamma}$ where $\bar{a}_n \in [\nu_\gamma]^{<\omega}$. Set $a_n = \sigma^\gamma \circ \pi_\gamma(\bar{a}_n)$ and $f_n = \sigma \circ \pi_\beta(\bar{f}_n)$, so that

$$\tau(x_n) = [a_n, f_n]^{Q^*_{\gamma+1}}_F .$$

For notational reasons, we shall sometimes regard the component measures E_c of an extender E as concentrating on order-preserving $t : c \to \mathrm{crit}(E)$, so that "for E a.e. $t : c \to \mathrm{crit}(E)$, $t \in X$" means that there is a set $Y \in E_c$ such that whenever $t : c \to \mathrm{crit}(E)$ is order preserving and $t''c \in Y$, then $t \in X$. Let us write $I(\beta, Q, \sigma)$ just in case σ is $r\Sigma_k$ elementary on its domain, σ is $r\,\Sigma_{k+1}$ elementary on $\mathrm{dom}\,\sigma \cap Y_\beta$, $\forall i < k(\rho_i(\mathcal{M}_\beta) \in \mathrm{dom}\,\sigma \Rightarrow \sigma(\rho_i(\mathcal{M}_\beta)) = \rho_i(Q))$, and $\sigma''\rho_k(\mathcal{M}_\beta) \subseteq \rho_k(Q)$. Thus, if $\pi : \mathcal{M}_\beta \to Q$, then π is a weak k -embedding from $\mathcal{M}_\beta$ into Q which is $r\,\Sigma_{k+1}$ elementary on $Y_\beta \Leftrightarrow (\forall \text{ finite } F \subseteq \mathcal{M}_\beta) I(\beta, Q, \pi \restriction F)$.

For $t : (b \cup a_0 \cdots \cup a_n) \to \kappa$ order preserving, let

$$\varphi_t^n(x_i) = f_i(t''a_i)$$

for all $i \leq n$.

Claim 8. Let $n < \omega$ and let $c = b \cup a_0 \cup \cdots \cup a_n$. Then there is a set $W_n \in G_c$ such that whenever $t : c \to \kappa$ is order preserving and $t''c \in W_n$,

(i) $I(\gamma + 1, Q^*_{\gamma+1}, \varphi_t^n)$
(ii) if $i^*_{\gamma+1}(y) \in \operatorname{dom} \varphi_t^n$, then $\varphi_t^n(i^*_{\gamma+1}(y)) = \sigma \circ \pi_\beta(y)$,
(iii) if $x_n < \lambda_\gamma$, then $\varphi_t^n(x_n) = \pi(t''b)(x_n)$, and
(iv) if $x_n = \lambda_\gamma$, then $\varphi_t^n(x_n) \geq \pi(t''b)(x_n)$.

Proof. We first show that (ii) holds for G a.e. $t : c \to \kappa$. Let $i^*_{\gamma+1}(y) = x_i = [\bar{a}_i, \bar{f}_i]_{E_\gamma}^{\mathcal{M}^*_{\gamma+1}}$, where $i \leq n$. Then $\bar{f}_i(\bar{u}) = y$ for $(E_\gamma)_{\bar{a}_i}$ a.e. $\bar{u}$. Since $\sigma \circ \pi_\beta$ and $\sigma^\gamma \circ \pi_\gamma$ agree on $P(\mu_0)$, this means that $f_i(\bar{u}) = \sigma \circ \pi_\beta(y)$ for F_{a_i} a.e. $\bar{u}$. The set of such $\bar{u}$ is in $\operatorname{ran}(\sigma^\gamma \circ \pi_\gamma)$, so $f_i(t''a_i) = \sigma \circ \pi_\beta(y)$ for G a.e. $t : c \to \kappa$. Since $\varphi_t^n(i^*_{\gamma+1}(y)) = f_i(t''a_i)$, we are done.

Next, we show (i) holds G a.e. First, let $\rho(v_0 \cdots v_n)$ be an $r\Sigma_k$ formula. Then

$$\begin{aligned} \mathcal{M}_{\gamma+1} \models \rho[x_0 \cdots x_n] \quad & \text{iff } Q'_{\gamma+1} \models \rho[\tau(x_0) \cdots \tau(x_n)] \\ & \text{iff for } F \text{ a.e. } t : \bigcup_{i \leq n} a_i \to \kappa\,, \\ & \quad Q^*_{\gamma+1} \models \rho[f_0(t''a_0) \cdots f_n(t''a_n)] \\ & \text{iff for } G \text{ a.e. } t : c \to \kappa\,, \\ & \quad Q^*_{\gamma+1} \models \rho[\varphi_t^n(x_0) \cdots \varphi_t^n(x_n)]\,. \end{aligned}$$

Notice, for the third equivalence above, that the appropriate set of $\bar{u}$ is in the range of $\sigma^\gamma \circ \pi_\gamma$, so that $F_{\bigcup_i a_i}$ and $G_{\bigcup_i a_i}$ give it the same measure. Second, we show φ_t^n is $r\Sigma_{k+1}$ elementary on $Y_{\gamma+1} \cap \{x_0 \cdots x_n\}$, for G a.e. $t : c \to \kappa$. Notice here that $Y_{\gamma+1} = i^{*\,\prime\prime}_{\gamma+1} Z$, where $\sigma \circ \pi_\beta$ is $r\Sigma_{k+1}$ elementary on Z. [If $\gamma + 1 \notin D^{\mathcal{T}}$ and $k = \deg(\gamma + 1) = \deg(\beta)$, then $Z = Y_\beta$, and $\sigma \circ \pi_\beta = \pi_\beta$ is $r\Sigma_{k+1}$ elementary on Y_β because $\mathcal{H}$ is a $\boldsymbol{Y}$-realization. Otherwise, Z is the universe of $\mathcal{M}^*_{\gamma+1}$, σ is a full k-embedding, and π_β is at least $r\Sigma_{k+1}$ as a map from $\mathcal{M}^*_{\gamma+1}$ to $\mathcal{J}_\eta^{Q_\beta}$, where $\omega\eta = \pi_\beta(\mathrm{OR} \cap \mathcal{M}^*_{\gamma+1})$.] Thus, if we set $Y_{\gamma+1} \cap \{x_0 \cdots x_n\} = \{i^*_{\gamma+1}(y_0), \cdots, i^*_{\gamma+1}(y_m)\}$, then we have for all $r\Sigma_{k+1}$ formulae ρ

$$\begin{aligned} \mathcal{M}_{\gamma+1} \models \rho[i^*_{\gamma+1}(y_0) \cdots i^*_{\gamma+1}(y_m)] & \text{ iff } \mathcal{M}^*_{\gamma+1} \models \rho[y_0 \cdots y_m] \\ & \text{iff } Q^*_{\gamma+1} \models \rho[\sigma \circ \pi_\beta(y_0) \cdots \sigma \circ \pi_\beta(y_m)] \\ & \text{iff for } G \text{ a.e. } t : c \to \kappa \\ & \quad Q^*_{\gamma+1} \models \rho[\varphi_t^n(i^*_{\gamma+1}(y_0)) \cdots \varphi_t^n(i^*_{\gamma+1}(y_m))]\,. \end{aligned}$$

This completes the proof of (i).

We now prove (iii). Let $x_n < \lambda_\gamma$, and assume first that $\lambda_\gamma = \nu_\gamma$. Since $x_n < \nu_\gamma$,

$$[\bar{a}_n, \bar{f}_n]_{E_\gamma}^{\mathcal{M}^*_{\gamma+1}} = [\{x_n\}, \mathrm{id}]_{E_\gamma}^{\mathcal{M}^*_{\gamma+1}},$$

so

$$[a_n, f_n]_F^{Q^*_{\gamma+1}} = [\{\sigma^\gamma \circ \pi_\gamma(x_n)\}, \mathrm{id}]_F^{Q^*_{\gamma+1}},$$

because of the agreement between $\sigma \circ \pi_\beta$ and $\sigma^\gamma \circ \pi_\gamma$. Letting $d = a_n \cup \{\sigma^\gamma \circ \pi_\gamma(x_n)\}$, this means that for F a.e. $t : d \to \kappa$, $f_n(t''a_n) = t(\sigma^\gamma \circ \pi_\gamma(x_n))$. Because the set of all $t''d$ for which this equation holds is in $\mathrm{ran}(\sigma^\gamma \circ \pi_\gamma)$, we get that

$$f_n(t''a_n) = t(\sigma^\gamma \circ \pi_\gamma(x_n)), \text{ for } G \text{ a.e. } t.$$

But also,

$$[\{\sigma^\gamma \circ \pi_\gamma(x_n)\}, \mathrm{id}]_G^N = \sigma_\gamma \circ \pi_\gamma(x_n) = [b, \lambda\bar{u} \cdot \pi(\bar{u})]_G^N\ (x_n),$$

and since x_n is countable in N, it is represented by the constantly x_n function in $\mathrm{Ult}(N, G)$. By Los' theorem for $\mathrm{Ult}(N, G)$,

$$\pi(t''b)(x_n) = t(\sigma^\gamma \circ \pi_\gamma(x_n)), \text{ for } G \text{ a.e. } t.$$

This finishes the proof of (iii) in case $\lambda_\gamma = \nu_\gamma$.

If $\nu_\gamma < \lambda_\gamma$, then $\nu_\gamma = \nu + 1$ where ν is the largest generator of E_γ, and $\lambda_\gamma = lh\ E_\gamma = (\nu^+)^{\mathrm{Ult}(\mathcal{M}^*_{\gamma+1}, E_\gamma)}$.

If $x_n < \nu_\gamma$, the proof in the first case applies, so assume $x_n \geq \nu_\gamma$. We then get a function $\bar{g} \in V_{\mu_0+1}^{\mathcal{M}^*_{\gamma+1}}$ such that

$$[\bar{a}_n, \bar{g}]_{E_\gamma}^{\mathcal{M}^*_{\gamma+1}} = \text{some wellorder of } \nu \text{ of order type } x_n.$$

Applying the shift lemma map τ to this fact, with $g = \sigma \circ \pi_\beta(\bar{g}) = \sigma^\gamma \circ \pi_\gamma(\bar{g})$,

$$[a_n, g]_F^{Q^*_{\gamma+1}} = \text{some wellorder of } \sigma^\gamma \circ \pi_\gamma(\nu) \text{ of order type } \sigma^\gamma \circ \pi_\gamma(x_n).$$

But now F_d agrees with G_d on all sets in $\mathrm{ran}(\sigma^\gamma \circ \pi_\gamma)$, whenever $d \in [\sigma^\gamma \circ \pi_\gamma(\nu + 1)]^{<\omega}$. This implies

$$[a_n, g]_F^{Q^*_{\gamma+1}} = [a_n, g]_G^N.$$

It follows that for G a.e. t, $g(t''a_n)$ is a wellorder of order type $t(\sigma^\gamma \circ \pi_\gamma(x_n))$. We also have that for F a.e. t, hence for G a.e. t, $g(t''a_n)$ has order type $f(t''a_n)$. So we get that $f(t''a_n) = t(\sigma^\gamma \circ \pi_\gamma(x_n))$ for G a.e. t. Now we can finish the proof of (iii) as in the first case.

We leave the proof of (iv) to the reader. The main point is that $[a_n, f_n]_G^N \geq \sigma^\gamma \circ \pi_\gamma(\lambda_\gamma)$.

(We may assume $\bar{f}_n \in V^{\mathcal{M}^*_{\gamma+1}}_{\mu_0+1}$, so that $f_n = \sigma^\gamma \circ \pi_\gamma(\bar{f}_n)$ is in $\mathrm{ran}(\sigma^\gamma \circ \pi_\gamma)$.) This follows from the agreement between F and G; the proof breaks into the cases $\nu_\gamma = \lambda_\gamma$ and $\nu_\gamma < \lambda_\gamma$ as did the proof of (iii).

This completes the proof of claim 8.

We can now finish the proof of 9.17 in case 1, the case that for some $\delta \leq \alpha$, α survives at $\gamma+1$. For $\beta \leq \eta \leq \gamma$, let σ^*_η be the complete resurrection embedding in $\mathcal{R}^*_\eta$ for Q^*_η from $\pi^*_\eta(\lambda_\eta)$. Then $\sigma^*_\eta \restriction \lambda_\eta = \sigma^\gamma \circ \pi_\gamma \restriction \lambda_\eta$, and $\sigma^*_\eta \circ \pi^*_\eta(\lambda_\eta) \leq \sigma^\gamma \circ \pi_\gamma(\lambda_\eta)$ for all $\eta \leq \gamma$; this one sees from the construction of $\mathcal{H}^*$. This agreement is a fact about $\mathcal{H}^*$ and $\sigma^\gamma \circ \pi_\gamma \restriction (\lambda_\gamma + 1)$ in $\mathrm{Ult}(N, G)$; by Los' theorem we get a set $X \in G_b$ such that for all $\bar{u} \in X$

$$(\sigma^*_\eta \circ \pi^*_\eta)(\bar{u}) \restriction \lambda_\eta = \pi(\bar{u}) \restriction \lambda_\eta$$

and

$$(\sigma^*_\eta \circ \pi^*_\eta)(\bar{u})(\lambda_\eta) \leq \pi(\bar{u})(\lambda_\eta)$$

for $\beta \leq \eta \leq \gamma$. Here $(\sigma^*_\eta \circ \pi^*_\eta)(\bar{u}) = \sigma^*_\eta(\bar{u}) \circ \pi^*_\eta(\bar{u})$, where $\mathcal{H}^*_\eta(\bar{u}) = (\mathcal{R}^*_\eta(\bar{u}), Q^*_\eta(\bar{u}), \pi^*_\eta(\bar{u}))$ and $\sigma^*_\eta(\bar{u})$ is the complete resurrection of $\pi^*_\eta(\bar{u})(\lambda_\eta)$ from $Q^*_\eta(\bar{u})$ in $\mathcal{R}^*_\eta(\bar{u})$. We can also arrange that for $\bar{u} \in X$,

$$\sigma^\gamma \circ \pi_\gamma \restriction \mu_0 = \pi(\bar{u}) \restriction \mu_0\,,$$

because $\sigma^\gamma \circ \pi_\gamma{}'' \mu_0$ is just a countable subset of $\kappa = \mathrm{crit}\ G$, and G is countably complete. Finally, we can arrange that for $\bar{u} \in X$, $\mathcal{R}^*_\eta(\bar{u})$ has $\omega \cdot \mathrm{rank}(\mathcal{U}(\eta, \mathcal{R}^*_\eta(\bar{u}), Q^*_\eta(\bar{u}), \pi^*_\eta(\bar{u})) + c(\eta, \gamma+2)$ cutoff points.

Now let W_n be as in claim 8, for all $n < \omega$, and let $t : b \cup \bigcup_{n<\omega} a_n \to \kappa$ be order preserving and such that $t''b \in X$ and $t''(b \cup a_0 \cdots \cup a_n) \in W_n$ for all n. Such a t exists because G is countably complete. Set $\varphi(x_n) = f_n(t''a_n)$ for all $n < \omega$, and

$$\mathcal{F} = \mathcal{H} \restriction \beta^\frown \mathcal{H}^*(t''b)^\frown \langle (\mathcal{R}_\beta, Q^*_{\gamma+1}, \varphi) \rangle\,.$$

It is easy to verify that $\mathcal{F}$ fulfills the requirements of 9.17 as a realizations of $\Phi(\mathcal{T} \restriction \gamma+2)$ in case 1.

Now let us prove 9.17 in case 2, the case that α is a break point at $\gamma+1$ and β does not survive at $\gamma+1$. From claim 8 and the countable completeness of G we get

Claim 9. For G_b a.e. $\bar{u}$, there is a $(\deg(\gamma+1), Y)$ embedding $\varphi : \mathcal{M}_{\gamma+1} \to Q^*_{\gamma+1}$ such that

(a) $\varphi \restriction \lambda_\gamma = \pi(\bar{u}) \restriction \lambda_\gamma$,
(b) $\varphi(\lambda_\gamma) \geq \pi(\bar{u})(\lambda_\gamma)$,
(c) $\varphi \circ i^*_{\gamma+1} = \sigma \circ \pi_\beta$.

Now if $\bar{u}$ and φ are as in claim 9, then $(\gamma+1, \varphi, Q^*_{\gamma+1})$ is a node of the tree $\mathcal{U}(\beta, \mathcal{R}^{\mathcal{H}}_\beta, Q^{\mathcal{H}}_\beta, \pi^{\mathcal{H}}_\beta)$. (The fact that β does not survive at $\gamma+1$ is relevant here.) Moreover, $\mathcal{U}(\gamma+1, \mathcal{R}^{\mathcal{H}}_\beta, Q^*_{\gamma+1}, \varphi)$ is isomorphic to the subtree of $\mathcal{U}(\beta, \mathcal{R}^{\mathcal{H}}_\beta, Q^{\mathcal{H}}_\beta, \pi^{\mathcal{H}}_\beta)$ consisting of nodes below $(\gamma+1, \varphi, Q^*_{\gamma+1})$. It follows that $\mathcal{R}^{\mathcal{H}}_\beta$ has an $\omega \cdot \mathrm{rank}(\mathcal{U}(\gamma+1, \mathcal{R}^{\mathcal{H}}_\beta, Q^*_{\gamma+1}, \varphi)) + c(\gamma+1, \gamma+2) + 1^{\mathrm{st}}$ cutoff point η. Working in $\mathcal{R}^{\mathcal{H}}_\beta$, we can form a Skolem hull of $V_\eta^{\mathcal{R}^{\mathcal{H}}_\beta}$ containing $V^{\mathcal{R}^{\mathcal{H}}_\beta}_{\pi(\bar{u})(\lambda_\gamma)} \cup \{Q^*_{\gamma+1}\varphi\}$, closed under ω-sequences and having size $< \kappa$. The collapse of this hull belongs to $V_\kappa^{\mathcal{R}^{\mathcal{H}}_\beta} = V_\kappa^N$. This gives us

Claim 10. For G_b a.e. $\bar{u}$, there is a triple $(\mathcal{R}, Q, \varphi)$ such that
(a) $(\mathcal{R}, Q, \varphi)$ is a $(\deg(\gamma+1), Y)$ realization of $\mathcal{M}_{\gamma+1}$,
(b) $(\mathcal{R}, Q, \varphi) \in V_\kappa^N$,
(c) $\mathcal{R}$ has $\omega \cdot \mathrm{rank}(\mathcal{U}(\gamma+1, \mathcal{R}, Q, \varphi)) + c(\gamma+1, \gamma+2)$ cutoff points,
(d) Q agrees with Q_γ below $\pi(\bar{u})(\lambda_\gamma)$, and $\mathcal{R}$ agrees with N below $\pi(\bar{u})(\lambda_\gamma)$,
and
(e) $\varphi \restriction \lambda_\gamma = \pi(\bar{u}) \restriction \lambda_\gamma$ and $\varphi(\lambda_\gamma) \geq \pi(\bar{u})(\lambda_\gamma)$.

By the axiom of choice in N, there is in N a function $f(\bar{u}) = (\mathcal{R}(\bar{u}), Q(\bar{u}), \varphi(\bar{u}))$ which picks, for each $\bar{u}$ in the relevant G_b-measure one set, a triple satisfying claim 10. Let

$$\mathcal{F} = \mathcal{H}^\frown [b, \lambda\bar{u} \cdot (\mathcal{R}(\bar{u}), Q(\bar{u}), \varphi(\bar{u}))]^N_G .$$

It is easy to see that $\mathcal{F}$ is a realization of the phalanx $\Phi(\mathcal{T} \restriction \gamma+2)$; the necessary agreement of models and embeddings comes from parts (d) and (e) of claim 10. Part (c) of claim 10 implies that $\mathcal{F}$ has enough room. As case 2 governed our definition of $\mathcal{H}$, $\mathcal{E} \restriction \alpha+1 = \mathcal{H} \restriction \alpha+1$ and $\mathcal{R}^{\mathcal{H}}_\gamma \subseteq \mathcal{R}^{\mathcal{E}}_\alpha$. It follows that $\mathcal{F} \restriction \alpha+1 = \mathcal{E} \restriction \alpha+1$, and since $\mathcal{R}^{\mathcal{F}}_{\gamma+1} \in \mathrm{Ult}(N, G) \subseteq \mathcal{R}^{\mathcal{H}}_\gamma$, we have $\mathcal{R}^{\mathcal{F}}_{\gamma+1} \in \mathcal{R}^{\mathcal{E}}_\alpha$. Thus $\mathcal{F}$ witnesses the truth of 9.17 (1).

This finishes the successor step in the inductive proof of 9.17.

Now let η be a limit ordinal, and $\alpha_0 \leq \alpha < \eta$. Let $\beta T \eta$, where β is large enough that $\alpha < \beta$, β survives at η, $D^{\mathcal{T}} \cap [\beta, \eta]_T = \phi$, and $\deg(\beta) = \deg(\eta)$. Let $\langle \beta_n \mid n \in \omega \rangle$ be such that $\beta_0 = \beta$, and $\beta_n T \beta_{n+1} T \eta$ for all n, and $\eta = \sup\{\beta_n \mid n \in \omega\}$. Let $\mathcal{E}$ be our given realization of $\Phi(\mathcal{T} \restriction \alpha+1)$.

Suppose first that α is a break point at η. Then α is a break point at β, so by induction we have a realization $\mathcal{F}_0$ of $\Phi(\mathcal{T} \restriction \beta+1)$ which has enough room. We also get $\mathcal{F}_0 \restriction \alpha+1 = \mathcal{E}$, and $\mathcal{R}^{\mathcal{F}_0}_\beta \in \mathcal{R}^{\mathcal{E}}_\alpha$. Now suppose $\mathcal{F}_n$ realizing $\Phi(\mathcal{T}) \restriction (\beta_n+1)$ is given. Since β_n survives at β_{n+1}, and between β_n and β_{n+1} there is no dropping in model or degree, our induction hypothesis gives a realization $\mathcal{F}_{n+1}$ of $\Phi(\mathcal{T} \restriction \beta_{n+1}+1)$ having enough room, and such that $\mathcal{F}_{n+1} \restriction \beta_n = \mathcal{F}_n \restriction \beta_n$, $\mathcal{R}^{\mathcal{F}_n}_{\beta_n} = \mathcal{R}^{\mathcal{F}_{n+1}}_{\beta_{n+1}}$ and $Q^{\mathcal{F}_n}_{\beta_n} = Q^{\mathcal{F}_{n+1}}_{\beta_{n+1}}$, and $\pi^{\mathcal{F}_n}_{\beta_n} = \pi^{\mathcal{F}_{n+1}}_{\beta_{n+1}} \circ i^{\mathcal{T}}_{\beta_n\beta_{n+1}}$. Let

$$\mathcal{F} = \bigcup_n \mathcal{F}_n \restriction \beta_n{}^\frown \langle \mathcal{R}^{\mathcal{F}_0}_\beta Q^{\mathcal{F}_0}_\beta, \pi \rangle ,$$

where for $x \in \mathcal{M}_\eta$ we define $\pi(x)$ by

$$x = i^{\mathcal{T}}_{\beta_n,\eta}(y) \Rightarrow \pi(x) = \pi^{\mathcal{F}_n}_{\beta_n}(y)\,.$$

It is easy to check that $\mathcal{F}$ witnesses 9.17 (1) for α and η.

Next, suppose $\delta_0 \leq \alpha$ is largest such that δ_0 survives at η. Let $\langle \beta_n \mid n < \omega\rangle$ be such that T-pred$(\beta_0) = \delta_0$, $\beta_n T \beta_{n+1} T \eta$ for all n, and $\sup_n \beta_n = \eta$. By induction hypothesis 9.17 (2) we get a realization $\mathcal{F}_0$ of $\Phi(\mathcal{T} \restriction \beta_0 + 1)$ which has enough room, and such that $\mathcal{F}_0 \restriction \delta_0 = \mathcal{E} \restriction \delta_0$ and $\mathcal{R}^{\mathcal{F}_0}_{\beta_0} = \mathcal{R}^{\mathcal{E}}_{\delta_0}$, and $Q^{\mathcal{F}_0}_{\beta_0}$ is related as required to $Q^{\mathcal{E}}_{\delta_0}$, and $\pi^{\mathcal{E}}_{\delta_0} = \pi^{\mathcal{F}_0}_{\beta_0} \circ i^{\mathcal{T}}_{\delta_0,\beta_0}$ if this is required. We can then use induction hypothesis 9.17 (2) repeatedly as in the last paragraph, and we easily get 9.17 (2) at η. This completes the proof of Lemma 9.17. □

We can now easily complete the proof of 9.14. Suppose first that θ is a limit ordinal. For $0 \leq i < \omega$, let α_{i+1} be defined by

$$n^*(\alpha_{i+1}) = \inf\{n^*(\beta) \mid \alpha_i < \beta < \theta\}\,.$$

(Recall that $\alpha_0 = lh(\mathcal{B}_0) - 1$.) Clearly, $\alpha_i < \alpha_{i+1}$ and α_{i+1} is a break point at θ, for all i. We may suppose that n^* was chosen so that $n^*(\alpha_0) = 0$, which means that α_0 is a break point at θ. But then 9.17 (1) gives us a sequence $\langle \mathcal{F}_i \mid i \in \omega\rangle$ such that $\mathcal{F}_0 = \mathcal{E}_0$, $\mathcal{F}_i$ is a realization of $\Phi(\mathcal{T} \restriction \alpha_i + 1)$, and $\mathcal{R}^{\mathcal{F}_{i+1}}_{\alpha_{i+1}} \in \mathcal{R}^{\mathcal{F}_i}_{\alpha_i}$, for all $i < \omega$. This is a contradiction.

Next, suppose $\theta = \gamma + 1$. We may suppose n^* is chosen so that $n^*(\gamma) = 0$, which implies that β survives at γ whenever $\beta T \gamma$. But then 9.17 (2) clearly implies that $\mathcal{M}^{\mathcal{T}}_\gamma$ is $\mathcal{E}_0$-realizable, as desired. □

Theorem 2.5 obviously follows from 9.14. (While 9.14 was only proved for normal trees, whereas 2.5 was stated for linear compositions of normal trees, we can nevertheless take care of such "almost normal" trees by applying 9.14 (2) repeatedly to their normal components.) The iterability of the exotic creatures of $\mathbb{C}$ which we used in the proof of 1.4 also follows immediately. This represents all the iterability we used in §1 - §5.

It remains only to prove Theorem 6.9, which states that if $K^c \models$ "There are no Woodin cardinals", then every K^c generated phalanx $\mathcal{B}$ such that $lh\ \mathcal{B} < \Omega$ is $\Omega + 1$-iterable. We shall now sketch the minor modifications of the proof of 9.14 which yield this result.

First, the reflection arguments of §2 show that it is enough to prove the following: let $\pi : \mathcal{M} \to K^c$ be elementary, where $\mathcal{M}$ is countable. Let $\mathcal{B}$ be a hereditarily countable phalanx which is $(\Sigma, \mathcal{M})$-generated, where Σ is the strategy of choosing unique cofinal wellfounded branches. Let $\mathcal{T}$ be a countable, normal, putative iteration tree on $\mathcal{B}$. Then either $\mathcal{T}$ has a cofinal wellfounded branch or $\mathcal{T}$ has a last, wellfounded model. So fix π, $\mathcal{M}$, and $\mathcal{B}$ with these properties; we shall show that there is a realization $\mathcal{E}$ of $\mathcal{B}$ such that $\forall \alpha < lh\ (\mathcal{B})$ ($\mathcal{R}^{\mathcal{E}}_\alpha$ has $\delta^{\mathcal{R}^{\mathcal{E}}_\alpha}$ cutoff points). The desired conclusion concerning $\mathcal{T}$ then follows from 9.14.

Since Ω is measurable, we can find $\xi < \Omega$ such that $\pi : \mathcal{M} \to \mathcal{N}_\xi$ is elementary. Let η be such that $(V_\eta, \in, \Omega)$ is a coarse premouse having $\Omega + \Omega$ cutoff points, and let $\mathcal{E}_0 = ((V_\eta, \in, \Omega), \mathcal{N}_\xi, \pi)$. Thus $\mathcal{E}_0$ is a realization of $\mathcal{M}$.

Now fix $\alpha < lh(\mathcal{B})$, and let $\mathcal{S}$ be a countable iteration tree on $\mathcal{M}$ such that $\mathcal{M}^{\mathcal{B}}_\alpha$ is an initial segment of $\mathcal{M}^{\mathcal{S}}_\gamma$, the last model of $\mathcal{S}$, and such that $\mathcal{S}$ has no maximal wellfounded branches. Such a tree $\mathcal{S}$ exists because $\mathcal{B}$ is $(\Sigma, \mathcal{M})$ generated. We have by definition 6.7: if $\alpha + 1 < lh(\mathcal{B})$, then $\forall\gamma(\nu(E^{\mathcal{S}}_\gamma) \geq \lambda(\alpha, \mathcal{B}))$, and if $\alpha + 1 = lh(\mathcal{B})$, then

$$\forall\gamma\forall\beta < \alpha(\nu(E^{\mathcal{S}}_\gamma) \geq \lambda(\beta, \mathcal{B}))\,.$$

We now apply a slight variant of 9.17 to find the desired realization $(\mathcal{R}^{\mathcal{E}}_\alpha, Q^{\mathcal{E}}_\alpha, \pi^{\mathcal{E}}_\alpha)$ of $\mathcal{M}^{\mathcal{B}}_\alpha$. Let $n^* : \gamma + 1 \to \omega$ be 1-1 and $n^*(0) = 0$. Let $n : \gamma + 1 \to \omega$ be defined from n^* as in the proof of 9.14, and interpret "survives" and "break point" relative to n as in 9.14. Thus 0 is a break point at γ. For $\beta < \gamma$, and $(\mathcal{R}, Q, \sigma)$ a $\deg^{\mathcal{S}}(\beta)$ realization of $\mathcal{M}^{\mathcal{S}}_\beta$, let $U(\beta, \mathcal{R}, Q, \sigma)$ be defined as in the proof of 9.14: it is the tree of attempts to build a maximal branch b of $\mathcal{S}$ and realize $\mathcal{M}^{\mathcal{S}}_b$ appropriately. Since $\mathcal{S}$ has no maximal wellfounded branches, $U(\beta, \mathcal{R}, Q, \sigma)$ is always wellfounded.

For $\mathcal{F}$ a realization of $\Phi(\mathcal{S} \restriction \xi)$, let us say that $\mathcal{F}$ has *more than enough room* just in case $\forall\beta < \xi(\mathcal{R}^{\mathcal{F}}_\beta$ has $\delta^{\mathcal{R}^{\mathcal{F}}_\beta} + \omega \cdot \operatorname{rank}(U(\beta, \mathcal{R}^{\mathcal{F}}, Q^{\mathcal{F}}_\beta, \pi^{\mathcal{F}}_\beta)) + c(\beta, \gamma)$ cutoff points), where of course $c(\beta, \gamma)$ is defined as in the proof of 9.14. So $\mathcal{E}_0$ has more than enough room. It is clear that the proof of 9.17 works equally well when "more than enough room" replaces "enough room" in its hypothesis and conclusion. Since 0 is a break point at γ, this version of 9.17 gives us a realization $\mathcal{F}$ of $\Phi(\mathcal{S})$ such that $\mathcal{E}_0 = \mathcal{F} \restriction 1$ and $\mathcal{R}^{\mathcal{F}}_\gamma$ has $\delta^{\mathcal{R}^{\mathcal{F}}_\gamma}$ cutoff points. Since $\mathcal{F}$ is a realization of $\Phi(\mathcal{S})$, $Q^{\mathcal{F}}_\gamma$ agrees with $Q^{\mathcal{F}}_0 = \mathcal{N}_\xi$ below $\sigma \circ \pi(\nu(E^{\mathcal{S}}_0))$ and $\pi^{\mathcal{F}}_\gamma \restriction \nu(E^{\mathcal{S}}_0) = \sigma \circ \pi \restriction \nu(E^{\mathcal{S}}_0) = \sigma \circ \pi \restriction \nu(E^{\mathcal{S}}_0)$, where σ is the appropriate complete resurrection embedding (bringing $\pi(E^{\mathcal{S}}_0)$ back to life). Now whenever $\beta \leq \alpha$ and $\lambda(\beta, \mathcal{B})$ is defined (i.e. $\beta + 1 < lh(\mathcal{B})$), $\lambda(\beta, \mathcal{B})$ is a cardinal of $\mathcal{M}$ and $\nu(E^{\mathcal{S}}_0) \geq \lambda(\beta, \mathcal{B})$. Thus $\pi(\lambda(\beta, \mathcal{B}))$ is a cardinal of K^c and $\sigma \restriction \pi(\lambda(\beta, \mathcal{B})) + 1$ is the identity. This implies that $\pi^{\mathcal{F}}_\gamma \restriction (\lambda(\beta, \mathcal{B}) + 1) = \pi \restriction (\lambda(\beta, \mathcal{B}) + 1)$, whenever $\beta \leq \alpha$ and $\lambda(\beta, \mathcal{B})$ exists. Let us set $\mathcal{R}^{\mathcal{E}}_\alpha = \mathcal{R}^{\mathcal{F}}_\gamma$, $\pi^{\mathcal{E}}_\alpha = \pi^{\mathcal{F}}_\gamma \restriction \mathcal{M}^{\mathcal{B}}_\alpha$, and $Q^{\mathcal{E}}_\alpha = Q^{\mathcal{F}}_\gamma$ if $\mathcal{M}^{\mathcal{B}}_\alpha = \mathcal{M}^{\mathcal{S}}_\gamma$, and $Q^{\mathcal{E}}_\alpha = \pi^{\mathcal{F}}_\gamma(\mathcal{M}^{\mathcal{B}}_\alpha)$ otherwise. Doing this for all $\alpha < lh(\mathcal{B})$, we obtain a realization $\mathcal{E}$ of $\mathcal{B}$ which has enough room; the agreement properties of $\mathcal{E}$ follow from the fact that if $\alpha + 1 < lh\ \mathcal{B}$, then $\pi^{\mathcal{E}}_\alpha \restriction (\lambda(\alpha, \mathcal{B}) + 1) = \pi \restriction (\lambda(\alpha, \mathcal{B}) + 1)$ and $Q^{\mathcal{E}}_\alpha$ agrees with $\mathcal{N}_\xi$ below $\pi(\lambda(\alpha, \mathcal{B}) + 1)$, and from the corresponding facts when $\alpha + 1 = lh(\mathcal{B})$.

From 9.14 we get that the tree $\mathcal{T}$ on $\mathcal{B}$ is well-behaved, and this completes the proof of 6.9.