§5. The construction of true K

The model K^c constructed in §1 depends too heavily on the universe within which it is constructed to serve our purposes. In this section we isolate a certain Skolem hull K of K^c, and prove that $K^V = K^{V[G]}$ whenever G is generic over V for a poset $\mathbb{P} \in V_\Omega$. The uniqueness result underlying this fact descends ultimately from Kunen's proof of the uniqueness of $L[\mu]$ ([Ku1]), and is based on the following lemma.

Lemma 5.1. *Let M and N be weasels which have the S-hull and S-definability properties at all $\beta < \alpha$. Let $(\mathcal{T},\mathcal{U})$ be a successful coiteration of M with N, let W be the common last model of $\mathcal{T}$ and $\mathcal{U}$, and let $i : M \to W$ and $j : N \to W$ be the iteration maps. Then $i \restriction \alpha = j \restriction \alpha =$ identity.*

Proof. Suppose not, and let $\kappa = \inf(\text{crit}(i),\ \text{crit}(j))$. Without loss of generality, let $\kappa = \text{crit}(i)$. We claim first that $\kappa = \text{crit}(j)$. For let

$$\Delta = \{\gamma < \Omega \mid i(\gamma) = j(\gamma) = \gamma\},$$

and recall that Δ is S-thick in M and N. Now $\kappa \notin H^W(\Delta)$, since otherwise κ is the range of i. On the other hand, N has the S-definability property at κ since $\kappa < \alpha$. Thus $\kappa \in H^N(\Delta)$, and if $\kappa < \text{crit}(j)$, then $\kappa \in H^W(\Delta)$. So $\kappa = \text{crit}(j)$.

We can now finish the proof as in 4.5. Let $A \subseteq \kappa$ and $A \in M$; we claim that $A \in N$ and $i(A) \cap \nu = j(A) \cap \nu$, where $\nu = \inf(i(\kappa), j(\kappa))$. For by the S-hull property of M at κ, we can find $\bar{\beta} \in \Delta^{<\omega}$ and a Skolem term τ such that $A = \tau^M(\bar{\beta}) \cap \kappa$. (Notice that $\kappa \subseteq \Delta$.) But then $i(A) = \tau^W(\bar{\beta}) \cap i(\kappa)$, so $A = \tau^W(\bar{\beta}) \cap \kappa = j(\tau^N(\bar{\beta})) \cap \kappa$. Since $\text{crit}(j) = \kappa$, this implies that $A = \tau^N(\bar{\beta}) \cap \kappa$, so that $A \in N$. Also $j(A) = \tau^W(\bar{\beta}) \cap j(\kappa)$, and therefore $i(A) \cap \nu = j(A) \cap \nu$ where $\nu = \inf(i(\kappa), j(\kappa))$.

A symmetric proof shows that if $A \subseteq \kappa$ and $A \in N$, then $A \in M$ and $i(A) \cap \nu = j(A) \cap \nu$. Let E and F be the first extenders used on the branches M-to-W and N-to-W of $\mathcal{T}$ and $\mathcal{U}$ respectively, and let $\theta = \inf(\nu(E),\ \nu(F))$, so that $\theta < \nu$. Then $i_E(A) \cap \theta = i(A) \cap \theta = j(A) \cap \theta = i_F(A) \cap \theta$ for A in $P(\kappa)^M$. It follows that $E \restriction \theta = F \restriction \theta$; on the other hand, since $(\mathcal{T},\mathcal{U})$ is a coiteration, no extender used in $\mathcal{T}$ is compatible with any extender used in $\mathcal{U}$. This contradiction completes the proof. □

Corollary 5.2. *Let M be an $\Omega{+}1$ iterable weasel which has the S-definability property at all $\beta < \alpha$; then M has the S-hull property at α.*

Proof. By induction we may suppose M has the S-hull property at all $\beta < \alpha$. Let $A \subseteq \alpha$, let Γ be S-thick in M, and let N be the transitive collapse of $H^M(\alpha \cup \Gamma)$. We must show that $A \in N$. Now N is $\Omega + 1$ iterable since it embeds in M, and Ω is S-thick in N. Also, N has the S hull and definability properties at all $\beta < \alpha$. Let $(\mathcal{T},\mathcal{U})$ be a successful coiteration of M with N, with iteration maps $i : M \to W$ and $j : N \to W$. By 5.1, $i \restriction \alpha = j \restriction \alpha =$

identity. Then $A = i(A) \cap \alpha$, so $A \in W$. Since $\text{crit}(j) \geq \alpha$, $A \in N$, as desired. □

Definition 5.3. *Let $\mathcal{M}$ be a set premouse, and let $S \subseteq \Omega$. We say that $\mathcal{M}$ is S-sound iff there is an $\Omega + 1$ iterable weasel W such that*
(1) $\mathcal{M} \trianglelefteq W$,
(2) *Ω is S-thick in W, and*
(3) *W has the S-definability property at all $\beta \in OR \cap \mathcal{M}$.*

Condition (3) of 5.3 is equivalent to: for every S-thick Γ, $\text{OR} \cap \mathcal{M} \subseteq H^W(\Gamma)$. This is simply because if β is least such that $\beta \notin H^W(\Gamma)$, then $\beta \notin H^W(\beta \cup \Gamma)$. Also, by 5.2, condition (3) implies that W has the S-hull property at all $\beta \leq \text{OR} \cap \mathcal{M}$.

Corollary 5.4. *Let $\mathcal{M}$ and $\mathcal{N}$ be S-sound; then either $\mathcal{M} \trianglelefteq \mathcal{N}$ or $\mathcal{N} \trianglelefteq \mathcal{M}$.*

Proof. Let W and R be weasels witnessing the S-soundness of $\mathcal{M}$ and $\mathcal{N}$ respectively. Let $i : W \to T$ and $j : R \to T$ be the iteration maps coming from a coiteration using $\Omega + 1$ iteration strategies. Then if $\alpha = \inf(\text{OR}^{\mathcal{M}}, \text{OR}^{\mathcal{N}})$, Lemma 5.1 implies $i \restriction \alpha = j \restriction \alpha = \text{identity}$. This means that $\mathcal{M} \trianglelefteq \mathcal{N}$ or $\mathcal{N} \trianglelefteq \mathcal{M}$. □

Let $S \subseteq \Omega$ be such that, for some $\Omega + 1$ iterable weasel W, Ω is S-thick in W. Clearly, there are many S-sound premice: $\mathcal{J}_\omega^W$ is an example, and $\mathcal{J}_\alpha^W$ for $\alpha = \omega_1^W$ is a slightly less trivial one. By 5.4 there is a proper premouse $\mathcal{R}$ such that the S-sound mice are precisely the proper initial segments of $\mathcal{R}$. We now give an alternative construction of $\mathcal{R}$, one which shows that it is embeddable in W.

Definition 5.5. *Suppose Ω is S-thick in W. Then we put*

$$x \in \mathit{Def}(W, S) \Leftrightarrow \forall \Gamma (\Gamma \text{ is } S\text{-thick in } W \Rightarrow x \in H^W(\Gamma)).$$

Clearly, $\text{Def}(W, S) \prec W$. (More precisely, $\text{Def}(W, S)$ is the universe of an elementary substructure of W. Recall here that the language of W includes a predicate $\dot{E}$ for its extender sequence. Thus a more careful statement would be that $(\text{Def}(W, S), \in \restriction \text{Def}(W, S), \dot{E}^W \cap \text{Def}(W, S))$ is an elementary submodel of W.)

We now show that, up to isomorphism, $\text{Def}(W, S)$ is independent of W.

Lemma 5.6. *Let Ω be S-thick in W, an let $i : W \to Q$ be the iteration map coming from an iteration tree on W; then $i'' \mathit{Def}(W, S) = \mathit{Def}(Q, S)$.*

Proof. Let $\Delta = \{\gamma < \Omega \mid i(\gamma) = \gamma\}$, so that Δ is S-thick in both W and Q. Suppose first $x \in \text{Def}(W, S)$. Let Γ be S-thick in Q; then $\Gamma \cap \Delta$ is S-thick in W, so $x = \tau^W(\bar{\beta})$ for some $\bar{\beta} \in (\Gamma \cap \Delta)^{<\omega}$ and term τ. But then $i(x) = \tau^Q(\bar{\beta})$, so $i(x) \in H^Q(\Gamma)$. As Γ was arbitrary, $i(x) \in \text{Def}(Q, S)$.

Suppose next that $y \in \text{Def}(Q, S)$. Since Δ is S-thick in Q, we can find $\bar{\beta} \in \Delta^{<\omega}$ so that $y = \tau^Q(\bar{\beta})$ for some term τ. Then $y = i(x)$, where $x = \tau^W(\bar{\beta})$.

Now let Γ be S-thick in W. Then $\Gamma \cap \Delta$ is S-thick in Q, and so $i(x) = \tau^Q(\bar{\alpha})$ for some term τ and $\bar{\alpha} \in (\Gamma \cap \Delta)^{<\omega}$. But then $x = \tau^W(\bar{\alpha})$, and since Γ was arbitrary, we have $x \in \mathrm{Def}(W, S)$. □

Corollary 5.7. *Let P and Q be $\Omega+1$ iterable weasels such that Ω is S-thick in each. Then $Def(P, S) \cong Def(Q, S)$.*

Proof. Once again, we are identifying $\mathrm{Def}(P, S)$ with the elementary submodel of P having universe $\mathrm{Def}(P, S)$. To prove 5.7, let $i : P \to W$ and $j : Q \to W$ be given by coiteration; then by 5.6 $\mathrm{Def}(P, S) \cong \mathrm{Def}(W, S) \cong \mathrm{Def}(Q, S)$. □

Definition 5.8. *Suppose there is an $\Omega + 1$ iterable weasel W such that Ω is S-thick in W; then $K(S)$ is the common transitive collapse of $Def(W, S)$ for all such weasels W.*

If there is no $\Omega + 1$ iterable weasel W such that Ω is S-thick in W, then $K(S)$ is undefined.

Lemma 5.9. *Suppose $K(S)$ is defined; then for any set premouse $\mathcal{M}$, $\mathcal{M}$ is S-sound iff $\mathcal{M} \trianglelefteq K(S)$.*

Proof. Let $\mathcal{M}$ be S-sound, as witnessed by the weasel W. Then $\mathrm{OR}^{\mathcal{M}} \subseteq \mathrm{Def}(W, S)$, as one can see by an easy induction on $\beta \in \mathrm{OR}^{\mathcal{M}}$. Thus $\mathcal{M} \subseteq \mathrm{Def}(W, S)$, and since $\mathcal{M}$ is transitive, $\mathcal{M} \trianglelefteq K(S)$.

Conversely, let $\mathcal{M} \trianglelefteq K(S)$. Let R be an $\Omega+1$ iterable weasel such that Ω is S-thick in R, and let $\pi : K(S) \to R$ be elementary with $\mathrm{ran}\ \pi = \mathrm{Def}(R, S)$. Let $\theta = \sup \pi'' \mathrm{OR}^{\mathcal{M}}$, and for each $\alpha \in \theta - \mathrm{ran}\ \pi$, let

$$\Gamma_\alpha = \text{some } S\text{-thick } \Gamma \text{ such that } \alpha \notin H^R(\Gamma)\,.$$

Then $\bigcap_{\alpha<\theta} \Gamma_\alpha$ is S-thick in W, so $\mathrm{Def}(R, S) \subseteq H^R(\bigcap_{\alpha<\theta} \Gamma_\alpha)$, while $H^R(\bigcap_{\alpha<\theta} \Gamma_\alpha) \cap \theta = \mathrm{Def}(R, S) \cap \theta$ by construction. Thus if we set

$$W = \text{transitive collapse of } H^R\left(\bigcap_{\alpha<\theta} \Gamma_\alpha\right)$$

then W is an $\Omega + 1$ iterable weasel with Ω S-thick in W, and $\mathcal{M} \trianglelefteq W$. It is easy to see that W has the S-definability property at all $\beta \in \mathrm{OR}^{\mathcal{M}}$: if not, then letting $\sigma : W \to R$ invert the collapse, we have that R fails to have the S definability property at $\sigma(\beta)$. Since $\beta \in \mathrm{OR}^{\mathcal{M}}$, $\sigma(\beta) = \pi(\beta)$, and since $\pi(\beta) \in \mathrm{Def}(R, S)$, this is a contradiction. Thus W witnesses that $\mathcal{M}$ is S-sound. □

As far as we know, it could happen that $K(S)$ is defined (that is, there is an $\Omega + 1$ iterable weasel W such that Ω is S-thick in W) and yet $K(S)$ is a set premouse, and hence not universal. We now show that if K^c satisfies "there are no Woodin cardinals", then $K(A_0)$, which exists by 2.12 and 3.12, is a universal weasel.

Theorem 5.10. *Suppose that $K^c \models$ there are no Woodin cardinals; then $K(A_0)$ is a weasel, and moreover $(\alpha^+)^{K(A_0)} = \alpha^+$ for μ_0-a.e. $\alpha < \Omega$, so that $K(A_0)$ is universal.*

Proof. We first show that $K(A_0)$ is a weasel, or equivalently, that $\mathrm{Def}(K^c, A_0)$ is unbounded in Ω. So suppose otherwise toward a contradiction. It is easy then to see that there are A_0-thick classes Γ_ξ, for $\xi < \Omega$, such that

$$\xi < \delta \Rightarrow \Gamma_\delta \subseteq \Gamma_\xi \,,$$

and letting

$$b_\xi = \text{least ordinal } \nu \in (H^{K^c}(\Gamma_\xi) - \mathrm{Def}(K^c, A_0)) \,,$$

we have that

$$(\mathrm{Def}(K^c, A_0) \cup \Omega) \subseteq b_0 \text{ and } \xi < \delta \Rightarrow b_\xi < b_\delta \,.$$

By Lemma 4.8, we can fix ν such that $0 < \nu < \Omega$, $\nu = \sup\{b_\xi \mid \xi < \nu\}$, and K^c has the A_0-definability property at ν. Let $c \in \nu^{<\omega}$ and $d \in \Gamma_{\nu+1}$ and τ a term be such that

$$\nu = \tau^{K^c}[c, d] \,.$$

Fix $\xi < \nu$ such that $c \in b_\xi^{<\omega}$, so that

$$\exists c \in b_\xi^{<\omega}(b_\xi < \tau^{K^c}[c, d] < b_{\nu+1}) \,.$$

This is an assertion about b_ξ, d, and $b_{\nu+1}$, all of which belong to $H^{K^c}(\Gamma_\xi)$. Thus we can find $c^* \in (b_\xi \cap H^{K^c}(\Gamma_\xi))^{<\omega}$ such that

$$b_\xi < \tau^{K^c}[c^*, d] < b_{\nu+1} \,.$$

But $b_\xi \cap H^{K^c}(\Gamma_\xi) = \mathrm{Def}(K^c, A_0) \cap \Omega$, so $c^* \in \mathrm{Def}(K^c, A_0)$. This implies $\tau^{K^c}[c^*, d] \in H^{K^c}(\Gamma_{\nu+1})$, and since $\mathrm{Def}(K^c, A_0) \subseteq b_0$, and $b_0 < \tau^{K^c}[c^*, d] < b_{\nu+1}$, this contradicts the definition of $b_{\nu+1}$.

Thus $\mathrm{Def}(K^c, A_0)$ is unbounded in Ω. We claim that, in fact, $\mathrm{Def}(K^c, A_0) \cap \Omega$ has μ_0- measure one. For this it is enough to show that if $\nu < \Omega$ is regular, $\mathrm{Def}(K^c, A_0)$ is unbounded in ν, and K^c has the A_0-definability property at ν, then $\nu \in \mathrm{Def}(K^c, A_0)$. So suppose ν is a counterexample to the last sentence.

For each $\eta \in (\nu + 1) - \mathrm{Def}(K^c, A_0)$, pick an A_0-thick class Γ_η such that $\eta \notin H^{K^c}(\Gamma_\eta)$, and let $\Gamma = \bigcap_\eta \Gamma_\eta$. Let b be the least ordinal in $H^{K^c}(\Gamma)$ which is strictly greater than ν. Fix $\xi \in \mathrm{Def}(K^c, A_0) \cap \nu$ and $d \in \Gamma^{<\omega}$ such that for some $c \in \xi^{<\omega}$ and term τ, $\nu = \tau^{K^c}[c, d]$. Then, as in the proof that $\mathrm{Def}(K^c, A_0)$ is unbounded, for each $\eta \in \mathrm{Def}(K^c, A_0) \cap \nu$ we can find $c_\eta \in \xi^{<\omega} \cap \mathrm{Def}(K^c, A_0)$ such that $\eta < \tau^{K^c}[c_\eta, d] < b$. As ν is regular, we can fix c^* so that $c_\eta = c^*$ for arbitrarily large $\eta < \nu$. But then $\nu \leq \tau^{K^c}[c^*, d] < b$. Since $c^* \in \mathrm{Def}(K^c, A_0) \subseteq H^{K^c}(\Gamma)$, this contradicts the definition of b.

Finally, we show that for μ_0-a.e. ν, Def(K^c, A_0) is unbounded in ν^+. This clearly implies that $(\nu^+)^{K(A_0)} = \nu^+$ for μ_0-a.e. ν, and so completes the proof of 5.10. So suppose not; then we can fix $\nu \in$ Def(K^c, A_0) such that $(v^+)^{K^c} = \nu^+$, K^c has the A_0-hull property at ν, and Def$(K^c, A_0) \cap \nu^+$ is bounded in ν^+. We have then an A_0-thick class Γ such that $H^{K^c}(\Gamma)$ is bounded in ν^+, say by $\delta < \nu^+$.

By the hull property we have a term τ and $d \in \Gamma^{<\omega}$ such that for some $c \in (\nu + 1)^{<\omega}$

$$\delta < \tau^{K^c}[c, d] < \nu^+ .$$

But now, set

$$\eta = \sup\{\tau^{K^c}[c^*, d] \mid c^* \in (\nu + 1)^{<\omega} \wedge \tau^{K^c}[c^*, d] < v^+\} .$$

Then $\delta < \eta < \nu^+$, and $\eta \in H^{K^c}(\Gamma)$ since $\nu, d \in H^{K^c}(\Gamma)$. This contradicts the choice of δ. □

It is very easy to show that, modulo the absoluteness of $\Omega + 1$ iterability, $K(S)$ is absolute under "set" forcing.

Theorem 5.11. *Suppose $K(S)$ is defined, as witnessed by the $\Omega + 1$-iterable weasel W such that Ω is S-thick in W. Let G be V-generic over $\mathbb{P}$, where $\mathbb{P} \in V_\Omega$, and suppose that $V[G] \models W$ is $\Omega + 1$-iterable. Then $V[G] \models$ "$K(S)$ exists, as witnessed by W", and $K(S)^{V[G]} = K(S)^V$.*

Proof. V and $V[G]$ have the same cardinals and cofinalities $> |\mathbb{P}|$; moreover, if $C \in V[G]$ and C is club in some regular $\nu > |\mathbb{P}|$, then $\exists D \in V$ ($D \subseteq C$ and D is club in ν). It follows that for any class $\Gamma \subseteq \Omega$ in $V[G]$

$$V[G] \models \Gamma \text{ is } S \text{ thick in } W \text{ iff } \exists \Delta \subseteq \Gamma (V \models \Delta \text{ is } S\text{ -thick in } W) .$$

This implies that Ω is S-thick in W in $V[G]$, and that Def$(W, S)^{V[G]} =$ Def$(W, S)^V$. Since W is $\Omega + 1$ iterable in $V[G]$ by hypothesis, we get that $K(S)^{V[G]}$ exists and $K(S)^{V[G]} = K(S)^V$. □

We doubt that one can show that $\Omega + 1$-iterability of W is absolute for "set" forcing in the abstract, although we have no counterexample here. It seems likely that one must appeal to the existence of a definable $\Omega + 1$ iteration strategy for W. This will come from a simplicity restriction on the iteration trees on W, which in turn will come from a smallness condition on W. At the one Woodin cardinal level, we can use the following lemma, whose proof is a slight extension of that of 2.4(a).

Lemma 5.12. *Let W be an $\Omega + 1$-iterable (respectively, $(\omega, \Omega + 1)$-iterable) proper premouse such that $W \models$ there are no Woodin cardinals, and let G be V-generic over $\mathbb{P}$, where $\mathbb{P} \in V_\Omega$. Then $V[G] \models W$ is $\Omega + 1$ iterable (respectively, $(\omega, \Omega + 1)$-iterable).*

Proof. We give the proof for $\Omega+1$-iterability. Using the weak compactness of Ω in $V[G]$, it is enough to show that $V[G]$ satisfies: whenever $\mathcal{T}$ is a putative normal, ω-maximal iteration tree on $\mathcal{J}_\alpha^W$, for some W-cardinal $\alpha < \Omega$, and $lh\ \mathcal{T} < \Omega$, then either $\mathcal{T}$ has a last, wellfounded model, or $\mathcal{T}$ has a cofinal wellfounded branch. So suppose $\mathcal{T}$ is a tree on $\mathcal{J}_\alpha^W$ which is a counterexample to this assertion, and let $\mathcal{T}, \mathcal{J}_\alpha^W \in V_\eta[G]$, where $\eta < \Omega$ is an inaccessible cardinal, and $\mathbb{P} \in V_\eta$. By the Löwenheim-Skolem theorem, we have in V a countable transitive M and elementary $\pi : M \to V_\eta$ such that $\mathcal{J}_\alpha^W, \mathbb{P} \in \text{ran}\ \pi$. Let $\pi(\langle \bar{W}, \bar{\mathbb{P}}\rangle) = \langle \mathcal{J}_\alpha^W, \mathbb{P}\rangle$; then M thinks that $\bar{\mathbb{P}}$ has a condition forcing the existence of a "bad" tree on $\bar{W}$. Since M is countable, we can find in V on M-generic filter $\bar{G}$ on $\bar{\mathbb{P}}$ such that $M[\bar{G}] \models \bar{\mathcal{T}}$ is a "bad" tree on $\bar{W}$. Notice that since $\bar{W}$ satisfies "There are no Woodin cardinals", $\bar{\mathcal{T}}$ is simple; moreover, since $\pi : \bar{W} \to \mathcal{J}_\alpha^W$ is elementary, $\bar{\mathcal{T}}$ is "good" in V. Thus $\bar{\mathcal{T}}$ cannot have a last, illfounded model, and $\bar{\mathcal{T}}$ has a unique cofinal wellfounded branch b in V. It is enough for a contradiction to show that $b \in M[\bar{G}]$, and for this it is enough to show $b \in M[\bar{G}][H]$, where H is $M[\bar{G}]$ generic for $\text{Col}(\omega, \max(|\bar{\mathcal{T}}|, |\bar{W}|)^{M[\bar{G}]})$. But now in $M[\bar{G}][H]$ there is a real x which codes $(\bar{\mathcal{T}}, \bar{W})$. Also, $x^\sharp \in M[\bar{G}][H]$, since M is closed under the sharp function on arbitrary sets because it embeds elementarily in V_η. It is a Σ^1_2 assertion about x that $\bar{\mathcal{T}}$ has a cofinal wellfounded branch, this assertion is true in V, and $x^\sharp \in M[\bar{G}][H]$, so this assertion is true in $M[\bar{G}][H]$. As b is unique, this means that $b \in M[\bar{G}][H]$. □

Putting together 5.11 and 5.12, we get

Theorem 5.13. *Suppose $K(S)$ is defined, as witnessed by a weasel W such that $W \models$ there are no Woodin cardinals. Let G be V-generic for $\mathbb{P}$, where $\mathbb{P} \in V_\Omega$. Then $V[G] \models$ "$K(S)$ is defined, as witnessed by W", and $K(S)^{V[G]} = K(S)^V$.*

Corollary 5.14. *Suppose $K^c \models$ there are no Woodin cardinals, and let G be V-generic over $\mathbb{P} \in V_\Omega$. Then $V[G] \models$ "$K(A_0)$ is defined, as witnessed by $(K^c)^V$; moreover $(\alpha^+)^{K(A_0)} = \alpha^+$ for μ_0- a.e. $\alpha < \Omega$".*

Let us observe in passing that if there is an $\Omega+1$ iterable weasel W such that Ω is S-thick in W, for some S, and $W \models$ there are no Woodin cardinals, then in fact $K^c \models$ there are no Woodin cardinals. [Sketch: If $K^c \models$ there is a Woodin cardinal, then its coherent sequence is of size $< \Omega$. Let $(\mathcal{T},\mathcal{U})$ be a terminal coiteration of K^c with W, using an iteration strategy on the W side and picking unique cofinal branches on the K^c side. $(\mathcal{T},\mathcal{U})$ cannot be successful, since otherwise the K^c side would have iterated past W, contrary to $(\alpha^+)^W = \alpha^+$ for stationary many α. Thus it must be that $\mathcal{T}$ has no cofinal wellfounded branch. The existence of generic branches for trees on K^c then implies $\delta(\mathcal{T})$, the sup of the lengths of the extenders used in $\mathcal{T}$, is Woodin in an iterate of W, a contradiction.] Thus we can add to the conclusion of 5.14: $(K^c)^{V[G]} \models$ there are no Woodin cardinals. We are not sure whether

Ω is $(A_0)^V$-thick in $(K^c)^{V[G]}$, however. We now show that, if there is an $(\omega, \Omega+1)$-iterable weasel, then there is at most one weasel of the form $K(S)$. First, let us note:

Lemma 5.15. *If there is an $(\omega, \Omega+1)$-iterable universal weasel, then every $\Omega+1$-iterable proper premouse is $(\omega, \Omega+1)$-iterable.*

Proof. Let W be universal and Σ an $(\omega, \Omega+1)$-iteration strategy for W. Let $\mathcal{M}$ be an $\Omega+1$ iterable premouse. By coiteration, we obtain a normal iteration tree $\mathcal{T}$ on W which is a play of round 1 of $\mathcal{G}^*(W, (\omega, \Omega+1))$ according to Σ, with last model $\mathcal{P}$, and an elementary $\pi : \mathcal{M} \to \mathcal{P}$. But then $\mathcal{P}$ is $(\omega, \Omega+1)$-iterable, and so by 2.9, so is $\mathcal{M}$. □

The next lemma says that, except possibly for its ordinal height, $K(S)$ is independent of S.

Lemma 5.16. *Suppose there is an $(\omega, \Omega+1)$-iterable universal weasel, and that S and T are stationary sets such that $K(S)$ and $K(T)$ exist. Then $K(S) \trianglelefteq K(T)$ or $K(T) \trianglelefteq K(S)$. In particular, if $K(S)$ and $K(T)$ are weasels, then $K(S) = K(T)$.*

Proof. Let $\mathcal{M}$ be S-sound, as witnessed by W, and let $\mathcal{N}$ be T-sound, as witnessed by R. We assume without loss of generality that $\mathrm{OR}^{\mathcal{M}} \leq \mathrm{OR}^{\mathcal{N}}$. W and R are $(\omega, \Omega+1)$-iterable by Lemma 5.15.

By Theorem 3.7 (1), for all but non-stationary many $\alpha \in S \cup T$, $(\alpha^+)^R = (\alpha^+)^W = \alpha^+$. Now let W^* be the (linear) iterate of W obtained by taking an ultrapower by the order zero total measure on α from W, for each $\alpha \in T - \mathrm{OR}^{\mathcal{M}}$ such that $W \models \alpha$ is measurable. Similarly, let R^* be obtained from R by taking an ultrapower by the order zero measure on α at each $\alpha \in S - \mathrm{OR}^{\mathcal{N}}$ such that $R \models \alpha$ is measurable. Then W^* and R^* still witness the S and T soundness of $\mathcal{M}$ and $\mathcal{N}$, respectively. Moreover, Ω is $S \cup T$ thick in each of W^* and R^*.

Let $i : W^* \to Q$ and $j : R^* \to Q$ come from coiteration. Let $\kappa = \min(\mathrm{crit}(i), \mathrm{crit}(j))$. It is enough to show that $\mathrm{OR}^{\mathcal{M}} \leq \kappa$, for then $\mathcal{M} \trianglelefteq \mathcal{N}$ as desired, so assume that $\kappa < \mathrm{OR}^{\mathcal{M}}$.

Suppose that $\kappa = \mathrm{crit}(i) < \mathrm{crit}(j)$. Since Ω is T-thick in R^* and W^*, and $\kappa \in \mathrm{Def}(R^*, T)$, we can find a term τ and common fixed points $\alpha_1 \cdots \alpha_k$ of i and j so that $\kappa = \tau^{R^*}[\bar{\alpha}]$. But then $\kappa = j(\kappa) = \tau^Q[\bar{\alpha}] = i(\tau^{W^*}[\bar{\alpha}])$, so $\kappa \in \mathrm{ran}(i)$, a contradiction. Similarly, we get $\mathrm{crit}(i) \leq \mathrm{crit}(j)$, so $\mathrm{crit}(j) = \mathrm{crit}(i) = \kappa$.

A similar argument with the hull property gives the usual contradiction. let $A \subseteq \kappa$ and $A \in W^*$. We have a term τ and common fixed points $\bar{\alpha}$ of i and j such that $A = \tau^{W^*}[\bar{\alpha}] \cap \kappa$, using here that W^* has the S-hull property as κ and Ω is S-thick in R^*. Then $i(A) = \tau^Q[\bar{\alpha}] \cap i(\kappa)$, so $\tau^Q[\bar{\alpha}] \cap \kappa = \tau^{R^*}[\bar{\alpha}] \cap \kappa = A$, and $j(A) = \tau^Q[\bar{\alpha}] \cap j(\kappa)$. Thus $i(A)$ and $j(A)$ agree below $\min(i(\kappa), j(\kappa))$. This implies that the extenders used first on the branches of the two trees in our coiteration which produced i and j are compatible with one another. This is a contradiction. □

Definition 5.17. *Suppose there is an $(\omega, \Omega+1)$ iterable universal weasel, and that $K(S)$ exists for some S; then we say that K exists, and define K to be the unique proper premouse $\mathcal{M}$ such that $\forall \mathcal{P}, S$ ($\mathcal{P}$ is S-sound $\Leftrightarrow \mathcal{P} \trianglelefteq \mathcal{M}$).*

We do know whether it is consistent with the definitions we have given that K exists, but is only a set premouse or a non-universal weasel. If we assume that $K^c \models$ there are no Woodin cardinals, then K exists by 2.12, 3.6, and 3.12; moreover K is universal by 5.10. We summarize what we have proved about K under this "no Woodin cardinals" assumption:

Theorem 5.18. *Suppose $K^c \models$ there are no Woodin cardinals; then*

(1) *K exists, and is $(\omega, \Omega + 1)$ iterable,*

(2) *$(\alpha^+)^K = \alpha^+$ for μ_0- a.e. $\alpha < \Omega$, and*

(3) *if G is V-generic/$\mathbb{P}$, for some $\mathbb{P} \in V_\Omega$, then $V[G] \models$ "K exists, is $(\omega, \Omega+1)$ iterable, and $(\alpha^+)^K = \alpha^{+'}$ for μ_0- a.e. $\alpha < \Omega$"; moreover $K^{V[G]} = K^V$.*