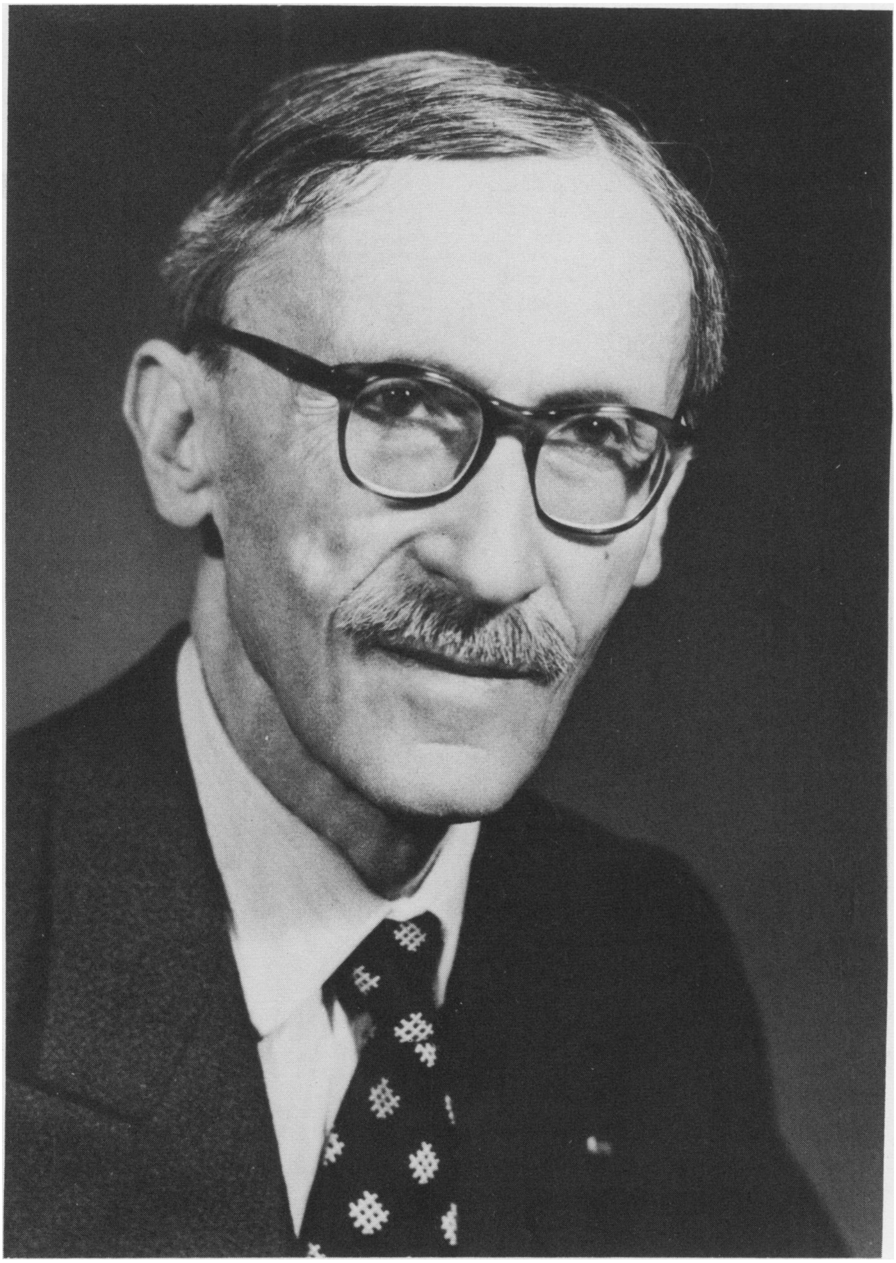

PAUL LÉVY
1886–1971

At a time when the composition of the present volume was already far advanced, we heard, with sorrow, that Paul Lévy died on December 15, 1971. The printing schedule did not permit us to present here a memorial article appropriate to the unique stature of Lévy in the probabilistic field. The following text offers only a few indications on Lévy's career and his work. A more extensive article, with a bibliography, is expected to appear in the first issue of the *Annals of Probability*.

We are greatly indebted to Mrs. Lévy and Professor Laurent Schwartz for communicating to us documents on Paul Lévy's career.

Ed.

Paul Lévy was born in Paris on September 15, 1886, in a family with a strong traditional link with mathematics. Both his father, Lucien Lévy, and his grandfather were professors of mathematics.

In 1904, Paul Lévy secured first place at the competitive examination for entrance to the École Normale Superieure. He also placed second in the École Polytechnique examination. For various reasons, Lévy elected to join the École Polytechnique, which then enjoyed the majestic presence of Henri Poincaré. This was a surprising choice for someone interested in mathematical research, but it may by chance have saved Lévy's life by placing him in a less vulnerable position during the first world war. After leaving Polytechnique, Lévy became an engineering student at the École des Mines (Paris) and finally, in 1910, Professor at the École des Mines of Saint Étienne. Paul Lévy's doctoral thesis was accepted in 1911 by a jury consisting of Poincaré, Hadamard, and E. Picard. He joined the École Polytechnique in 1913 and replaced G. Humbert as Professeur d'Analyse in 1920. Lévy kept this position until 1959.

In spite of poor health, Lévy continued an uninterrupted stream of mathematical activity almost up to the time of his death on December 15, 1971. This activity resulted in the publication of about 260 scientific papers. In addition, Lévy published four scientific monographs as follows:

(i) *Leçons d'Analyse Fonctionnelle*, Gauthier Villars, Paris, 1922 (439 pages), 2nd edition, 1951, with the title, *Problèmes Concrets d'Analyse Fonctionnelle*.

(ii) *Calcul des Probabilités*, Gauthier Villars, Paris, 1925 (350 pages).

(iii) *Théorie de l'Addition des Variables Aléatoires*, Gauthier Villars, Paris, 1937 (17 + 328 pages), 2nd edition, 1954.

(iv) *Processus Stochastiques et Mouvement Brownien*, Gauthier Villars, Paris, 1948 (438 pages), 2nd edition, 1965.

Paul Lévy also published in 1935 and in 1964 Notices on his scientific work and in 1970 an autobiographical volume, *Quelques Aspects de la Pensée d'un Mathematicien* (Paris, A. Blanchard, 1970, 222 pages). This last book gives a charming and very candid description of Lévy's mathematical thoughts from his independent discovery in 1902 of von Koch's tangentless continuous curve up to his hesitant philosophical speculations on Paul Cohen's proof of the indecidability of the continuum hypothesis.

Paul Lévy's own work has had immense influence especially on the developments in the field of probability and stochastic processes. His magnificent *Théorie de l'Addition des Variables Aléatoires* was published 35 years ago. Since then the field of probability has undergone a period of unbridled expansion. In spite of this the reader who will take the trouble to ponder Lévy's writings of the thirties will still find them permeated with unbelievable and yet unexhausted treasures. These writings, in beautiful fluent classical French, do not have the dry formal structure younger mathematicians have been trained to expect. Lévy, with his powerful intuition, seemed to be able to "see" the sample functions of stochastic processes or the fluctuations in a sequence of random variables. They were intimate friends which he described, pausing only from time to time to state a more formal proposition. This does not make his proofs any less rigorous than the more usual ones, but it may prevent the hurried reader from assimilating the richness of the thought and even prompted Fréchet to comment that "Your results are more or less complex according to one's own perspective."

A description of the specifics of Lévy's work would be worth a volume itself. We shall give here only a few comments on some of the highlights.

Paul Lévy's early work, represented for instance by his doctoral thesis in 1911 and culminating in the 1922 volume on *Analyse Fonctionnelle*, revolves around the extension to an infinity of dimensions of the classical theorems relative to first and second order partial differential equations. This was influenced by Volterra's style of study of "functions of lines" and Hadamard's questions concerning the manner in which Green's functions depend on the contours of the domain. Paul Lévy was able to extend many of the results relative to first order partial differential equations, but the second order equations led to problems of a very different nature. The study of the Laplacian in Hilbert space led both Gâteaux and Lévy to introduce independently the idea of mean values taken on balls or other convex subsets of Hilbert space. Gâteaux, who was killed at the beginning of World War I, left unpublished manuscripts. Hadamard gave Lévy the task of rewriting them for publication. Some of the unsolved questions raised by these papers were combined with Lévy's own ideas and at least partially solved in the 1922 volume. This work of Lévy is not well known today, perhaps because functional analysis took around that time a rather more abstract direction with the introduction of Banach spaces and the ensuing emphasis on the general theory of linear operations, and perhaps because the extension of the Laplacian to Hilbert space by Lévy and Gâteaux turned out to be different from that needed in quantum field theory. The rather deep aspects of the geometry of Hilbert space studied by Lévy still remain to be fully explored. One can mention, as part of the incidental contributions of Lévy, his 1919 description of Lebesgue measure on the infinite dimensional cube carried out without knowledge of

Fréchet's 1915 paper on general σ-additive set functions and independently of the concurrent work of P. J. Daniell.

In 1919 Lévy was requested to give three lectures on the Calculus of Probabilities. This incident was to change radically the direction of Lévy's work and the field of probability itself. The general shape of the field at that time was not particularly resplendent, and the texts available to Lévy were even worse, ignoring in particular the works of Chebyshev and Liapounov. In this state of affairs, Lévy set out to prove the proposition suggested by Laplace and Poincaré that an error which is a sum of many independent terms will have a distribution close to the Gaussian, unless the maximum term yields a substantial contribution. With the exception of a conjecture which was to be proved by Cramér in 1936, Lévy achieved his stated goal in 1934.

As a first step, Lévy had to rethink what was meant by random variables and their distributions. For the description of these latter he introduces, independently of von Mises, the idea of cumulative distribution function and also an essentially measure-theoretic description relying on ordered countable families of finite partitions. At about the same time Lévy starts using, under the name of characteristic function, the Fourier transform of a probability measure, gives an inversion formula, and proves that the relation between Fourier transform and probability measure is bicontinuous. This theorem, now very familiar, is better appreciated if one remembers that it does not extend to signed measures, that the correspondence is not at all uniformly continuous, and that the only inversion formulas available at that time were encumbered by various restrictions.

With this tool, Lévy proceeds to prove various versions of the central limit theorem and a number of propositions relative to symmetric or asymmetric stable laws. Some of the results including refinements of the previous work of G. Pólya and a version of Lindeberg's proof of the central limit theorem, can be found in Lévy's *Calcul des Probabilités* of 1925. The book also contains a chapter on applications to the kinetic theory of gases and even some discussion of the use of trimmed means for estimation when the errors are not in the domain of attraction of the Gaussian distribution.

It is curious to note that in 1925 Lévy discusses attraction to stable distributions, but considers convergence to the Gaussian distribution exclusively for sums of variables which possess second moments. This restriction is notably absent ten years later in Lévy's work and in Feller's work.

After 1925 Lévy continues to elaborate on the behavior of sums of independent variables, rediscovering and improving in particular Khinchin's law of the iterated logarithm. However, the most spectacular contributions to the subject can be found in three papers: the 1931 paper in *Studia Mathematica*, the voluminous paper in *Journal de Mathématiques*, Vol. 14 (1935), and the two part paper in the *Bulletin des Sciences Mathématiques*, Vol. 59 (1935).

The method, used by Lévy in 1931, of bounding the oscillations of successive sums of independent random variables by the dispersion of the last term and through symmetrization remains one of the most powerful available in this domain. In the independent case, the 1935 papers give necessary and sufficient conditions for a sum of variables to have a distribution close to the Gaussian under the assumption that individually the terms are small. An equivalent result was obtained independently and published at approximately the same date by W. Feller. However, Lévy's description of the problem goes much deeper. He attempts to see what happens if the summands are not individually small and on this occasion formulates and elaborates the consequences of the conjecture that if $X + Y$ is Gaussian so are X and Y. Cramér proved the validity of the conjecture in 1936, thereby clearing up the situation and prompting Lévy to write his memorable monograph of 1937.

This seemed to complete the program Lévy had set for himself in 1919. However, in the meantime, Lévy had enlarged the scope of the investigation to the case of dependent variables. The main results available in this domain were those of S. Bernstein (1927). Lévy attacked the problem with such a flood of new ideas that even today the dust has not settled and it is not unusual to find papers which reprove particular cases of Lévy's results of 1935. It is in these 1935 papers that the convergence theorems for series of independent variables are extended to the martingale case and that the central limit theorem is extended to martingales.

For the latter, Lévy considers martingales differences and then conditional variances. He measures "time" according to the sums of these conditional variances and first proves a theorem concerning martingales stopped at fixed "time." He then proceeds to show that the result remains valid for other stopping rules, provided that the variables so obtained do not differ too much from the fixed "time" sums. For a very recent result in this direction, the reader may consult a paper by A. Dvoretzky in the present *Proceedings* (Volume II). (The names "martingale" and "stopping times" do not occur in these papers. Doob, to whom many of the fundamental results on martingales are due, borrowed that name from J. Ville's analysis of gambling systems.)

A glance at any modern text on probability or any of the standard journals will convince the reader that martingales have now penetrated the bulk of new developments.

Since Paul Lévy had first relied on characteristic functions for the proof of limit theorems, it is worth mentioning that curiously enough the 1937 volume makes very little use of this tool in proving the usual limit theorems. The only place where the Fourier transform appears essentially is in the Cramér-Lévy theorem. Paul Lévy mentioned several times that this was one of the very few instances where he could not obtain a proof by following his intuition. The fact that a nonvanishing entire characteristic function of order two must be Gaussian carries little probabilistic flavor and at the present time no intuitive proof exists.

On the contrary, Lévy's proof of the general central limit theorem retains a lot of intuitive appeal. To make it rigorous, Lévy had first shown that the space of probability measures on the line may be metrized in such a way that if two variables X and Y are such that $\Pr\{|X - Y| > \varepsilon\} \leqq \varepsilon$, then they have distributions differing by at most ε. Prohorov extended this result to separable metric spaces in 1953 and Strassen showed in 1965 that the property Lévy requested of his distance can in fact be taken as a definition of Prohorov's distance.

When looking at sums of independent variables Lévy separates the values they take into a set of small values and a set of large values. Through the use of concentration inequalities, he shows that it is legitimate to treat the two sets as if they were independent. Kolmogorov revived and improved this method around 1955. He and his school have shown that it leads to many deep approximation results.

In 1938, prompted by a question of J. Marcinkiewicz, Lévy returns to the study of the Brownian motion process and publishes two fundamental memoirs on the subject in *Compositio Mathematica* and in the *American Journal of Mathematics*. In these memoirs, one finds the description of the sets of zeros of ordinary Brownian motion, the distributions of first passage times, and their relation to the increasing stable process of exponent 1/2, the arcsine law for percentage of time spent above a level, and a number of other related results including the first definitions and theorems on what is now called the "local time" of the process. The *Compositio* memoir deals only with one dimensional processes.

The *American Journal of Mathematics* memoir considers a particle undergoing Brownian motion in the plane. There, Lévy introduces stochastic integrals to give relations satisfied by the area between the Brownian curve and its chord, and gives results on the measure of the Brownian curve. Many of these results appear in his book of 1948.

Lévy's ideas on "local times" and on stochastic integrals were soon noticed by K. Itô whose book with H. P. McKean (*Diffusion Processes and Their Sample Paths*, Springer-Verlag, Berlin, 1965) can be consulted for more recent development.

Lévy obtained in the early fifties several other groups of results on the measure of n dimensional Brownian motion and on the canonical representations of Laplacian random functions. Part of this, together with a mention of the beautiful results of Dvoretzky, Erdös, and Kakutani, was added to the 1965 edition of the book.

The preceding memoirs refer to stochastic processes which may be multidimensional, or even Hilbert valued, but where the indexing set T is the real line. Around 1945, Lévy starts the study of a very different object. Now the variables are real valued, but the index set T becomes an n dimensional space, or a sphere, or a Hilbert space. The underlying space T has then a metric structure and Lévy requires that the expectation $E|X(s) - X(t)|^2$ be the distance between s and t. The work described in the 1948 volume concerns mostly continuity properties of such a Gaussian process, including iterated logarithm laws.

By 1955 Lévy undertakes a deeper study of the processes and of their averages on spheres of varying radius and shows in particular that if T is infinite dimensional Hilbert space, the process is already determined by its values in arbitrarily small balls. The 1965 edition of *Processus Stochastiques* gives a summary of the main results obtained by Lévy between 1955 and 1963, including a modification of an integral representation formula of Chentsov and mention of improvements on Lévy's results due to T. Hida and H. P. McKean.

Another important part of Lévy's work started in 1950 as a result of a conversation with K. L. Chung. The 1951 memoir in the *Annales de l'École Normale Superieure* gives a classification of the states of Markov processes with a countable state space, together with constructions of various Markov processes exhibiting all sorts of curious and seemingly pathological behavior. It is a peculiar trait of Lévy's psychology that since his mind would follow the development in time of the trajectories, he could not conceive that one would dare call "Markov process" a process which was not strongly Markovian. This should be kept in mind while reading this memoir, which in spite of difficulties pointed out by K. L. Chung still gives a wonderful intuitive view of the situation.

The above does not exhaust Lévy's contributions to Probability, nor does it give justice to Lévy's mathematical work in other directions. Paul Lévy seems to have been the first to represent a stochastic process by a curve in Hilbert space. He also initiated the study of composition of random variables under laws other than addition and was led in this way to look at diffusion processes on a circle or a sphere. In pure analysis his early work includes several notes on the theory of functions of real or complex variables. Perhaps one of his most striking contributions to analysis is the famous theorem, often called theorem of Wiener-Lévy, according to which if a Fourier series converges absolutely to, say f, and if ϕ is analytic on the range of f, then the series of $\phi(f)$ also converges.

It is obviously not possible to give a complete description of the work here. The interested reader may find it pleasant and profitable to read Paul Lévy's own candid description in his 1970 essay on "Quelques aspects de la pensée d'un mathematicien." The probabilistically inclined reader will certainly find much food for thought in Paul Lévy's original papers. Their author is no longer with us, but his works will remain one of the monuments of the Calculus of Probability.

L. Le Cam